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Maximal saddle solution of a nonlinear elliptic equation involving the p-Laplacian

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Abstract. A saddle solution is called maximal saddle solution if its absolute value is not smaller than those absolute values of any solutions that vanish on the Simons cone $\mathcal{C} = \{s = t\}$ and have the same sign as $s - t$. We prove the existence of a maximal saddle solution of the nonlinear elliptic equation involving the p-Laplacian, by using the method of monotone iteration,

$$-\Delta_p u = f(u) \quad \text{in} \quad R^{2m},$$

where $2m \geq p > 2$.

Keywords. p-Laplacian; maximal saddle solution; monotone iteration methods.

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1. Introduction and main results

In this paper we study the maximal saddle solution of nonlinear elliptic equation involving the p-Laplacian operator

$$-\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in} \quad R^{2m}, \tag{1.1}$$

where $2m \geq p > 2$. Nonlinearity $f(u)$ satisfies several conditions (1.7)–(1.10) as stated at the end of this section.

Saddle solutions were first studied by Dang *et al.* [6] for equation $-\Delta u = f(u)$ in R^2, with f odd, bistable and $f(u)/u$ decreasing for $u \in (0, 1)$. They proved the existence and uniqueness of a saddle solution. They also established monotonicity properties and the asymptotic behavior of the saddle solution. Schatzman [10] studied in detail its instability, which was already indicated in a partial result of [6]. The nondegeneracy of the saddle solution was proved by Kowalczyk and Liu in [9]. Alama *et al.* [1] studied vector-valued saddle solutions in R^2. The article [2] concerns scalar saddle type solutions in R^2 changing sign on more nodal lines than $x_1 = \pm x_2$. Recently, Cabré and Terra [3] proved the existence of saddle solutions of bistable diffusion equations $-\Delta u = f(u)$ in all even dimensions and the instability of saddle solutions in R^4. Cabré and Terra in [4] also obtained the instability of saddle solutions in R^6, by establishing the so-called maximal saddle solution and its monotonicity. They also proved that there exists a minimal saddle solution in [4]. In the recent paper [5], Cabré obtained the uniqueness of saddle solutions of the above bistable diffusion equations in all even dimensions and the stability of saddle solutions in R^{2m}, whenever $2m \geq 14$. Extending the Laplace operator to p-Laplace operator, Lai *et al.* [8] proved the existence of saddle solutions of (1.1) in R^{2m} with dimensions $2m \geq 2p$. In [14] we established the existence of saddle solutions of (1.1) in R^{2m} with more dimensions $2m \geq p$. In this paper, we will show that there is a maximal saddle solution of (1.1) in even-dimensional spaces R^{2m} with $2m \geq p$ by using monotone iteration methods.

Before introducing the notion of saddle solution, we give the weak solution notion of (1.1).

DEFINITION 1.1

We say that u is a bounded weak solution of (1.1) if $u \in W^{1,p}_{\rm loc}(R^{2m}) \cap L^\infty(R^{2m})$, and

$$\int_{R^{2m}} [|\nabla u|^{p-2}\nabla u \cdot \nabla \eta - f(u)\eta] \mathrm{d}x = 0, \quad \text{for all } \eta \in W^{1,p}_0(R^{2m}). \tag{1.2}$$

Saddle solution notion is closely related to the Simons cone. The Simons cone is defined by

$$\mathcal{C} = \{x \in R^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\}.$$

We define two radial variables s and t by

$$s = \sqrt{x_1^2 + \cdots + x_m^2} \geq 0, \quad t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2} \geq 0. \tag{1.3}$$

Then the Simons cone is given by $\mathcal{C} = \{s = t\}$. It is easy to verify that $\mathcal{C}$ has zero mean curvature at every $x \in \mathcal{C}\backslash\{0\}$, in every even dimension $2m \geq 2$. The following is the precise notion of saddle solution, see [3].

DEFINITION 1.2

Let $f \in C^1(R)$ be odd. We say that u is a saddle solution of (1.1), if u is a bounded weak solution of (1.1) and

(1) u depends only on the variables s and t. We write $u = u(s, t)$;
(2) $u > 0$ in $\mathcal{H} := \{s > t\}$;
(3) $u(s, t) = -u(t, s)$ in R^{2m}.

Condition (3) implies that every saddle solution vanishes on the Simons cone $\mathcal{C} = \partial\mathcal{H} = \{s = t\}$.

Equation (1.1) is the Euler–Lagrange equation associated to the energy functional

$$J(v, \Omega) = \int_\Omega \left(\frac{|\nabla v|^p}{p} + F(v)\right) \mathrm{d}x, \tag{1.4}$$

where $F' = -f$ and $\Omega \subset R^{2m}$ is a bounded domain. From the energy J we may give the definition of semi-stability.

DEFINITION 1.3

Suppose $f \in C^1(R)$ and $p > 2$. Let $\Omega \subset R^{2m}$ be an open set (bounded or unbounded). We say that a bounded weak solution u of $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u)$ in Ω is semi-stable in Ω if the second variation of energy is nonnegative for all ξ with compact support in Ω. That is, if

$$Q_{u,\Omega}(\xi) := \int_{\Omega} (|\nabla u|^{p-2}|\nabla \xi|^2 + (p-2)|\nabla u|^{p-4}|\nabla u \cdot \nabla \xi|^2 \\ - f'(u)\xi^2)\mathrm{d}x \geq 0 \quad \text{for all } \xi \in W_0^{1,p}(\Omega). \tag{1.5}$$

To state our main results in this paper, given a C^1 nonlinearity $f : R \to R$ and $M > 0$, define

$$F(u) = \int_u^M f. \tag{1.6}$$

We have that $F \in C^2(R)$ and $F' = -f$. As in [8], for $p > 2$, we assume that there exist $0 < c < 1 < C$, $\tilde{C} > 0$ and some $w^\star \in (0, M)$ such that F is nonnegative in R. For any $w \in [0, M]$,

$$cw^p \leq F(-M + w) \leq Cw^p, \tag{1.7}$$

$$f \text{ is odd in } R, \quad f'(0) > 0 \text{ and } f \text{ is positive in } (0, M), \tag{1.8}$$

$$\text{for any } w \in [0, w^\star), \quad f(-M + w) \geq -\tilde{C}w^{p-1}, \tag{1.9}$$

$$\frac{f'(w)}{f(w)} \leq \frac{p-1}{w} \quad \text{in } (0, M). \tag{1.10}$$

Note that if conditions (1.7)–(1.9) hold, then F is even in R, $F(M) = F(-M) = 0$ and $F > 0$ in $(-M, M)$, as well as $f(0) = f(\pm M) = 0$ is available. The potentials F are of 'double-well type'. We also point out that, under the conditions (1.7)–(1.9), the nonlinearity f satisfies $f'(M) = 0$ and the following property:

$$\text{there exist a } u_0 \in (0, M) \text{ such that } f'(u_0) = \min_{u_0 \in [0,M]} f'(u) < 0. \tag{1.11}$$

Hence the concavity of f in $(0, M)$ does not hold as in the case of $p = 2$ (see [3]). The condition (1.10) implies that $\frac{f(w)}{w^{p-1}}$ is decreasing in $(0, M)$. Several of our assumptions (1.7)–(1.10) are satisfied for the right nonlinearity in equation $-\Delta_p u = u(1 - u^2)|1 - u^2|^{p-2}$. In this case, $M = 1$ and $F(u) = \frac{|1-u^2|^p}{2p}$, which is the typical double well for the p-Laplacian case, as considered, for instance in [12].

The following results were proven in [8, 14].

PROPOSITION 1.4

Let $p > 2$ and $f \in C^1(R)$ satisfy (1.7)–(1.10) for some constant $M > 0$, where F is defined by (1.6). Then for every even dimension space R^{2m} with $2m \geq p > 2$, there exists a saddle solution u of equation (1.1) satisfying $|u| < M$ in R^{2m}.

Moreover, in the even-dimensional spaces $2m \geq 2p > 4$, the saddle solution u holds energy estimate

$$J(u, B_R) = \int_{B_R} \left(\frac{|\nabla u|^p}{p} + F(u) \right) \mathrm{d}x \leq CR^{2m-1} \quad \textit{for all } R > 1, \tag{1.12}$$

where C is a constant independent of R and B_R denotes the open ball of radius R centered at 0. Furthermore, the second variation of energy $Q_{u,R^{2m}}(\xi)$ at u, as defined in (1.5), is nonnegative for all functions $\xi \in C^1(R^{2m})$ with compact support in R^{2m} and vanishing on the Simons cone $\mathcal{C} = \{s = t\}$.

PROPOSITION 1.5

Let $p > 2$ and $f \in C^1(R)$ satisfy (1.7) and (1.8). If u is a bounded weak solution of $-\Delta_p u = f(u)$ in R^{2m} that vanishes on the Simons cone $\mathcal{C} = \{s = t\}$, then

$$|u(x)| \le |g(\mathrm{dist}(x, \mathcal{C}))| = \left| g\left(\frac{s-t}{\sqrt{2}}\right)\right| \quad \text{for all } x \in R^{2m}, \tag{1.13}$$

where g is the monotone solution of $-(|u'|^{p-2}u')' = f(u)$ in R vanishing at 0 (see Lemma 2.4). *In particular, this statement holds true for every saddle solution.*

In addition, the function $g\left(\frac{s-t}{\sqrt{2}}\right)$ is a supersolution of $-\Delta_p u = f(u)$ in the set $\mathcal{H} = \{s > t\}$.

PROPOSITION 1.6

Let $p > 2$. Assume that $f \in C^1(R)$ satisfies (1.7)–(1.10) and that u_1 and u_2 are any two saddle solutions of (1.1) in even dimension space R^{2m} with $2m \ge p > 2$. Then there exist two saddle solutions u_* and u^* of (1.1) such that

$$u_* \le \min\{u_1, u_2\} \le \max\{u_1, u_2\} \le u^* \quad \text{in} \quad \mathcal{H}. \tag{1.14}$$

As a consequence, we also have

$$|u_*| \le \min\{|u_1|, |u_2|\} \le \max\{|u_1|, |u_2|\} \le |u^*| \quad \text{in} \quad R^{2m}. \tag{1.15}$$

Our main results in this paper are as follows.

Theorem 1.7. *Let $p > 2$ and $f \in C^1(R)$ satisfy (1.7)–(1.10). Then for every even dimension space R^{2m} with $2m \ge p > 2$, there exists a maximal saddle solution $\bar{u}$ of equation (1.1) in the sense: for every bounded weak solution u of $-\Delta_p u = f(u)$ in R^{2m} that vanishes on the Simons cone $\{s = t\}$ and such that u has the same sign as $s - t$, we have*

$$0 < u \le \bar{u} \quad \text{in } \mathcal{H}. \tag{1.16}$$

As a consequence, we also have

$$|u| \le |\bar{u}| \quad \text{in } R^{2m}. \tag{1.17}$$

Moreover $\bar{u} \equiv u^*$, where u^* is given in Proposition 1.6.

2. Preliminaries

This section contain several properties which will be used to prove Theorem 1.7.

We first state a weak comparison theorem for solutions of differential inequalities involving the p-Laplacian operator, see [7].

Theorem 2.1. *Suppose Ω is bounded in R^n and $u, v \in W^{1,\infty}(\Omega)$, $p > 1$, weakly satisfy*

$$-\Delta_p u + h(x, u) \le -\Delta_p v + h(x, v) \quad \text{in } \Omega,$$

where $h \in C(\bar{\Omega} \times R)$ is such that for each $x \in \Omega$, $h(x, \tau)$ is nondecreasing in τ for $|\tau| \le \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$. Let $\Omega' \subseteq \Omega$ be open and suppose $u \le v$ on $\partial\Omega'$, then $u \le v$ in Ω'.

We also need a strong maximum principle proved by Vazquez [13].

Theorem 2.2. *Let Ω be an open connected (not necessarily) bounded set in R^n and suppose that $u \in C^1(\Omega)$, $u \ge 0$ in Ω, weakly satisfies*

$$-\Delta_p u + cu^q = g \ge 0 \quad \text{in } \Omega$$

with $q \ge p - 1$, $c \ge 0$ and $g \in L^\infty_{\text{loc}}$. If u is not identically zero, then $u > 0$ in Ω.

From Theorem 2.2, one can get the following result, see footnote 6 in [11].

Theorem 2.3. *Let $f \in C^1(R)$ satisfy (1.9). Function u weakly solves*

$$-\Delta_p u = f(u) \quad \text{in} \quad \Omega,$$

where Ω is an connected open set in R^n. If $|u| \le M$ and $|u| \not\equiv M$, then $|u| < M$.

The existence of solution to nonlinear equation involving the p-Laplacian in dimension one can be seen in [8].

Lemma 2.4. *Let $F \in C^2(R)$. There exists a bounded function $g \in C^2(R)$ satisfying*

$$(|g'|^{p-2}g')' = F'(g) \quad \text{and} \quad g' > 0 \text{ in } R$$

if and only if there exist two real numbers $m_1 < m_2$ for which F satisfies

$$F'(m_1) = F'(m_2) = 0$$

and

$$F > F(m_1) = F(m_2) \text{ in } (m_1, m_2).$$

In such cases we have $m_1 = \lim_{\tau\to-\infty} g(\tau)$ and $m_2 = \lim_{\tau\to\infty} g(\tau)$. Moreover, the solution $g = g(\tau)$ is unique up to translations of the independent variable τ.

Adding a constant to F, assume that

$$F(m_1) = F(m_2) = 0,$$

then we have that

$$\frac{p-1}{p}|g'|^p = F(g) \quad \text{in } R.$$

Finally we introduce the semi-stability result of some solutions to $-\Delta_p u = f(u)$, see [14].

Lemma 2.5. *Let $f \in C^1(R)$ satisfy (1.7)–(1.10) and $\Omega \subset R^{2m}$ be a open set (bounded or unbounded). Let u be a positive solution of $-\Delta_p u = f(u)$ in Ω such that $0 < u < M$ in Ω. Then u is a semi-stable solution in Ω in the sense of Definition* 1.3.

3. Existence of a maximal saddle solution

We use the method of monotone iteration to prove Theorem 1.7, following the main ideas in [4].

For $R > 0$, set

$$\mathcal{H}_{\mathcal{R}} := \mathcal{H} \cap B_R = \{x \in R^{2m} : s > t\} \cap B_R,$$

where B_R is the open ball in R^{2m} of radius R centered at 0.

We define

$$h(\tau) := f(\tau) - f'(u_0)\tau \quad \text{for } \tau \in [0, M], \tag{3.1}$$

where the nonlinearity f is assumed as in (1.7)–(1.10) and u_0 is the number defined in (1.11). Then h is nonnegative and nondecreasing in $(0, M)$, since $h'(\tau) = f'(\tau) - f'(u_0) \geq 0$ in $(0, M)$ and $h(0) = 0$.

We first establish the existence of a maximal solution in $\mathcal{H}_{\mathcal{R}}$ by using a monotone iteration method.

Lemma 3.1. *Let $f \in C^1(R)$ satisfy (1.7)–(1.9). Then there exists a positive solution $\bar{u}_R$ of*

$$\begin{cases} -\Delta_p \bar{u}_R = f(\bar{u}_R) & \text{in } \mathcal{H}_{\mathcal{R}}, \\ \bar{u}_R = g\left(\dfrac{s-t}{\sqrt{2}}\right) & \text{on } \partial\mathcal{H}_{\mathcal{R}}, \end{cases} \tag{3.2}$$

with $0 < \bar{u}_R < M$ in $\mathcal{H}_{\mathcal{R}}$, which is maximal in the sense: we have that $\bar{u}_R \geq u$ in $\mathcal{H}_{\mathcal{R}}$ for every positive solution u of $-\Delta_p u = f(u)$ in $\mathcal{H}_{\mathcal{R}}$ satisfying $u \leq g\left(\frac{s-t}{\sqrt{2}}\right)$ in all of $\overline{\mathcal{H}_{\mathcal{R}}}$. Moreover, $\bar{u}_R$ depends only on s and t.

Furthermore, if f also satisfies (1.10), then $\bar{u}_R$ is semi-stable in $\mathcal{H}_{\mathcal{R}}$.

Proof. Let g be the solution defined in Lemma 2.4. We write the equation in (3.2) as $[-\Delta_p - f'(u_0)]u = h(u)$, where h is defined in (3.1).

We define a sequence of functions $\bar{u}_{R,k}$ by setting $\bar{u}_{R,0} = g\left(\frac{s-t}{\sqrt{2}}\right)$ and solving the following problems

$$\begin{cases} [-\Delta_p - f'(u_0)]\bar{u}_{R,k+1} = h(\bar{u}_{R,k}) & \text{in } \mathcal{H}_{\mathcal{R}}, \\ \bar{u}_{R,k+1} = g\left(\dfrac{s-t}{\sqrt{2}}\right) & \text{on } \partial\mathcal{H}_{\mathcal{R}}. \end{cases} \tag{3.3}$$

From Theorem 2.1, problem (3.3) admits a unique weak solution $\bar{u}_{R,k+1}(x)$. Moreover (and here we argue by induction), $\bar{u}_{R,k+1}$ depends only on s and t, since the problem and its data are invariant by orthogonal transformations in the first (respectively, in the last) m variables x_i.

Claim. The sequence $\bar{u}_{R,k}$ is nonincreasing in k, namely

$$g\left(\frac{s-t}{\sqrt{2}}\right) = \bar{u}_{R,0} \geq \bar{u}_{R,1} \geq \cdots \geq \bar{u}_{R,k} \geq \bar{u}_{R,k+1} \geq \cdots \geq 0 \quad \text{in } \mathcal{H}_{\mathcal{R}}.$$

Indeed, we have

$$[-\Delta_p - f'(u_0)]\bar{u}_{R,1} = f(\bar{u}_{R,0}) - f'(u_0)\bar{u}_{R,0} \le [-\Delta_p - f'(u_0)]\bar{u}_{R,0} \text{ in } \mathcal{H}_{\mathcal{R}},$$

since $\bar{u}_{R,0} = g\left(\frac{s-t}{\sqrt{2}}\right)$ is a supersolution of $-\Delta_p u = f(u)$ in the set $\mathcal{H} = \{s > t\}$, as stated in Proposition 1.5. By Theorem 2.1, we have $\bar{u}_{R,1} \le \bar{u}_{R,0}$. Similarly, we have

$$[-\Delta_p - f'(u_0)]\bar{u}_{R,1} = f(\bar{u}_{R,0}) - f'(u_0)\bar{u}_{R,0} \ge 0 = [-\Delta_p - f'(u_0)]0 \text{ in } \mathcal{H}_{\mathcal{R}},$$

which yields $\bar{u}_{R,1} \ge 0$.

Assume now that $0 \le \bar{u}_{R,k} \le \bar{u}_{R,k-1}$ in $\mathcal{H}_{\mathcal{R}}$, for some $k \ge 1$. Then, since h is nondecreasing in $(0, M)$, we have

$$[-\Delta_p - f'(u_0)]\bar{u}_{R,k+1} = h(\bar{u}_{R,k}) \le h(\bar{u}_{R,k-1}) = [-\Delta_p - f'(u_0)]\bar{u}_{R,k} \text{ in } \mathcal{H}_{\mathcal{R}}.$$

Again Theorem 2.1 gives $\bar{u}_{R,k+1} \le \bar{u}_{R,k}$. Similarly $\bar{u}_{R,k+1} \ge 0$.

Now, by monotone convergence this sequence converges to a nonnegative weak solution $\bar{u}_R = \bar{u}_R(s,t)$ of (3.2). Since $\bar{u}_R \not\equiv 0$, the strong maximum principle Theorem 2.2 yields $\bar{u}_R > 0$ in $\mathcal{H}_{\mathcal{R}}$. On the other hand, the facts $0 \le \bar{u}_R \le \bar{u}_{R,0} \le M$, $\bar{u}_R \not\equiv M$ and Theorem 2.3 lead to $\bar{u}_R < M$ in $\mathcal{H}_{\mathcal{R}}$.

Now we need to prove that the solution $\bar{u}_R$ is maximal. For every solution $0 < u \le g\left(\frac{s-t}{\sqrt{2}}\right)$ in $\mathcal{H}_{\mathcal{R}}$, assuming that $0 < u \le \bar{u}_{R,k}$ for some $k \ge 0$ (which evidently holds for $k = 0$), we have

$$[-\Delta_p - f'(u_0)]u = h(u) \le h(\bar{u}_{R,k}) = [-\Delta_p - f'(u_0)]\bar{u}_{R,k+1} \quad \text{in } \mathcal{H}_{\mathcal{R}},$$

where we have used the fact that h is nondecreasing in $(0, M)$. Note that $u \le g\left(\frac{s-t}{\sqrt{2}}\right) = \bar{u}_{R,k+1}$ on $\partial\mathcal{H}_{\mathcal{R}}$. Hence $u \le \bar{u}_{R,k+1}$ in $\mathcal{H}_{\mathcal{R}}$. By induction, $u \le \bar{u}_{R,k}$ for all k, which gives $u \le \bar{u}_R$ in $\mathcal{H}_{\mathcal{R}}$.

Finally, the semi-stability of $\bar{u}_R$ in $\mathcal{H}_{\mathcal{R}}$ is the direct result of Lemma 2.5.

Lemma 3.2. *Let $f \in C^1(R)$ satisfy* (1.7)–(1.10). *Then there exists a positive solution $\bar{u}$ of*

$$\begin{cases} -\Delta_p \bar{u} = f(\bar{u}) & \text{in } \mathcal{H}, \\ \bar{u} = 0 & \text{on } \mathcal{C} = \partial\mathcal{H}, \end{cases} \tag{3.4}$$

with $0 < \bar{u} < M$ in $\mathcal{H}$, which is maximal in the sense: we have that $\bar{u} \ge u$ in $\mathcal{H}$ for every bounded weak solution of $-\Delta_p u = f(u)$ in R^{2m} that vanishes on the Simons cone and has the same sign as $s - t$. In addition, $\bar{u}$ depends only on s and t.

Proof. By elliptic estimates and a compactness argument, the limit of solutions of Lemma 3.1, $\lim_{R\to\infty} \bar{u}_R$ exists (up to subsequences) in every compact set of $\overline{\mathcal{H}}$. Making a Cantor diagonal argument, we obtain a sequence $\bar{u}_{R_j}$ converging in $C^1_{\text{loc}}(\mathcal{H})$ to a weak solution $\bar{u}$. Clearly, $0 \le \bar{u} \le M$ in $\mathcal{H}$, $\bar{u} = 0$ on $\mathcal{C} = \partial\mathcal{H}$ and $\bar{u}$ depends only on s and t. From Theorem 2.3, we get $0 \le \bar{u} < M$ in $\mathcal{H}$.

Next we establish the maximality of $\bar{u}$. Let u be a solution as stated in the statement. From Proposition 1.5, we have $u \le g\left(\frac{s-t}{\sqrt{2}}\right)$ in $\mathcal{H}$. Therefore from Lemma 3.1, we get $u \le \bar{u}_R$ in $\mathcal{H}_{\mathcal{R}}$ for all R. Hence $u \le \bar{u}$ in $\mathcal{H}$.

Finally, we claim that $\bar{u} \not\equiv 0$ in $\mathcal{H}$. Indeed, from the maximality of $\bar{u}$ and the existence of saddle solution u in Proposition 1.4, we have that $\bar{u} \ge u > 0$ in $\mathcal{H}$.

Remark 3.3. An alternative proof $\bar{u} \not\equiv 0$ in Lemma 3.2 is to use the semi-stability of $\bar{u}_R$ in $\mathcal{H}_\mathcal{R}$ as given in Lemma 3.1. Namely we have $Q_{\bar{u}_R,\mathcal{H}_\mathcal{R}}(\xi) \geq 0$ for all $\xi \in W_0^{1,p}(\mathcal{H}_\mathcal{R})$. Now, letting $R \to \infty$, we get

$$Q_{\bar{u},\mathcal{H}}(\xi) \geq 0 \quad \text{for all} \quad \xi \in W_0^{1,p}(\mathcal{H}). \tag{3.5}$$

Suppose $\bar{u} \equiv 0$ in $\mathcal{H}$. Then $Q_{\bar{u},\mathcal{H}}(\xi) = Q_{0,\mathcal{H}}(\xi) = -\int_\mathcal{H} f'(0)\xi^2 \mathrm{d}x$. This and the condition $f'(0) > 0$ in (1.8) contradict with (3.5). Hence $\bar{u} \not\equiv 0$ in $\mathcal{H}$.

Now we finish the proof of Theorem 1.7.

Proof. Since f is odd, by odd reflection with respect to $\mathcal{C}$ and Lemma 3.2 we obtain a weak solution $\bar{u}$ in $R^{2m}\backslash\{0\}$ such that

$$|\bar{u}| \geq |u| \quad \text{in } R^{2m}\backslash\{0\}, \tag{3.6}$$

where u is any bounded weak solution of $-\Delta_p u = f(u)$ in R^{2m} that vanishes on the Simons cone $\{s = t\}$ and such that u has the same sign as $s - t$.

Then we need to show that for every even dimension $2m \geq p > 2$, $\bar{u}$ is also a solution in R^{2m}. As in [14], it is enough to show that $\bar{u}$ is also a solution in B_R for some $R > 0$. Since $\bar{u}$ is a solution in $B_R\backslash\{0\}$, we have

$$\int_{B_R} [|\nabla\bar{u}|^{p-2}\nabla\bar{u}\cdot\nabla\eta - f(\bar{u})\eta]\mathrm{d}x = 0, \quad \text{for all } \eta \in W_0^{1,p}(B_R\backslash\{0\}). \tag{3.7}$$

We set $\eta = v(1 - \xi_\varepsilon)$ in (3.7), where $v \in C_c^\infty(B_R)$ and ξ_ε is identically 1 in $B_{\frac{\varepsilon}{2}}(0)$ and vanishes outside of $B_\varepsilon(0)$. Then (3.7) becomes

$$\int_{B_R}(1-\xi_\varepsilon)|\nabla\bar{u}|^{p-2}\nabla\bar{u}\cdot\nabla v\mathrm{d}x - \int_{B_\varepsilon} v|\nabla\bar{u}|^{p-2}\nabla\bar{u}\cdot\nabla\xi_\varepsilon\mathrm{d}x = \int_{B_R} f(\bar{u})v(1-\xi_\varepsilon)\mathrm{d}x. \tag{3.8}$$

We need to show that the second integral on the left-hand side of (3.8) goes to zero as $\varepsilon \to 0$. Indeed, by Hölder inequality, we have

$$\int_{B_\varepsilon} v|\nabla\bar{u}|^{p-2}\nabla\bar{u}\cdot\nabla\xi_\varepsilon\mathrm{d}x \leq C\left(\int_{B_\varepsilon}|\nabla\bar{u}|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{B_\varepsilon}|\nabla\xi_\varepsilon|^p\mathrm{d}x\right)^{\frac{1}{p}}.$$

Since $|\nabla\xi_\varepsilon|^p \leq C/\varepsilon^p$, $|B_\varepsilon| \leq C_R\varepsilon^{2m}$ and $2m \geq p$, the second factor in the right-hand side is bounded independently of ε. At the same time, the first factor in the right-hand side tends to zero as $\varepsilon \to 0$ since $|\nabla\bar{u}|^p$ is integrable in B_R. Hence $\bar{u}$ is also a solution in B_R. Clearly $\bar{u}$ is a saddle solution in R^{2m} and inequality (1.17) holds in whole R^{2m}.

Finally, we show $\bar{u} \equiv u^*$, where u^* is given in Proposition 1.6. From (1.17), we have $|\bar{u}| \geq |u^*|$. On the other hand, since $\bar{u}$ is a saddle solution of (1.1), Proposition 1.6 gives that $|\bar{u}| \leq |u^*|$. Hence $|\bar{u}| \equiv |u^*|$, which yields $\bar{u} \equiv u^*$. □

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