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A note on conjugacy classes of finite groups

HEMANT KALRA and DEEPAK GUMBER

School of Mathematics and Computer Applications, Thapar University,
Patiala 147 004, India
E-mail: happykalra26@gmail.com; dkgumber@yahoo.com

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Abstract. Let G be a finite group and let x^G denote the conjugacy class of an element x of G. We classify all finite groups G in the following three cases: (i) Each non-trivial conjugacy class of G together with the identity element 1 is a subgroup of G, (ii) union of any two distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G, and (iii) union of any three distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G.

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1. Introduction

Throughout this paper, unless or otherwise stated, G denotes a finite group. The influence of arithmetic structure of conjugacy classes of G, like conjugacy class sizes, the number of conjugacy classes or the number of conjugacy class sizes, on the structure of G is an extensively studied question in group theory. Many authors have studied the influence of some other kind of behaviours of conjugacy classes on the structure of the group. For example, Dade and Yadav [2] have classified all finite groups in which the product of any two non-inverse conjugacy classes is again a conjugacy class. For more details, reader can refer to an excellent survey article by Camina and Camina [1].

We say that G satisfies (i), $i \geq 1$, if union of any i distinct non-trivial conjugacy classes of G together with 1 is a subgroup of G. In this short note, we classify all finite groups G satisfying (i) for $1 \leq i \leq 3$.

An element x of G is called a real element if there exists another element g in G such that $x^g = x^{-1}$. A group G is called a rational group if every element x of G is conjugate to x^m, where m is a natural number coprime to $|x|$. By $\mathrm{Re}(G)$ we denote the set of all real elements of G and by $\Pi(G)$ we denote the set of all the primes dividing the order of G. By $F(G)$ we denote the fitting subgroup of G and $O_p(G)$ denotes the unique maximal normal p-subgroup of G. All other unexplained notations, if any, are standard.

2. Classifications

It is not very hard to see that an abelian group G satisfies (1) if and only if G is an elementary abelian 2-group. The restriction (1) is of course very harsh on non-abelian groups G, and as such we do not get any non-abelian group with this restriction. We prove the statement in the following theorem.

Theorem 2.1. *There is no non-abelian group G satisfying* (1).

Proof. Suppose that G is a non-abelian group satisfying (1). It is easily seen that $|G|$ is even. Let $x \in G - Z(G)$. Since $|x^G|$ and $|x^G| + 1$ both divide the order of G, G cannot be a 2-group. Let $x \in G$ be such that $|x| = mn$, where $m, n \geq 2$. Then two different ordered elements x and x^m are in the same conjugacy class x^G, a contradiction. Thus each element of G is of prime order. This immediately implies that no two elements of G of different order can commute with each other. Let $x \in G$ and let $|x| = p$, a prime. We prove that $C_G(x)$ is a Sylow p-subgroup of G. Suppose that $|G| = p^m t$, where p does not divide t. If $y \in C_G(x)$, then $|y| = p$ and thus $|C_G(x)| = p^n$, where $1 \leq n \leq m$. If $|C_G(x)| < p^m$, then $|x^G| = p^l t$, where $l \geq 1$. Thus p divides $|x^G|$ and $|x^G| + 1$, a contradiction and hence $C_G(x)$ is a Sylow p-subgroup of G. If $|1 \cup x^G| = p^k$, where $k \leq m$, then $p^k - 1 = t$. Let p_1, p_2 be two different primes which divide the order of G and let $|G| = p_1^{m_1} p_2^{m_2} s$, where p_1 and p_2 do not divide s and $s \geq 1$. We choose elements $x_1, x_2 \in G$ such that $|x_1| = p_1$ and $|x_2| = p_2$. Then as before, we get the equations $p_1^{n_1} - 1 = p_2^{m_2} s$ and $p_2^{n_2} - 1 = p_1^{m_1} s$, where $n_1 \leq m_1$ and $n_2 \leq m_2$. Thus

$$p_1^{n_1} - 1 = p_2^{m_2} s = p_2^{m_2 - n_2} s(p_1^{m_1} s + 1) \geq p_1^{m_1} + 1 > p_1^{n_1} - 1,$$

a final contradiction. □

It is easy to see that if G is isomorphic to C_3, the cyclic group of order 3, then G satisfies (2). Conversely, suppose that G is an abelian group satisfying (2). Of course then $|G| \geq 3$. Let $x, y \in G - \{1\}$ be two distinct elements. Then $\{1, x, y\}$ is a subgroup of G such that $y = x^2$. If $z \in G - \{1, x\}$, then $\{1, x, z\}$ is a subgroup of G such that $z = x^2 = y$. Thus G is a cyclic group of order 3. The following theorem settles the non-abelian case.

Theorem 2.2. *A non-abelian group G satisfies* (2) *if and only if G is isomorphic to S_3.*

Proof. If $G \approx S_3$, then G satisfies (2). Conversely suppose that G is a non-abelian group satisfying (2). Assume $|G| = p^n$, where p is an odd prime. Then $|Z(G)| \geq 3$. By the preceding paragraph, $|Z(G)| = 3$ and hence $|G| = 3^n$. Choose x a non-central and x_1 a central element of G. Let $H_0 = 1 \cup x^G \cup x_1^G$. Then $|H_0| = 1 + 1 + 3^a, \ 1 \leq a < n$, is not divisible by 3, a contradiction. Thus G cannot be a p-group.

Let $y \in G$ be of composite order. We can write $|y| = pm$, where p is a prime and $m \geq 2$. Choose an element $y_1 \in G$ of order q, where q is a prime different from p. Then $y^m \in H_1 = 1 \cup y^G \cup y_1^G$, because $H_1 \leq G$. But on the other hand, neither y^m belongs to y^G nor to y_1^G. It thus follows that every element of G is of prime order.

Suppose $|G|$ is odd and let $p \neq q \in \Pi(G)$. Let x_2 and y_2 be two elements of G such that $|x_2| = p$ and $|y_2| = q$. Since $|x_2^G|$ is odd, x_2^{-1} cannot be in x_2^G and hence x_2^{-1} is not in the subgroup $H_2 = 1 \cup x_2^G \cup y_2^G$. This is a contradiction and thus $|G|$ is even.

We claim that $|G|$ is divisible by exactly two distinct primes. On the contrary assume that $|G| = p^l q^m r^n s$, where p, q, r are distinct primes, $l, m, n, s \geq 1$ and $(p^l q^m r^n, s) = 1$. First suppose that all same ordered elements in G are conjugate. Let $x_3, y_3, z_3 \in G$ be of order p, q and r respectively. Since every element of G is of prime order, $C_G(x_3)$ is a p-group and therefore $|x_3^G| = p^{l_1} q^m r^n s$, where $0 \leq l_1 < l$. Similarly $|y_3^G| = p^l q^{m_1} r^n s$

and $|z_3^G| = p^l q^m r^{n_1} s$, where $0 \le m_1 < m$ and $0 \le n_1 < n$. Let $H_3 = 1 \cup x_3^G \cup y_3^G$, $K_3 = 1 \cup y_3^G \cup z_3^G$ and $L_3 = 1 \cup z_3^G \cup x_3^G$. Then $H_3 \cap K_3 = 1 \cup y_3^G$ is a Sylow q-subgroup of G and $H_3 \cap L_3 = 1 \cup x_3^G$ is a Sylow p-subgroup of G. Therefore $q^m = p^l q^{m_1} r^n s + 1$ and $p^l = p^{l_1} q^m r^n s + 1$. It follows that

$$q^m = p^l q^{m_1} r^n s + 1 = (p^{l_1} q^m r^n s + 1) q^{m_1} r^n s + 1 > q^m,$$

a contradiction. We therefore suppose that there exists a prime $p \in \Pi(G)$ such that all the elements of order p are not conjugate to each other. Let x_4, y_4 be two non-conjugate elements of order p and let $z_4, w_4 \in G$ be of order q and r respectively. Then $|x_4^G| = p^{l_2} q^m r^n s$, $|y_4^G| = p^{l_3} q^m r^n s$, $|z_4^G| = p^l q^{m_2} r^n s$ and $|w_4^G| = p^l q^m r^{n_2} s$, where $0 \le l_2, l_3 < l$, $0 \le m_2 < m$ and $0 \le n_2 < n$. Let $H_4 = 1 \cup x_4^G \cup z_4^G$ and $K_4 = 1 \cup y_4^G \cup w_4^G$. Then H_4 and K_4 are disjoint normal subgroups of G and $|H_4 K_4| = |H_4||K_4| > |G|$. This is a contradiction and hence the claim.

Let $|G| = 2^a p^b$, where p is an odd prime and $a, b \ge 1$. Let $x \in G$ be of order p, $y \in G$ be of order 2, and $H_5 = 1 \cup x^G \cup y^G$. Then x^n is conjugate to x for $1 \le n \le p-1$. Thus G is a rational group and hence G splits over $\gamma_2(G)$ with a Sylow 2-subgroup H as complement and $\gamma_2(G)$ is a Sylow 3-subgroup of G by Proposition 21 of [6]. For $1 \ne h \in H$, the mapping $\tau_h : \gamma_2(G) \to \gamma_2(G)$ sending g to $h^{-1} g h$ is a fixed point free automorphism of order 2. Therefore $\gamma_2(G)$ is abelian and $\tau_h(g) = g^{-1}$ for all $g \in \gamma_2(G)$ by Theorem 10.1.4 of [5]. It then follows that $|g^G| = 2$ for each $g \in \gamma_2(G)$ and thus $|G| = 2.3^b$. If $|\gamma_2(G)| > 3$, then we can find $g_1, g_2 \in \gamma_2(G)$ such that $< g_1 > \ne < g_2 >$. Then $K_5 = 1 \cup g_1^G \cup g_2^G$ is a subgroup of G of order 5, a contradiction. Thus $|\gamma_2(G)| = 3$ and hence $G \approx S_3$. □

Observe that if G satisfies (3), then $|G|$ is even. It is easily seen that if $G \approx C_4$ or $G \approx C_2 \times C_2$, then G satisfies (3). Conversely suppose that G is an abelian group satisfying (3). Let p be an odd prime dividing the order of G and let $x_1 \in G$ be of order p. Let $x_2 \in G$ be of order 2. Let $H_1 = 1 \cup x_1^G \cup x_2^G \cup (x_1^2)^G$. Then H_1 is a subgroup of G containing x_1, x_2 but not $x_1 x_2$. Thus G is a 2-group of order bigger than 3. Suppose $d(G) \ge 3$ and let y_1, y_2, y_3 be three generators of G. Let $H_2 = 1 \cup y_1^G \cup y_2^G \cup y_3^G$. Then H_2 is a subgroup of G containing y_1, y_2 but not $y_1 y_2$. Thus $d(G) \le 2$. Suppose $d(G) = 1$ and let $G = < x >$. Let $H = 1 \cup x^G \cup (x^2)^G \cup (x^3)^G$. Then H is a subgroup of G of order 4 containing G. Thus G is a cyclic group of order 4. Now suppose $d(G) = 2$ and let $G = < y, z >$. If $|y| > 2$, then $|y| \ge 4$. Let $K = 1 \cup y^G \cup (y^2)^G \cup z^G$. Then K is a subgroup of G containing y but not y^3. Thus $|y| = 2$ and similarly $|z| = 2$. Therefore $G \approx C_2 \times C_2$. For non-abelian groups, we have the following theorem.

Theorem 2.3. *A non-abelian group G satisfies* (3) *if and only if either G is isomorphic to D_5 or to A_4.*

Proof. It is easily seen that if G is isomorphic to D_5 or to A_4, then G satisfies (3). For the converse part, we proceed in a number of steps.

Step 1. G cannot be a 2-group.

Proof. Contrarily assume that G is a 2-group. Suppose G has 3 or more non-central conjugacy classes and let x_1^G, x_2^G and x_3^G be three of them. Then $H = 1 \cup x_1^G \cup x_2^G \cup x_3^G$ is a subgroup of G of odd order, which is not possible. Thus G has at most two non-central conjugacy classes. Suppose $|Z(G)| > 2$ and let $z_1 \ne z_2 \in Z(G) - 1$, $w \in G - Z(G)$.

Then $K = 1 \cup z_1^G \cup z_2^G \cup w^G$ is a subgroup of G of odd order, which is not possible. Therefore $|Z(G)| = 2$ and $G = 1 \cup x_1^G \cup x_2^G \cup z_1^G$, where $x_1, x_2 \in G - Z(G)$ and $z_1 \in Z(G) - 1$. We show that $Z(G) = \gamma_2(G)$. If not, then either $\gamma_2(G) = 1 \cup x_1^G \cup z_1^G$ or $\gamma_2(G) = 1 \cup x_2^G \cup z_1^G$. In either of the cases, $\gamma_2(G)$ is a maximal subgroup of G. This implies that $G/\gamma_2(G)$ and hence G is cyclic, which is not so. Thus $Z(G) = \gamma_2(G)$. Therefore $|x_1^G| = |x_2^G| = 2$ and $|G| = 6$, a contradiction. Hence G cannot be a 2-group.

Step 2. $|Z(G)| = 1$.

Proof. Assume $|Z(G)| > 1$ and let p be the largest prime dividing $|Z(G)|$. Suppose $p \geq 3$. Let $x \in Z(G)$ be of order p, $y \in G$ be of order 2 and let $H = 1 \cup x^G \cup (x^2)^G \cup y^G$. Then H is a subgroup of G containing x and y but not containing xy, a contradiction. Thus $p = 2$ and $Z(G)$ is a 2-group. Let S be a Sylow 2-subgroup of G containing $Z(G)$. Let $z \in Z(G)$ be of order 2, $w \neq z \in S$ and $u \in G$ be of order q, where $q \in \Pi(G)$ is an odd prime. Then $K = 1 \cup z^G \cup w^G \cup u^G$ is a subgroup of G not containing zu. Thus $S = Z(G)$ is the unique Sylow 2-subgroup of G of order 2. Therefore $|G/Z(G)|$ is odd and by Lemma 2.4 of [4], $\mathrm{Re}(G) = \mathrm{Re}(Z(G))$. Let $L = 1 \cup u^G \cup z^G \cup (uz)^G$. Now $u^{-1} \in L$ is of order q, therefore $u^{-1} \in u^G$. Thus u is a real element of $G - Z(G)$, a contradiction to $\mathrm{Re}(G) = \mathrm{Re}(Z(G))$. Hence $|Z(G)| = 1$.

Step 3. $|G|$ is divisible by exactly 2 distinct primes.

Proof. Suppose $|G|$ is divisible by at least three distinct primes. We show that order of every element of G is a prime. Let $x_1 \in G$ be of order $p_1 l$ and let $x_2, x_3 \in G$ be of orders p_2 and p_3 respectively, where p_1, p_2, p_3 are distinct primes and $l \geq 2$. Then $H = 1 \cup x_1^G \cup x_2^G \cup x_3^G$ is a subgroup of G containing x_1 but not x_1^l. This is a contradiction and thus every element of G is of prime order. Let $x \in G$ be arbitrary and let $|x| = p$, a prime. Let $y_1, y_2 \in G$ be of orders q_1, q_2 respectively, where $q_1 \neq q_2$ are primes different from p. Let $K = 1 \cup x^G \cup y_1^G \cup y_2^G$. Then x^m is conjugate to x whenever m is relatively prime to p. Thus G is a rational group with elementary abelian Sylow 2-subgroup and therefore G is a $\{2, 3\}$ group by Proposition 21 of [6]. This is a contradiction to $|\Pi(G)| \geq 3$ and hence $|G|$ is divisible by exactly 2 distinct primes.

Step 4. Order of every element of G is prime.

Proof. By step 3, we can assume that $|G| = 2^a p^b$, where p is an odd prime and $a, b \geq 1$. Let $x \in G$ be arbitrary and let $|x| = m$. If G contains an element of composite order, then we can choose two non-conjugate elements x_2, x_3 of orders different from m. Let $H_1 = 1 \cup x^G \cup x_2^G \cup x_3^G$. Then x^r is conjugate to x whenever r is relatively prime to m. Thus G is a rational group and hence $G/\gamma_2(G)$ is an elementary abelian 2-group by Proposition 1 and Corollary 2 of [7]. Let $|G/\gamma_2(G)| = 2^n$, where $n \leq a$, and let H be a Sylow 2-subgroup of G. Then $|\gamma_2(G)| = 2^{a-n} p^b$, $|\gamma_2(G) \cap H| = 2^{a-n}$ and

$$|\gamma_2(G)H| = \frac{|\gamma_2(G)||H|}{|\gamma_2(G) \cap H|} = 2^a p^b = |G|.$$

Thus $G = \gamma_2(G)H$.

First suppose that H is abelian. Then G splits over $\gamma_2(G)$ with H as complement and $\gamma_2(G)$ is a Sylow 3-subgroup of G by Proposition 21 of [6]. Let h_1 and h_2 be two elements of H and let $h_1 = h_2^g$ for some $g \in G$. Then $h_2^{-1}h_2^g \in H \cap \gamma_2(G) = 1$ and thus $h_1 = h_2$. Therefore distinct elements of H are not conjugate in G. Let $|H| > 2$, $h_3 \in H$ be of order 2, and let $h_4 (\neq h_3)$ be a non-trivial element of H. Let $x_4 \in G$ be a 3-element and let $K = 1 \cup h_3^G \cup h_4^G \cup x_4^G$. Then K is a subgroup of G containing h_3 and h_4 but not h_3h_4, a contradiction. Thus $|H| = 2$ and hence $|G| = 2.3^b$. Let h be the non-trivial element of H and let $\tau_h : \gamma_2(G) \to \gamma_2(G)$ be defined by $\tau_h(x) = h^{-1}xh$. It is easy to see that τ_h is an automorphism of $\gamma_2(G)$ of order 2. Suppose $y \in \gamma_2(G)$ is such that $\tau_h(y) = y$. Then $y \in C_G(H) = Z(H) = H$, by Corollary 16A of [6]. Thus $y \in \gamma_2(G) \cap H = 1$ and hence τ_h is a fixed-point free automorphism of $\gamma_2(G)$. It then follows from Theorem 10.1.4 of [5] that $\gamma_2(G)$ is abelian. Suppose $\gamma_2(G)$ contains an element, say w, of order bigger than 3. Since G is rational and $< w >=< w^k >$ for more than two values of k, $|w^G| > 2$. On the other hand, since $\gamma_2(G)$ is abelian, $\gamma_2(G) \leq C_G(w)$ and thus $|w^G| = 2$. A contradiction and thus $\gamma_2(G)$ is an elementary abelian 3-group. If $|\gamma_2(G)| > 3$, then we can find $y_1, y_2 \in \gamma_2(G)$ such that $< y_1 > \neq < y_2 >$. Let $K_1 = 1 \cup y_1^G \cup y_2^G \cup h^G$. Then $y_1, y_2 \in K_1$ but $y_1y_2 \notin K_1$, because $y_1^G = \{y_1, y_1^{-1}\}$ and $y_2^G = \{y_2, y_2^{-1}\}$. Thus $|\gamma_2(G)| = 3$ and $G \approx S_3$, a contradiction because S_3 has only 2 non-trivial conjugacy classes.

Now suppose that H is non-abelian. Let P be a Sylow p-subgroup of G. Let h_5, h_6 be two non-conjugate elements of $Z(H)$ and let $K_2 = 1 \cup h_5^G \cup h_6^G \cup z_1^G$, where $z_1 \in Z(P)$. Since $|K_2|$ is odd, K_2 is a p-group containing elements of order 2, which is not possible. Thus all the elements in $Z(H)$ are conjugate and hence $Z(H)$ is an elementary abelian 2-group. We show that all involutions of H are conjugate. Suppose there exists an involution h in H which is not conjugate to some element of $Z(H)$. Let $h_7 \in H$ be of order bigger than 2. Let $K_3 = 1 \cup h^G \cup h_7^G \cup z_1^G$. Then K_3 is an odd ordered subgroup of G containing even ordered elements. A contradiction and thus all involutions of H and hence of G are conjugate. Thus $H \approx Q_8$ by Proposition 1 of [3] and $G \approx Q_8E_p$ by Proposition 35 of [6], where $p = 3$ or 5. Let $e_1, e_2 \in E_p - 1$ be two non-conjugate elements and let $h_8 \in Q_8$ be of order 4. Let $K_4 = 1 \cup e_1^G \cup e_2^G \cup h_8^G$. Then K_4 is an odd ordered subgroup of G containing even ordered elements. A contradiction and thus all non-trivial elements of E_p are conjugate. Since $|E_p| = p^b$, $(p^b - 1)|8$ and thus $|G| = 24,\ 40$ or 72. Using GAP [9] it can be seen that the groups with GAP IDs [24, 12], [72, 15], [72, 39], [72, 40], [72, 41], [72, 43] and [72, 44] are the only groups with trivial center. None of these groups satisfy the hypothesis. This is a contradiction and hence every element of G is of prime order.

Step 5. $G \approx D_5$ or A_4.

Proof. By steps 3 and 4, $|G| = 2^a p^b$, $a, b \geq 1$ and order of every element of G is either 2 or p. Let H be a Sylow 2-subgroup and P be a Sylow p-subgroup of G. First suppose that $a = 1$. For $h \in H$, the map $\lambda_h : P \to P$ defined by $\lambda_h(x) = h^{-1}xh$ is an automorphism of P of order 2. If λ_h fixes some element, say $x_1 \in P$, then G contains an element x_1h of composite order, which is not so. Thus λ_h is a fixed point free automorphism of P of order 2. Therefore P is abelian by Theorem 10.1.4 of [5]. By hypothesis, all the elements of P are not conjugate. Let $y_1, y_2 \in P$ be two non-conjugate elements. Since P is abelian, $|y_1^G| = |y_2^G| = 2$. If $K = 1 \cup y_1^G \cup y_2^G \cup h^G$, then $|K| = p^b + 5$ is more than half of $|G|$. Thus $p^b + 5 = 2p^b$ and hence $p = 5,\ b = 1$ and $G \approx D_5$.

Now suppose that $a > 1$. Since every element of G is of prime order, $F(G) = O_2(G)$ or $F(G) = O_p(G)$. Suppose $F(G) = O_p(G)$. Let S be an abelian non-cyclic subgroup of H. Since G is solvable, $C_G(F(G)) \leq F(G)$. Thus $C_{(F(G))}(x) = 1$ for every $1 \neq$

$x \in S$, because otherwise G contains an element of composite order. It thus follows from Theorem 5.3.16 of [5] that $F(G) =< C_{F(G))}(x)|x \in S, x \neq 1 >= 1$, a contradiction and hence every abelian subgroup of H is cyclic. Thus either H is cyclic or $H \approx Q_g$ by Theorem 4.4.4 of [8]. Since H is elementary abelian, $H \approx C_2$ and hence $a = 1$, which is not so. Therefore $F(G) = O_2(G)$. If $F(G) = O_2(G) < H$, then since H is elementary abelian, $H \leq C_G(F(G)) \leq F(G) < H$, a contradiction. Thus $F(G) = O_2(G) = H$ and hence H is a normal subgroup of G. Using similar arguments, as used above on H, we can see that P is cyclic of order p. Therefore $|G| = 2^a p, a > 1$. Suppose $h_1, h_2 \in H$ are non-conjugate and let $y_1 \in P$. Let $K = 1 \cup h_1^G \cup h_2^G \cup y_1^G$. Then K is a subgroup of G of odd order containing an element of order 2, a contradiction. Thus all the involutions in G are conjugate and hence $|x^G| = 2^a - 1$, where x is an involution. On the other hand, since $H = C_G(x)$, $|x^G| = p$. By hypothesis, G contains at least two non-trivial non-conjugate elements y_1, y_2 of order p. Let $K_1 = 1 \cup x^G \cup y_1^G \cup y_2^G$. Then

$$2^a - 1 + 2^a + 2^a + 1 = 2^n p,\ \ n \leq a.$$

It follows that $p = 3$ and $a = n$, and since $2^a - 1 = p = 3$, $a = 2$. Thus $|G| = 12$. There are three non-abelian groups of order 12 viz. A_4, D_6 and $C_3 \rtimes C_4$, and it can be easily seen that only A_4 satisfies the hypothesis. Thus $G \approx A_4$. □

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