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On approximation of Lie groups by discrete subgroups

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Abstract. A locally compact group G is said to be approximated by discrete subgroups (in the sense of Tôyama) if there is a sequence of discrete subgroups of G that converges to G in the Chabauty topology (or equivalently, in the Vietoris topology). The notion of approximation of Lie groups by discrete subgroups was introduced by Tôyama in *Kodai Math. Sem. Rep.* **1** (1949) 36–37 and investigated in detail by Kuranishi in *Nagoya Math. J.* **2** (1951) 63–71. It is known as a theorem of Tôyama that any connected Lie group approximated by discrete subgroups is nilpotent. The converse, in general, does not hold. For example, a connected simply connected nilpotent Lie group is approximated by discrete subgroups if and only if G has a rational structure. On the other hand, if Γ is a discrete uniform subgroup of a connected, simply connected nilpotent Lie group G then G is approximated by discrete subgroups Γ_n containing Γ. The proof of the above result is by induction on the dimension of G, and gives an algorithm for inductively determining Γ_n. The purpose of this paper is to give another proof in which we present an explicit formula for the sequence $(\Gamma_n)_{n\geq 0}$ in terms of Γ. Several applications are given.

Keywords. Nilpotent Lie group; rational structure; discrete uniform subgroup; lattice subgroup; Chabauty topology; Vietoris topology.

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1. Introduction and statement of results

The following definition is introduced by Tôyama in [24].

DEFINITION 1.1

A locally compact group G is said to be approximated by discrete subgroups if there is a sequence of discrete subgroups $(H_n)_{n\geq 0}$ of G satisfying the following condition:

(A) For any open set O of G, there exists an integer k such that $O \cap H_n \neq \emptyset$, for every $n \geq k$.

Let $\mathfrak{X}$ be the class of locally compact groups approximated by discrete subgroups. A locally compact group belonging to the class $\mathfrak{X}$ will often be called $\mathfrak{X}$-group.

The first results in this direction were obtained by Tôyama.

Theorem 1.2 (Theorem 1 of [24]). *Any connected Lie group with discrete center is not a $\mathfrak{X}$-group.*

Theorem 1.3 (Theorem 2 of [24]). *Every non-commutative compact Lie group is not a $\mathfrak{X}$-group.*

In 1951, Kuranishi proved the following remarkable theorem (Corollary, page 64 of [13]).

Theorem 1.4. *Let G be a Lie group approximated by discrete subgroups. Then G is nilpotent.*

Remark 1.5. A version of Theorem 1.4 was also noted earlier in Theorem 2 of [26].

Let $\mathscr{S}(G)$ denote the space of discrete co-compact subgroup of a Lie group G. We have the following result of Kuranishi (Theorem 2 of [13]).

Theorem 1.6 (**The approximation theorem**). *Let G be a connected, simply connected nilpotent Lie group such that $\mathscr{S}(G) \neq \emptyset$. Then the following statements hold*:

(1) *G is a $\mathfrak{X}$-group.*
(2) *Let $\Gamma \in \mathscr{S}(G)$. Then G is approximated by discrete subgroups $(\Gamma_n)_{n\geq 0}$ containing Γ.*

The next theorem can be easily deduced from Theorem 4 of [13], which was announced without complete proof. This paper gives a simple proof of this result.

Theorem 1.7. *Let G be a connected simply connected nilpotent Lie group. If G is a $\mathfrak{X}$-group, then $\mathscr{S}(G) \neq \emptyset$.*

On the other hand, the proof of statement (2) of Theorem 1.6 is by induction on the dimension of G, and gives an algorithm for inductively determining Γ_n. In this paper, we are interested in the following problem.

Problem 1.8. *Given a connected simply connected nilpotent Lie group with discrete uniform subgroup Γ, construct an explicit sequence $(\Gamma_n)_{n\in\mathbb{N}}$ which satisfies the condition of* (2) *of the approximation theorem.*

The main result of this paper is the following:

Theorem 1.9. *Let G be a connected simply connected nilpotent Lie group and Γ a discrete uniform subgroup of G. For every $n \in \mathbb{N}^*$, let $\Gamma^{\frac{1}{n}}$ be the subgroup of G generated by the set $\{x \in G : x^n \in \Gamma\}$. Then*

(1) *for every $n \in \mathbb{N}^*$, $\Gamma^{\frac{1}{n}} \in \mathscr{S}(G)$;*
(2) *the group G is approximated by $(\Gamma^{\frac{1}{n}})_{n\geq 1}$.*

In the case when the group G is not simply connected, we obtain the following result.

Theorem 1.10. *Let G be a connected nilpotent Lie group. Let $\tilde{G}$ be the universal covering group of G, and Z be a discrete normal subgroup of $\tilde{G}$ such that $G \cong \tilde{G}/Z$. Let π be the canonical projection of $\tilde{G}$ onto G. Let $\Gamma \in \mathscr{S}(\tilde{G})$ such that $Z \cap \Gamma \in \mathscr{S}(Z)$. Then*

(1) *for every $n \geq 1$, $\pi(\Gamma^{\frac{1}{n}}) \in \mathscr{S}(G)$;*
(2) *the group G is approximated by the sequence $(\pi(\Gamma^{\frac{1}{n}}))_{n\geq 1}$.*

Remark 1.11. The hypothesis of the existence of Γ in $\mathscr{S}(\tilde{G})$ such that $Z \cap \Gamma \in \mathscr{S}(Z)$ is necessary for the group G to be $\mathfrak{X}$-group (see Theorem 4 of [13]).

Notations. In this paper we adopt the following notation: Let G be a group. We use the notation $H \leq G$ to mean that H is a subgroup of G. The index of a subgroup H in G is denoted by $[G : H]$. The subgroup of G generated by a subset A is denote by $\langle A \rangle$. Let H, K be two subgroups of G. We denote by $[H, K]$ the subgroup of G that is generated by $\{hkh^{-1}k^{-1}\}$. The derived series $G = \mathcal{D}^0(G) \geq \mathcal{D}^1(G) \geq \cdots$ of G is defined inductively by setting $\mathcal{D}^{n+1}(G) = [\mathcal{D}^n(G), \mathcal{D}^n(G)]$, for every $n \geq 0$. Let $\mathcal{D}(G)$ denote $\mathcal{D}^1(G)$.

2. Reformulation of Definition 1.1

2.1 *Topologies of closed subgroups*

This subsection follows from [5, 10, 11] and [19]. Various topologies on closed subgroups of a locally compact topological group are considered. For the interrelationships between these topologies, see [19].

2.1.1 *Chabauty–Fell topology.* Let X be a locally compact topological space. We denote by $\mathscr{F}(X)$ the space of closed subsets of X equipped with the Chabauty–Fell topology: the open are any meetings of finite intersections of parts of the form:

$$\mathcal{O}_1(K) = \{H \in \mathscr{F}(X) : H \cap K = \emptyset\},$$

$$\mathcal{O}_2(U) = \{H \in \mathscr{F}(X) : H \cap U \neq \emptyset\},$$

where $K \subseteq X$ is compact and U is an open set of X. This is a compact space.

PROPOSITION 2.1 (Lemma E.1.1 of [1])

If X is a locally compact topological space with a countable basis of open sets, the space $\mathscr{F}(X)$ has a countable basis too and moreover it is metrizable.

PROPOSITION 2.2

Let X_1 and X_2 be two locally compact topological spaces. Let $\phi : X_1 \longrightarrow X_2$ be an open continuous mapping. Then the mapping

$$\mathscr{F}(X_2) \longrightarrow \mathscr{F}(X_1), \quad F \longmapsto \phi^{-1}(F)$$

is continuous.

As a consequence of Proposition 2.2 we have the following corollary, where we use $\simeq$ as notation for 'is homeomorphic to'.

COROLLARY 2.3

If $X_1 \simeq X_2$, then $\mathscr{F}(X_1) \simeq \mathscr{F}(X_2)$. More precisely, if $\phi : X_1 \longrightarrow X_2$ is a homeomorphism of topological groups, then

$$\phi^* : \mathscr{F}(X_1) \longrightarrow \mathscr{F}(X_2), F \longmapsto \phi(F),$$

is a homeomorphism.

If $X = G$ is a locally compact group, the space $\mathscr{C}(G)$ of closed subgroups of G is closed in $\mathscr{F}(G)$; in this specific case this topology was introduced by Chabauty [5].

Moreover we can define the topology of geometric convergence on $\mathscr{C}(G)$ (Definition 9.1.1, p. 225 of [23]).

DEFINITION 2.4 (Geometric convergence)

A sequence $(\Gamma_n)_{n\geq 0}$ of closed subgroups converges geometrically to $\Gamma \in \mathscr{C}(G)$ if the two conditions below are satisfied:

(GC1) Let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be an increasing map and let $(x_{\varphi(n)})_{n\geq 0}$ be a sequence in G such that $x_{\varphi(n)} \in \Gamma_{\varphi(n)}$ for any $n \geq 0$. If $(x_{\varphi(n)})_{n\geq 0}$ converges to some $x \in G$, then $x \in \Gamma$.

(GC2) Any point in Γ is the limit of a sequence $(x_n)_{n\geq 0}$ with $x_n \in \Gamma_n$ for each $n \geq 0$.

For the proof of the next result, we refer to Lemma 2 of [12] or Proposition E 1.2 of [1].

Theorem 2.5. *The convergence in the Chabauty topology is equivalent to the geometric convergence.*

The following elementary lemma will be of use in the sequel.

Lemma 2.6 (*Squeeze theorem for sequences*). *Let G be a locally compact group. Let $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ be three sequences of $\mathscr{C}(G)$. Suppose that for all n sufficiently large $A_n \leq B_n \leq C_n$. If $A_n \longrightarrow F$ and $C_n \longrightarrow F$, then $B_n \longrightarrow F$.*

2.1.2 *Vietoris topology.* Let G be a locally compact group, $\mathscr{C}(G)$ the space of closed subgroups of G. The Vietoris topology on $\mathscr{C}(G)$ is the topology on $\mathscr{C}(G)$ which has families of the form

$$\mathcal{D}_1(U) = \{H \in \mathscr{C}(G) : H \subset U\},$$

$$\mathcal{D}_2(V) = \{H \in \mathscr{C}(G) : H \cap V \neq \emptyset\},$$

where U and V run over the open sets of G, as a prebase.

2.1.3 *Topological limits.*

DEFINITION 2.7

Let X be a topological space and $(A_i)_{i\geq 0}$ be a sequence of closed subsets of X. Then $\liminf(A_i)$, $\limsup(A_i)$ and $\lim(A_i)$ are defined as

(1) $x \in \liminf(A_i)$ if and only if for each open neighborhood U of x, $U \cap A_i \neq \emptyset$ for all but finitely many naturel numbers i;
(2) $x \in \limsup(A_i)$ if and only if for each open neighborhood U of x, $U \cap A_i \neq \emptyset$ for infinitely many naturel numbers i;
(3) if $\liminf(A_i) = \limsup(A_i) = A$, then we say that $A = \lim(A_i)$. $\lim(A_i)$ is called the topological limit of the sequence $(A_i)_{i\geq 0}$.

2.2 *Reformulation of Definition* 1.1

Recall the following definition:

DEFINITION 2.8

Let $(A_n)_{n\geq 0}$ be a sequence of $\mathscr{S}(G)$ and g in G. We say that g is a sequential limit point of $(A_n)_{n\geq 0}$ if there exists a sequence $(a_n)_{n\geq 0}$ such that for all $n \in \mathbb{N}$, $a_n \in A_n$ and $\lim_{n\longrightarrow +\infty} a_n = g$.

Definition 1.1 can be reformulated as follows:

PROPOSITION 2.9

Let G be a locally compact group. The following assertions are equivalent:

(1) *the group G is approximated by the discrete subgroups* $(\Gamma_n)_{n\geq 0}$;
(2) *the sequence* $(\Gamma_n)_{n\geq 0}$ *converges to G in the Chabauty topology*;
(3) *the sequence* $(\Gamma_n)_{n\geq 0}$ *converges to G in the Vietoris topology*;
(4) *the group G is the topological limit of the sequence* $(\Gamma_n)_{n\geq 0}$;
(5) *G is the set of all sequential limit points of* $(\Gamma_n)_{n\geq 0}$.

Proof. The proof is trivial. □

3. Rational structures and uniform subgroups

In this section we present some results on discrete uniform subgroups of nilpotent Lie groups.

3.1 *Rational structures*

Let G be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a *rational structure* if there is a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $(X_1, \ldots, X_n)$ with rational structure constants.

3.2 *Uniform subgroups*

A discrete subgroup Γ is called *uniform* in G if the quotient space G/Γ is compact. We denote by $\mathscr{S}(G)$ the space of all discrete uniform subgroups of G.

A proof of the next result can be found in Theorem 7 of [16] or in Theorem 2.12 of [21].

Theorem 3.1 (The Malcev rationality criterion). *Let G be a simply connected nilpotent Lie group. Then G admits a discrete uniform subgroup Γ if and only if G admits rational structure.*

More precisely, if G has a uniform subgroup Γ, then $\mathfrak{g}$ has a rational structure such that

$$\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Q},\Gamma} = \mathbb{Q}\text{-span}\,\{\log(\Gamma)\}\,.$$

Conversely, if $\mathfrak{g}$ has a rational structure given by some $\mathbb{Q}$-algebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$, then G has a uniform subgroup Γ such that $\log(\Gamma) \subset \mathfrak{g}_{\mathbb{Q}}$ (see [7] or [16]).

3.3 *Strong Malcev basis*

DEFINITION 3.2 [7]

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathscr{B} = (X_1, \ldots, X_n)$ be a basis of $\mathfrak{g}$. We say that $\mathscr{B}$ is a strong Malcev basis for $\mathfrak{g}$ if $\mathfrak{g}_i = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_i\}$ is an ideal of $\mathfrak{g}$ for each $1 \le i \le n$.

DEFINITION 3.3

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \in \mathscr{S}(G)$. A strong Malcev basis $(X_1, \ldots, X_n)$ for $\mathfrak{g}$ is said to be *strongly based on* Γ if

$$\Gamma = \exp(\mathbb{Z}X_1) \cdots \exp(\mathbb{Z}X_n).$$

PROPOSITION 3.4 (Theorem 5.1.6 of [7], [17])

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma \in \mathscr{S}(G)$. Then $\mathfrak{g}$ has a strong Malcev basis strongly based on Γ.

3.4 *Rational subgroups*

DEFINITION 3.5 (Rational subgroup)

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We suppose that $\mathfrak{g}$ has a rational structure given by $\mathfrak{g}_{\mathbb{Q},\Gamma} = \mathbb{Q}\text{-span}\{\log(\Gamma)\}$, where $\Gamma \in \mathscr{S}(G)$.

(1) Let $\mathfrak{h}$ be an $\mathbb{R}$-subspace of $\mathfrak{g}$. We say that $\mathfrak{h}$ is Γ-*rational* if $\mathfrak{h} = \mathbb{R}\text{-span}\left\{\mathfrak{h}_{\mathbb{Q},\Gamma}\right\}$, where $\mathfrak{h}_{\mathbb{Q},\Gamma} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q},\Gamma}$.
(2) A connected, closed subgroup H of G is called Γ-*rational* if its Lie algebra $\mathfrak{h}$ is rational.
(3) A connected, closed subgroup H of G is called *rational* if H is Γ-*rational* for certain discrete uniform subgroup Γ of G.

Remark 3.6. The $\mathbb{R}$-span and the intersection of rational subspaces are rational (Lemma 5.1.2 of [7]).

DEFINITION 3.7 (Subgroup with good Γ-heredity [20])

Let Γ be a discrete uniform subgroup in a locally compact group G and H a closed subgroup in G. We say that H is a subgroup with good Γ-heredity if the intersection $\Gamma \cap H$ is a discrete uniform subgroup of H.

Theorem 3.8 (Lemma A.5 of [8]). *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, let Γ be a discrete uniform subgroup of G, and give $\mathfrak{g}$ the rational structure $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}$-span $\{\log(\Gamma)\}$. Let H be a Lie subgroup of G. Then the following statements are equivalent*:

(1) *H is Γ-rational*;
(2) *H is a subgroup with good Γ-heredity*;
(3) *the group H is Γ-closed* (*i.e.*, *the set $H\Gamma$ is closed in G*).

The proof of the following theorem can be found in Theorem 4.7 of [20].

Theorem 3.9. *Suppose that Γ is a discrete uniform subgroup in a locally compact group G, H a closed normal subgroup of G, $p : G \longrightarrow G/H$ the canonical homomorphism. The subgroup $p(\Gamma)$ is a discrete uniform subgroup in G/H if and only if H is a subgroup with good Γ-heredity.*

Two subgroups of a group are called *commensurable* if their intersection has finite index in both of them. Being commensurable is an equivalence relation on the set of subgroups of a given group. In the next results we show that the property of being a subgroup with good Γ-heredity is invariant under the commensurability class of Γ.

PROPOSITION 3.10

Let G be a locally compact group. Let Γ_1 and Γ_2 be two discrete uniform subgroups of G such that $\Gamma_1 \leq \Gamma_2$. Let H be a closed subgroup of G. Then the following statements are equivalent:

(1) *H is a subgroup with good Γ_1-heredity*;
(2) *H is a subgroup with good Γ_2-heredity.*

Proof. Since Γ_1 is a finite index subgroup of Γ_2, the subgroup $H \cap \Gamma_1 = \Gamma_1 \cap (H \cap \Gamma_2)$ has finite index in $H \cap \Gamma_2$. This completes the proof. □

As an immediate consequence of Proposition 3.10 we get the following:

COROLLARY 3.11

Let G be a locally compact group. For a discrete uniform subgroup Γ of G, the property of being a subgroup with good Γ-heredity is invariant under the commensurability class of Γ.

4. Proof of the results

4.1 *Proof of Theorem* 1.7

DEFINITION 4.1

A sequence $(H_n)_{n\in\mathbb{N}}$ of discrete subgroups of G is called uniformly uniform if there exists a compact set K of G such that $G = KH_n$ for all $n \in \mathbb{N}$.

The proof of the following proposition is given in Lemma 5.7 of [26].

PROPOSITION 4.2

Let G be a compactly generated locally compact group, and $(H_n)_{n\in\mathbb{N}}$ a sequence of subgroups of G converging in the Chabauty topology to a closed subgroup H of G with G/H compact. Then there exists $n_0 \in \mathbb{N}$ such that the sequence $(H_n)_{n\geq n_0}$ is uniformly uniform.

Proof of Theorem 1.7. This follows from Proposition 2.9 and Proposition 4.2. □

4.2 *Proof of Theorem* 1.9

If H is a subgroup of a group G, let $H^{\frac{1}{n}}$, $n \in \mathbb{N}^*$ the subgroup generated by all elements g with g^n in H:

$$H^{\frac{1}{n}} = \langle\{g \in G :\ g^n \in H\}\rangle.$$

An interesting property of the subgroup $H^{\frac{1}{n}}$ is the following (Theorem F of [27]).

Theorem 4.3. *Let G be a connected nilpotent Lie group and H a finitely generated subgroup of G. Then the index $[H^{\frac{1}{n}} : H]$ is finite, and $\neq 1$ unless $n = 1$ or $H = \{e\}$.*

As a consequence of Theorem 4.3, we have

Lemma 4.4. *Let G be a connected nilpotent Lie group. If $H \in \mathscr{S}(G)$, then $H^{\frac{1}{n}} \in \mathscr{S}(G)$, for every $n \geq 1$.*

For the proof we need the following lemma (Corollary 2 of [21]).

Lemma 4.5. *A discrete subgroup of a connected nilpotent Lie group is finitely generated.*

Proof of Lemma 4.4. The subgroup H is finitely generated and therefore the index $[H^{\frac{1}{n}} : H]$ is finite. Thus $H^{\frac{1}{n}}$ is a finite disjoint union of discrete closed sets, hence it is discrete. Since G/H is compact, $G/H^{\frac{1}{n}}$ is also compact. The lemma is proved. □

DEFINITION 4.6 (Co-exponential basis)

Let G be a Lie group and H a connected subgroup of G. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G and H respectively. A basis $(X_1, \ldots, X_p)$, $p = \dim(\mathfrak{g}/\mathfrak{h})$ is said to be co-exponential to $\mathfrak{h}$ in $\mathfrak{g}$ if the map

$$\phi_{\mathfrak{g},\mathfrak{h}} : H \times \mathbb{R}^p \longrightarrow G,\ (h, (t_1, \ldots, t_p))) \longmapsto h \cdot \exp t_p X_p \cdots \exp t_1 X_1$$

is a diffeomorphism.

It is well-known that every connected subgroup of a connected simply connected solvable Lie group admits a co-exponential basis (see [14]).

The following plays a basic role in the proof of Theorem 1.9.

PROPOSITION 4.7

Let G be a connected, simply connected nilpotent Lie group and H a connected closed subgroup of G. Let $(X_1, \ldots, X_p)$ be a co-exponential basis for H in G. Then the mapping

$$\begin{array}{rcl} \Phi \ : \ \mathscr{F}(H) \times \mathscr{F}(\mathbb{R})^p & \longrightarrow & \mathscr{F}(G) \\ (L, K_1, \ldots, K_p) & \longmapsto & L \exp(K_1 X_1) \cdots \exp(K_p X_p) \end{array}$$

is continuous, where $\mathscr{F}(H)$, $\mathscr{F}(\mathbb{R})$ and $\mathscr{F}(G)$ are endowed with the Chabauty–Fell topology.

Proof. As the mapping

$$\phi_{\mathfrak{g},\mathfrak{h}} : H \times \mathbb{R}^p \longrightarrow G, \ (x, t_1, \ldots, t_p) \longmapsto x \exp(t_1 X_1) \cdots \exp(t_p X_p)$$

is a diffeomorphism, then the mapping

$$\phi^*_{\mathfrak{g},\mathfrak{h}} : \mathscr{F}(H \times \mathbb{R}^p) \longrightarrow \mathscr{F}(G), \quad F \longmapsto \phi_{\mathfrak{g},\mathfrak{h}}(F)$$

is a homeomorphism (Corollary 2.3). On the other hand, the inclusion mapping

$$i : \mathscr{F}(H) \times \mathscr{F}(\mathbb{R})^p \longrightarrow \mathscr{F}(H \times \mathbb{R}^p)$$

is continuous. Consequently, the mapping Φ is continuous as the composite of the continuous mappings $\phi^*_{\mathfrak{g},\mathfrak{h}}$ and i:

$$\begin{array}{ccc} \mathscr{F}(H) \times \mathscr{F}(\mathbb{R})^p & \xrightarrow{\quad i \quad} & \mathscr{F}(H \times \mathbb{R}^p) \\ & \searrow^{\Phi} \quad \swarrow_{\phi^*_{\mathfrak{g},\mathfrak{h}}} & \\ & \mathscr{F}(G) & \end{array}$$

□

Example 4.8. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non null positive reals. Then $\mathbb{R}$ is approximated by $(a_n\mathbb{Z})_{n\geq 0}$ if and only if $(a_n)_{n\in\mathbb{N}}$ converges to 0. There are various ways to do this. For example, it suffices to apply the following fact: The mapping

$$\phi_{\mathbb{R}} : [0, \infty] \longrightarrow \mathscr{C}(\mathbb{R})$$

defined by $\alpha \longmapsto \frac{1}{\alpha}\mathbb{Z}$ where $\frac{1}{0}\mathbb{Z} = \{0\}$ and $\frac{1}{\infty}\mathbb{Z} = \mathbb{R}$ is a homeomorphism. For a proof, see Proposition 2.8 of [22].

Let us now prove Theorem 1.9.

Proof of Theorem 1.9. The proof is by induction on the dimension of G. If $\dim G = 1$, then $G = \mathbb{R}$ and $\Gamma = a\mathbb{Z}$, where $a \in \mathbb{R}^*_+$. We have $\Gamma^{\frac{1}{n}} = \frac{1}{n}a\mathbb{Z}$ for all $n \geq 1$. By Example 4.8, $\mathbb{R}$ is approximated by $(\frac{1}{n}a\mathbb{Z})_{n\geq 1}$.

Suppose then that the result has been proved for all the groups of dimensions less than $\dim G$. Let $\Gamma \in \mathscr{S}(G)$ and $(X_1, \ldots, X_m)$ be a strong Malcev basis for $\mathfrak{g}$ strongly based on Γ. Let

$$G_0 = \exp(\mathbb{R}X_1) \cdots \exp(\mathbb{R}X_{m-1}).$$

It is easy to verify that

$$\Gamma^{\frac{1}{n}} = (G_0 \cap \Gamma^{\frac{1}{n}}) \exp\left(\frac{1}{n}\mathbb{Z}X_m\right).$$

As G_0 is Γ-rational, then $G_0 \cap \Gamma \in \mathscr{S}(G_0)$. Then by the assumption of induction G_0 is approximated by $((G_0 \cap \Gamma)^{\frac{1}{n}})_{n\geq 1}$:

$$\lim (G_0 \cap \Gamma)^{\frac{1}{n}} = G_0. \tag{1}$$

On the other hand, we have the following inclusion:

$$(G_0 \cap \Gamma)^{\frac{1}{n}} \subset G_0 \cap \Gamma^{\frac{1}{n}}. \tag{2}$$

Using (1), (2) and Lemma 2.6, we deduce that the sequence $(G_0 \cap \Gamma^{\frac{1}{n}})_n$ converges to G_0. On the other hand, the sequence $(\frac{1}{n}\mathbb{Z})_{n\geq 1}$ converges to $\mathbb{R}$ (Example 4.8). By Proposition 4.7 the desired conclusion then follows. □

The next example shows that the inclusion in (2) can be strict.

Example 4.9. Let $\mathfrak{h}_3$ be the three dimensional Heisenberg algebra with basis X, Y, Z with

$$[X, Y] = Z$$

and the non-defined brackets being equal to zero or obtained by antisymmetry. Let $G = H_3$ be the connected simply connected nilpotent Lie group associated to $\mathfrak{h}_3$. Let

$$\Gamma = \exp(\mathbb{Z}Z)\exp(\mathbb{Z}Y)\exp(\mathbb{Z}X)$$

and

$$G_0 = \exp(\mathbb{R}Z)\exp(\mathbb{R}Y).$$

It is clear that

$$\Gamma^{\frac{1}{n}} = \exp\left(\frac{1}{n^2}\mathbb{Z}Z\right)\exp\left(\frac{1}{n}\mathbb{Z}Y\right)\exp\left(\frac{1}{n}\mathbb{Z}X\right).$$

Then

$$G_0 \cap \Gamma^{\frac{1}{n}} = \exp\left(\frac{1}{n^2}\mathbb{Z}Z\right)\exp\left(\frac{1}{n}\mathbb{Z}Y\right).$$

On the other hand, we have

$$(G_0 \cap \Gamma)^{\frac{1}{n}} = (\exp(\mathbb{Z}Z)\exp(\mathbb{Z}Y))^{\frac{1}{n}} = \exp\left(\frac{1}{n}\mathbb{Z}Z\right)\exp\left(\frac{1}{n}\mathbb{Z}Y\right).$$

Consequently

$$G_0 \cap \Gamma^{\frac{1}{n}} \underset{\neq}{\subsetneq} (G_0 \cap \Gamma)^{\frac{1}{n}}.$$

PROPOSITION 4.10

Let G be a connected, simply connected nilpotent Lie group with discrete uniform subgroup Γ. If H is Γ-rational subgroup of G then H is approximated by the discrete uniform subgroups $H \cap \Gamma^{\frac{1}{n}}$.

Proof. Let $n \geq 1$. Since the index of Γ in $\Gamma^{\frac{1}{n}}$ is finite the Γ and $\Gamma^{\frac{1}{n}}$ are commensurable and therefore define the same rational structure on $\mathfrak{g}$ (Theorem 5.1.12 of [7]). Consequently H is also $\Gamma^{\frac{1}{n}}$-rational. Then $H \cap \Gamma^{\frac{1}{n}}$ is a discrete uniform subgroup of H, for every $n \geq 1$. □

4.3 *Proof of Theorem* 1.10

Proof. Since Γ is a subgroup of finite index in $\Gamma^{\frac{1}{n}}$ then Z is a subgroup with good $\Gamma^{\frac{1}{n}}$-heredity. Then Theorem 3.9 implies that $\pi(\Gamma^{\frac{1}{n}}) \in \mathscr{S}(G)$. Now, let U be an open set of G. Since the group $\widetilde{G}$ is approximated by the sequence $(\Gamma^{\frac{1}{n}})_{n \geq 1}$, there exists $n_0 \in \mathbb{N}$ such that $\pi^{-1}(U) \cap \Gamma^{\frac{1}{n}} \neq \emptyset$ for every $n \geq n_0$. Hence $U \cap \pi(\Gamma^{\frac{1}{n}}) \neq \emptyset$ for every $n \geq n_0$ and therefore the condition (A) is satisfied. □

5. Approximation and lattice subgroups

Firstly, we recall the following definition (Definition 1 of [18]).

DEFINITION 5.1

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. A lattice subgroup is a uniform subgroup Γ of G such that $\Lambda = \log \Gamma$ is an additive subgroup of $\mathfrak{g}$.

The following theorem is due to Moore (Theorem 2 of [18]).

Theorem 5.2. *Let Γ be a discrete uniform subgroup of a connected simply connected nilpotent Lie group G. Then there exist lattice subgroups Γ_1 and Γ_2 of G such that*

$$\Gamma_1 \leq \Gamma \leq \Gamma_2.$$

We extend Definition 5.1 to the non-simply connected case.

DEFINITION 5.3

Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. A lattice subgroup is a uniform subgroup Γ of G such that $\Lambda = \exp^{-1}(\Gamma)$ is an additive subgroup of $\mathfrak{g}$.

Remark 5.4. Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$ and

$$\mathfrak{d} = \{X \in \mathfrak{g} : \ X \in \mathfrak{z}(\mathfrak{g}), \ \exp(X) = e\}. \tag{3}$$

By Theorem 3.6.1 of [25], $\mathfrak{d}$ is a discrete additive subgroup of $\mathfrak{g}$, and the exponential map induces an analytic diffeomorphism of the manifold $\mathfrak{g}/\mathfrak{d}$ onto G. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ p\ } & \mathfrak{g}/\mathfrak{d}, \\ & {\scriptstyle \exp}\searrow \quad \swarrow{\scriptstyle \widetilde{\exp}} & \\ & G & \end{array} \tag{4}$$

where $\widetilde{\exp} : \mathfrak{g}/\mathfrak{d} \longrightarrow G$ is an analytic diffeomorphism, and $p : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{d}$ is the canonical projection. As a consequence, we deduce that

$$\{X \in \mathfrak{g} : \ \exp(X) = e\} = \mathfrak{d},$$

and therefore, if Γ is a lattice subgroup of G, then $\exp^{-1}(\Gamma)$ is a discrete subgroup of $\mathfrak{g}$.

PROPOSITION 5.5

Let G be a connected nilpotent Lie group and Γ a discrete uniform subgroup of G. Let $\tilde{G}$ be the universal covering group of G and let $\pi : \tilde{G} \longrightarrow G$ be the covering map. The following statements are equivalent:

(1) Γ *is a lattice subgroup of* G;
(2) $\pi^{-1}(\Gamma)$ *is a lattice subgroup of* $\tilde{G}$.

Proof. This follows from the fact that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Lie}(\tilde{G}) & \xrightarrow{\ \exp_{\tilde{G}}\ } & \tilde{G} \\ {\scriptstyle d\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g} & \xrightarrow[\ \exp_G\ ]{} & G \end{array},$$

where the mapping $d\pi : \mathrm{Lie}(\tilde{G}) \longrightarrow \mathfrak{g}$ is a Lie algebras isomorphism. □

PROPOSITION 5.6

Let G be a connected nilpotent Lie group. Let $\tilde{G}$ be the universal covering group of G, and Z be a discrete normal subgroup of $\tilde{G}$ such that $G \cong \tilde{G}/Z$. Let π be the canonical projection of $\tilde{G}$ onto G. If Γ is a lattice subgroup of $\tilde{G}$ such that Z is with good Γ-heredity then $\pi(\Gamma)$ is a lattice subgroup of G.

Proof. It is clear that $\pi(\Gamma)$ is a discrete uniform subgroup of G (see Theorem 3.9). Now, let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathfrak{d}$ be the subgroup

$$\mathfrak{d} = \{X \in \mathfrak{g} : \ X \in \mathfrak{z}(\mathfrak{g}), \exp_G(X) = e\},$$

as in (3). We have

$$\exp_G^{-1}(\pi(\Gamma)) = d\pi \left(\exp_{\tilde{G}}^{-1}(\Gamma)\right) + \mathfrak{d},$$

and therefore $\exp_G^{-1}(\pi(\Gamma))$ is an additive subgroup of $\mathfrak{g}$. This completes the proof. □

The following theorem is a generalization of Moore's theorem (Theorem 5.2).

Theorem 5.7. *Let Γ be a discrete uniform subgroup of a connected nilpotent Lie group G. Then there exist lattice subgroups Γ_1 and Γ_2 of G such that*

$$\Gamma_1 \leq \Gamma \leq \Gamma_2.$$

Proof. Let $G = \tilde{G}/Z$, where $\tilde{G}$ is the universal covering group of G. Let $\pi : \tilde{G} \longrightarrow G$ be the covering map. $\pi^{-1}(\Gamma)$ is a discrete uniform subgroup of $\tilde{G}$. By Theorem 5.2, there exist two lattice subgroups K_1 and K_2 of $\tilde{G}$ such that

$$K_1 \leq \pi^{-1}(\Gamma) \leq K_2. \tag{5}$$

On the other hand, since $\pi(\pi^{-1}(\Gamma)) = \Gamma$ then Z is a subgroup with good $\pi^{-1}(\Gamma)$-heredity (Theorem 3.9). It follows that Z is also with good K_i-heredity, for $i = 1, 2$ (see Proposition 3.10). Proposition 5.6 implies that $\pi(K_1)$ and $\pi(K_2)$ are two lattice subgroups of G. Finally, using (5) we get

$$\Gamma_1 = \pi(K_1) \leq \Gamma \leq \pi(K_2) = \Gamma_2.$$

□

PROPOSITION 5.8

Let G be a connected nilpotent Lie group. If G is a $\mathfrak{X}$-group then G is approximated by lattice subgroups.

Proof. Let $(\Gamma_n)_{n\geq 0}$ be a sequence of discrete subgroups of G converging to G. By Proposition 4.2, we can suppose that $\Gamma_n \in \mathscr{S}(G)$, $\forall n \geq 0$. For every $n \geq 0$ there exists a lattice subgroup Γ'_n such that $\Gamma_n \leq \Gamma'_n$ (Theorem 5.7). Using Lemma 2.6 we deduce that G is approximated by $(\Gamma'_n)_{n\geq 0}$. □

PROPOSITION 5.9

Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $(\Gamma_n)_{n\geq 0}$ be a sequence of lattice subgroups of G. The following statements are equivalent:

(1) *G is approximated by $(\Gamma_n)_{n\geq 0}$;*
(2) *$\mathfrak{g}$ is approximated by $(\exp^{-1}(\Gamma_n))_{n\geq 0}$.*

Proof. The exponential map is open, because it is the composite of two open maps (see (4)). Then, the following map

$$\phi : \mathscr{F}(G) \longrightarrow \mathscr{F}(\mathfrak{g}), \quad F \longmapsto \exp^{-1}(F)$$

is continuous (Proposition 2.2). It follows that (1) implies (2). Conversely, let $g \in G$. Then there exists $X \in \mathfrak{g}$ such that $g = \exp(X)$ because the exponential map is surjective (Theorem 3.6.1 of [25]). Let (X_n) be a sequence of $\mathfrak{g}$ such that for every $n \geq 0$, $X_n \in \exp^{-1}(\Gamma_n)$ and

$$\lim_{n\longrightarrow+\infty} X_n = X.$$

It follows that, for every $n \geq 0$ we have $\exp(X_n) \in \Gamma_n$ and $\lim_{n\longrightarrow+\infty} \exp(X_n) = g$. Consequently, the group G is the set of all sequential limit points of $(\Gamma_n)_{n\geq 0}$. This completes the proof of the proposition. □

6. Some applications

The approximations of Lie groups by discrete subgroups have some interesting applications.

6.1 *Examples of countable dense subgroups*

In [15], Macias-Virgos considered the following question: Given a dense subgroup D of a connected simply connected nilpotent Lie group G, is there a discrete uniform subgroup Γ of G such that $\Gamma \leq D$? In this paper we are interested in a kind of converse of this question: Determine a proper dense subgroup of a connected simply connected nilpotent Lie group G containing a given discrete uniform subgroup Γ of G.

First we establish the following proposition:

PROPOSITION 6.1

Let G be a locally compact group. If G is approximated by $(H_n)_{n\geq 0}$ then for every $n \in \mathbb{N}$, the set

$$\bigcup_{k\geq n} H_k$$

is dense in G.

Proof. By Proposition 2.9, the group G is the set of all sequential limit points of $(H_n)_{n\geq 0}$. This completes the proof. □

In the next proposition, we give a simple method for producing countable dense subgroups in a connected, simply connected nilpotent Lie group.

PROPOSITION 6.2

Let G be a connected nilpotent Lie group with discrete uniform subgroup Γ. Let $p \in \mathbb{N}^*$. Then

$$\Gamma^{\frac{1}{p^\infty}} = \bigcup_{n\in\mathbb{N}} \Gamma^{\frac{1}{p^n}}$$

is a countable dense subgroup of G containing Γ.

Proof. $(\Gamma^{\frac{1}{p^n}})_{n\in\mathbb{N}}$ is an increasing subsequence of $(\Gamma^{\frac{1}{n}})_{n\in\mathbb{N}^*}$. Then $\cup_{n\in\mathbb{N}}\Gamma^{\frac{1}{p^n}}$ is a subgroup of G. On the other hand, by Lemmas 4.4 and 4.5, $\cup_{n\in\mathbb{N}}\Gamma^{\frac{1}{p^n}}$ is countable. In the case when G is simply connected, the density follows from Theorem 1.9 and Proposition 6.1. Now, if G is not simply connected, then it is a quotient of its universal cover $\tilde{G}$ by a discrete normal subgroup Z. Let $\pi : \tilde{G} \longrightarrow G = \tilde{G}/Z$ be the canonical projection. Set

$L = \pi^{-1}(\Gamma)$. Note that L is a discrete uniform subgroup in $\tilde{G}$. Then $L^{\frac{1}{p^\infty}}$ is dense in $\tilde{G}$. Using

$$\pi\left(L^{\frac{1}{p^\infty}}\right) \leq \Gamma^{\frac{1}{p^\infty}},$$

we deduce that $\Gamma^{\frac{1}{p^\infty}}$ is dense in G. This completes the proof. □

6.2 *Density theorems*

In this subsection we give an example of a density theorem for uniform subgroups: a property of Γ passes over to G.

For the proof of the next proposition, we refer to Proposition 3.4 of [4]. But it can also be proved directly by applying the geometric convergence.

PROPOSITION 6.3

Let $\mathscr{A}(G)$ be the space of closed abelian subgroups of a Lie group G. Then $\mathscr{A}(G)$ is a closed set of $\mathscr{C}(G)$.

Proof. By Proposition 2.1, the space $\mathscr{C}(G)$ is metrizable. Let $(A_n)_{n\geq 0}$ be a sequence of closed abelian subgroups of G converging to A. Let $a, b \in A$. By $(GC2)$ there exist two sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ converging respectively to a and b such that for every $n \geq 0$, $a_n, b_n \in A_n$. Since $\lim a_n b_n = ab$ and $a_n b_n = b_n a_n$ $\forall n \in \mathbb{N}$, then $ab = ba$ and therefore $A \in \mathscr{A}(G)$. □

For $n \in \mathbb{N}^*$, let P_n be the n-th power map:

$$\begin{aligned} P_n \ : \ G &\longrightarrow G \\ x &\longmapsto x^n. \end{aligned}$$

Theorem 6.4 (Theorem C of [6]). *Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ and let n be an integer with $n \geq 2$. Then the following conditions are equivalent*:

(1) $P_n : G \longrightarrow G$ *is surjective.*
(2) $P_n : G \longrightarrow G$ *is a diffeomorphism.*
(3) $P_n : G \longrightarrow G$ *is injective.*
(4) $\exp : \mathfrak{g} \longrightarrow G$ *is a diffeomorphism.*

Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. It is well known that the exponential map

$$\exp : \mathfrak{g} \longrightarrow G$$

is a diffeomorphism [9]. We then obtain the following:

COROLLARY 6.5

Let G be a connected simply connected nilpotent Lie group. Let $n \in \mathbb{N}^*$. The n-th power map P_n is a diffeomorphism. Furthermore, for every connected closed subgroup H of G we have $P_n(H) = H$.

Proof. As $P_n(H)$ is a closed connected subgroup of H with the same dimension, then $P_n(H) = H$. □

Remark 6.6. Corollary 6.5 could be proved easily using the fact that for every $x = \exp(X) \in H$ we have

$$\exp\left(\frac{1}{n}X\right) \in H$$

and

$$x = \left(\exp\left(\frac{1}{n}X\right)\right)^n \in P_n(H).$$

The following is without doubt a known fact. In this paper, we give another proof using Theorem 1.9.

PROPOSITION 6.7

Let G be a connected simply connected nilpotent Lie group. If $\mathscr{A}(G) \cap \mathscr{S}(G) \neq \emptyset$, then $G \in \mathscr{A}(G)$.

Proof. Let $\Gamma \in \mathscr{A}(G) \cap \mathscr{S}(G)$. Let $g_1, g_2 \in \{g \in G : g^n \in \Gamma\}$. Since Γ is abelian, then $g_1^n g_2^n = g_2^n g_1^n$ and therefore $g_2^{-n} g_1^n g_2^n = g_1^n$. Clearly

$$(g_2^{-n} g_1 g_2^n)^n = g_1^n.$$

By Corollary 6.5, we obtain $g_2^{-n} g_1 g_2^n = g_1$. Then $g_2^n = g_1^{-1} g_2^n g_1 = (g_1^{-1} g_2 g_1)^n$ and hence $g_2 = g_1^{-1} g_2 g_1$. Consequently, for every $g_1, g_2 \in \{g \in G : g^n \in \Gamma\}$ we have

$$g_1 g_2 = g_2 g_1.$$

As $\Gamma^{\frac{1}{n}}$ is generated by the set $\{g \in G : g^n \in \Gamma\}$ then $\Gamma^{\frac{1}{n}}$ is abelian, for every $n \geq 1$. On the other hand, by Theorem 1.9, the sequence $(\Gamma^{\frac{1}{n}})_{n\geq 1}$ converges to G. Finally, Proposition 6.3 implies that $G \in \mathscr{A}(G)$. □

The following proposition can be found in Corollary 2.5 of [20].

PROPOSITION 6.8

Let G be a connected simply connected nilpotent Lie group. If $\Gamma \in \mathscr{S}(G)$, then $\mathcal{D}^k(\Gamma) \in \mathscr{S}(\mathcal{D}^k(G))$ for all $k \in \mathbb{N}$.

A metabelian group is a group G that possesses a normal subgroup N such that N and G/N are both abelian. Equivalently, G is metabelian if and only if the commutator subgroup $\mathcal{D}(G)$ is abelian. Equivalently again, G is metabelian if and only if it is solvable of length at most 2. Let $\mathscr{M}(G)$ be the space of all metabelian closed subgroups of a Lie group G:

$$\mathscr{M}(G) = \{H \in \mathscr{C}(G) : H \text{ is metabelian}\}.$$

PROPOSITION 6.9

Let G be a Lie group. The space $\mathscr{M}(G)$ is closed in $\mathscr{C}(G)$.

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of metabelian subgroups of G converging to H. Let $a, b, x, y \in H$. Then there exist sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ which converge respectively to a, b, x, y and that verify for all $n \in \mathbb{N}$, $a_n, b_n, x_n, y_n \in H_n$. We have

$$\lim[a_n, b_n][x_n, y_n] = [a, b][x, y] \tag{6}$$

and

$$\lim[x_n, y_n][a_n, b_n] = [x, y][a, b]. \tag{7}$$

On the other hand, for all n we have $H_n \in \mathscr{M}(G)$, then $\mathcal{D}(H_n)$ is abelian. In particular, for all n we have

$$[a_n, b_n][x_n, y_n] = [x_n, y_n][a_n, b_n]. \tag{8}$$

From (6), (7) and (8) we deduce that

$$[a, b][x, y] = [x, y][a, b].$$

Then $\mathcal{D}(H)$ is abelian and therefore $H \in \mathscr{M}(G)$. □

PROPOSITION 6.10

Let G be a connected simply connected nilpotent Lie group. If $\mathscr{M}(G) \cap \mathscr{S}(G) \neq \emptyset$, then $G \in \mathscr{M}(G)$.

Proof. Let $\Gamma \in \mathscr{S}(G) \cap \mathscr{M}(G)$. By Proposition 6.8, $\mathcal{D}(\Gamma)$ is an abelian discrete uniform subgroup of $\mathcal{D}(G)$. Then $\mathcal{D}(G)$ is abelian (Proposition 6.7) and therefore $G \in \mathscr{M}(G)$. □

Using Proposition 6.8 one can prove the following proposition which would imply Proposition 6.7 and Proposition 6.10.

PROPOSITION 6.11

If G is a connected nilpotent Lie group and $\Gamma \in \mathscr{S}(G)$ is such that $\mathcal{D}^k(\Gamma) = \{e\}$, then $\mathcal{D}^{k+1}(G) = \{e\}$. Further, if G is simply connected, $\mathcal{D}^k(G) = \{e\}$.

Proof. When G is simply connected, Proposition 6.8 shows that $\mathcal{D}^k(\Gamma)$ is a discrete uniform subgroup of $\mathcal{D}^k(G)$. As $\mathcal{D}^k(\Gamma) = \{e\}$ we get that $\mathcal{D}^k(G) = \{e\}$. Now, if G is not simply connected, then it is a quotient of its universal cover $\tilde{G}$ by a discrete normal subgroup Z. Let $\pi : \tilde{G} \longrightarrow G = \tilde{G}/Z$ be the canonical projection. Set $L = \pi^{-1}(\Gamma)$. Note that L is a discrete uniform subgroup in $\tilde{G}$ and that $Z \leq L$. As $\mathcal{D}^k(\Gamma) = \{e\}$ we get that $\mathcal{D}^k(L) = Z$ and therefore $\mathcal{D}^{k+1}(L) = \{e\}$, because Z is central. It follows that $\mathcal{D}^{k+1}(\tilde{G}) = \{e\}$ and hence $\mathcal{D}^{k+1}(G) = \{e\}$. □

The next example shows that in general

$$\mathcal{D}^k(\Gamma) = \{e\} \Longrightarrow \mathcal{D}^k(G) = \{e\}$$

need not be true.

Example 6.12. Consider the 3-dimensional Heisenberg nilpotent Lie algebra

$$\mathfrak{g} = \mathbb{R}\text{-span}\{X, Y, Z\}$$

as in Example 4.9. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Let

$$\Gamma = \exp(\mathbb{Z}Z)\exp(\mathbb{Z}Y)\exp(\mathbb{Z}X)$$

be a discrete uniform subgroup of G. We put

$$G' = G/\mathscr{Z}(\Gamma) \quad \text{and} \quad \Gamma' = \Gamma/\mathscr{Z}(\Gamma),$$

where $\mathscr{Z}(\Gamma) = \exp(\mathbb{Z}Z)$. It is clear that Γ' is a discrete uniform subgroup of G'. On the other hand, we have

$$\mathcal{D}(G') = \exp(\mathbb{R}Z)/\mathscr{Z}(\Gamma) \quad \text{and} \quad \mathcal{D}(\Gamma') = \{e\}.$$

6.3 *Syndetic hull of discrete subgroups*

DEFINITION 6.13 (Definition 5.1 of [28])

Let Γ be a closed subgroup of a connected Lie group G. A syndetic hull of Γ in G is a connected closed subgroup B of G containing Γ such that B/Γ is compact.

Theorem 6.14 [7, 21]. *Let Γ be a discrete subgroups of a connected, simply connected, nilpotent Lie group G. Then Γ has a unique syndetic hull of Γ in G.*

We denote by $\mathcal{S}_h(\Gamma)$ the syndetic hull of Γ in G.

PROPOSITION 6.15

Let Γ be a discrete subgroup of a connected simply connected, nilpotent Lie group G. Then $(\Gamma^{\frac{1}{n}})_{n\geq 1}$ converges to $\mathcal{S}_h(\Gamma)$.

Proof. Let $g \in G$ be such that $g^n \in \Gamma$. Then $g^n \in \mathcal{S}_h(\Gamma)$. By Corollary 6.5, we obtain $g \in \mathcal{S}_h(\Gamma)$. Since $\Gamma^{\frac{1}{n}} = \langle\{g \in G : \ g^n \in \Gamma\}\rangle$ then $\Gamma^{\frac{1}{n}} \leq \mathcal{S}_h(\Gamma)$ for all $n \geq 1$. By Theorem 1.9, $\mathcal{S}_h(\Gamma)$ is approximated by $(\Gamma^{\frac{1}{n}})_{n\geq 1}$. □

As an immediate consequence of Theorem 1.9, Theorem 3.8 and Proposition 6.15 we have the following:

COROLLARY 6.16

Let G be a connected simply connected nilpotent Lie group and Γ a discrete uniform subgroup of G. Let H be a connected closed subgroup of G. The following statements are equivalent:

(1) *H is Γ-rational*;
(2) *the sequence $\left((H \cap \Gamma)^{\frac{1}{n}}\right)_{n\geq 1}$ converges to H.*

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