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Real parabolic vector bundles over a real curve

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Abstract. We define real parabolic structures on real vector bundles over a real curve. Let (X, σ_X) be a real curve, and let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. Let $N \geq 2$ be an integer. We construct an N-fold cyclic cover $p : Y \to X$ in the category of real curves, ramified precisely over each point of S, and with the property that for any element g of the Galois group Γ, and any $y \in Y$, one has $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$. We established an equivalence between the category of real parabolic vector bundles on (X, σ_X) with real parabolic structure over S, all of whose weights are integral multiples of $1/N$, and the category of real Γ-equivariant vector bundles on (Y, σ_Y).

Keywords. Real parabolic bundles; real curve.

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1. Introduction

The notion of parabolic vector bundles over a compact Riemann surface was introduced by Seshadri [4] and their moduli studied in [2]. Here we consider real vector bundles over a real curve and define parabolic structures on real vector bundles.

By a real curve, we mean a pair (X, σ_X), where X is a compact Riemann surface and σ_X is an anti-holomorphic involution on X. A real vector bundle over a real curve (X, σ_X) is a pair (E, σ^E), where $\pi : E \to X$ is a holomorphic vector bundle and σ^E is an anti-holomorphic involution on E such that $\pi \circ \sigma^E = \sigma_X \circ \pi$ and for all $x \in X$, the map $\sigma^E|_{E(x)} : E(x) \to E(\sigma_X(x))$ is $\mathbb{C}$-antilinear:

$$\sigma^E(\lambda \cdot \eta) = \bar{\lambda} \cdot \sigma^E(\eta), \text{ for all } \lambda \in \mathbb{C} \text{ and all } \eta \in E(x),$$

where $E(x)$ denotes the fibre of E at x.

Let (X, σ_X) be a real curve, and let (E, σ^E) be a real vector bundle over (X, σ_X). Let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. By a real parabolic structure on (E, σ^E) over S we mean for each $x \in S$, a strictly decreasing weighted flag in $E(x)$ which is preserved by σ^E and the weights over x and $\sigma_X(x)$ are same (see §3 for the precise definition). Let $N \geq 2$ be a positive integer. In §4, we construct a N-fold cyclic cover $p : Y \to X$ in the category of real curves, ramified precisely over each point of S, and with the property that for any element g of the Galois group Γ of p, and any $y \in Y$, one has $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$. Let $\mathbf{RP}(X, N)$ be the category of real parabolic vector bundles on (X, σ_X) with parabolic structure over S, whose all the weights are integral multiples of $1/N$. Let $\mathbf{RE}_\Gamma(Y)$ be the category whose objects are all Γ-equivariant real vector bundles on (Y, σ_Y) and morphisms are morphisms of Γ-equivariant real vector bundles. In §5, we prove the following (see Theorem 5.3):

Theorem 1.1. *There is a canonical functor* $\Psi : \mathbf{RP}(X, N) \to \mathbf{RE}_{\boldsymbol{\Gamma}}(Y)$ *which is an equivalence of categories.*

2. Preliminaries

By a real curve, we mean a pair (X, σ_X), where X is a compact Riemann surface and σ_X is an anti-holomorphic involution on X. Let $\sigma_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ be the conjugate map $z \mapsto \bar{z}$.

PROPOSITION 2.1

A continuous involution $\sigma : X \to X$ *on a Riemann surface* X *is an anti-holomorphic involution if and only if for every open subset* U *of* X, *the map*

$$\tilde{\sigma} = \tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$$

defined by $f \mapsto \sigma_{\mathbb{C}} \circ f \circ \sigma$ *is an isomorphism of rings.*

Proof. If $\sigma : X \to X$ is an anti-holomorphic involution, then the map $\tilde{\sigma}_U$ defined as above will be an isomorphism. Conversely, suppose that the map

$$\tilde{\sigma} = \tilde{\sigma}_U : \mathcal{O}_X(U) \to \mathcal{O}_X(\sigma(U))$$

defined by $f \mapsto \sigma_{\mathbb{C}} \circ f \circ \sigma$ is an isomorphism. For every pair of charts $\psi_1 : U_1 \to V_1 \subset \mathbb{C}$ and $\psi_2 : U_2 \to V_2 \subset \mathbb{C}$ on X with $\sigma(U_1) \subset U_2$, the map

$$\psi_2 \circ \sigma \circ \psi_1^{-1} : V_1 \to V_2$$

is anti-holomorphic, since $\sigma_{\mathbb{C}} \circ \psi_2 \circ \sigma$ is holomorphic. This proves that the map $\sigma : X \to X$ is an anti-holomorphic involution. □

Real vector bundles. Let (X, σ_X) be a real curve. A *real holomorphic vector bundle* $E \to X$ is a holomorphic vector bundle, together with an anti-holomorphic involution σ^E of E making the diagram

$$\begin{array}{ccc} E & \xrightarrow{\sigma^E} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow[\sigma_X]{} & X \end{array}$$

commutative, and such that, for all $x \in X$, the map $\sigma^E|_{E(x)} : E(x) \to E(\sigma(x))$ is $\mathbb{C}$-antilinear:

$$\sigma^E(\lambda \cdot \eta) = \bar{\lambda} \cdot \sigma^E(\eta), \text{ for all } \lambda \in \mathbb{C} \text{ and all } \eta \in E(x).$$

A homomorphism between two real bundles (E, σ^E) and $(E', \sigma^{E'})$ is a homomorphism

$$f : E \to E'$$

of holomorphic vector bundles over X such that $f \circ \sigma^E = \sigma^{E'} \circ f$.

A holomorphic subbundle F of a real holomorphic vector bundle E is said to be real subbundle of E if $\sigma^E(F) = F$.

Real $\mathcal{O}_X$-modules. Let (X, σ) be a real curve, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define an $\mathcal{O}_X$-module $\mathcal{F}^\sigma$ as follows. For any open subset U of X, $\mathcal{F}^\sigma(U) = \mathcal{F}(\sigma(U))$, and for every $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}^\sigma(U)$, $f \cdot s = \tilde{\sigma}_U(f)s$. It is easy to check that $\mathcal{F}^\sigma$ is an $\mathcal{O}_X$-module.

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Define $\phi^\sigma : \mathcal{F}^\sigma \to \mathcal{G}^\sigma$ as follows: For every open subset U of X,

$$\phi^\sigma_U : \mathcal{F}^\sigma(U) \to \mathcal{G}^\sigma(U), \quad \phi^\sigma_U = \phi_{\sigma(U)}$$

If $f \in \mathcal{O}_X(U)$, and $s \in \mathcal{F}^\sigma(U)$ then

$$\phi^\sigma_U(f \cdot s) = \phi_{\sigma(U)}\big(\sigma_U(f)(s)\big) = \sigma_U(f)\phi_{\sigma(U)}(s),$$

since $\phi_{\sigma(U)}$ is an $\mathcal{O}_X(\sigma(U))$-linear. Therefore, $\phi^\sigma(f \cdot s) = f \cdot \phi^\sigma(s)$. It follows that ϕ^σ is a homomorphism of $\mathcal{O}_X$-modules.

DEFINITION 2.2

A real structure on an $\mathcal{O}_X$-module $\mathcal{F}$ is an $\mathcal{O}_X$-module homomorphism $\sigma^{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^\sigma$ such that $(\sigma^{\mathcal{F}})^\sigma \circ \sigma^{\mathcal{F}} = \mathbf{1}_{\mathcal{F}}$. By a *real $\mathcal{O}_X$–module*, we mean a pair $(\mathcal{F}, \sigma^{\mathcal{F}})$, where $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\sigma^{\mathcal{F}}$ is a real structure on an $\mathcal{O}_X$-module $\mathcal{F}$.

Let $(\mathcal{F}, \sigma^{\mathcal{F}})$ and $(\mathcal{G}, \sigma^{\mathcal{G}})$ be two real $\mathcal{O}_X$-modules. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is said to be a morphism of real $\mathcal{O}_X$-modules if $\sigma^{\mathcal{G}} \circ \phi = \phi^\sigma \circ \sigma^{\mathcal{F}}$.

We recall that a vector bundle $\mathcal{F}$ on X is called *semistable* if for every non-zero subbundle $\mathcal{F}'$ of $\mathcal{F}$, the inequality

$$\mu(\mathcal{F}') \leq \mu(\mathcal{F})$$

is valid. If strict inequality is valid for all non-zero proper subbundle $\mathcal{F}'$, then $\mathcal{F}$ is called *stable*. If $\mathcal{F}$ is a direct sum of stable vector bundles having the same slope, then $\mathcal{F}$ is called *polystable*.

DEFINITION 2.3

A real holomorphic vector bundle (E, σ^E) over a real curve is said to be *real semistable* (respectively, *real stable*) if for every proper real coherent subsheaf $\mathcal{F}$ of E, we have

$$\mu(\mathcal{F}) \leq \mu(E) \quad \text{(respectively,} \quad \mu(\mathcal{F}) < \mu(E)).$$

Let (E, σ^E) be a real holomorphic vector bundle over a real curve (X, σ). Then (E, σ^E) is said to be *real polystable* if $E = \oplus_{i=1}^n E_i$, where $(E_i, \sigma^E|_{E_i})$ is real stable subbundle of E satisfying $\mu(E_i) = \mu(E)$ for $i = 1, \ldots, n$.

Remark 2.4. Let E be a real holomorphic vector bundle over a real curve X. Then the corresponding locally free $\mathcal{O}_X$-module $\mathcal{E}$ is a real $\mathcal{O}_X$-module. Conversely, if $\mathcal{E}$ is a locally free real $\mathcal{O}_X$-module then the corresponding holomorphic vector bundle is a real holomorphic vector bundle.

3. Real parabolic vector bundles

DEFINITION 3.1

Let (E, σ^E) be a real vector bundle over a real curve (X, σ_X). Let S be a finite subset of X such that $\sigma(S) = S$. The points in S are called the real parabolic points.

By *real quasi-parabolic structure* on (E, σ^E) over S, we mean for each $x \in S$, there is a strictly decreasing flag

$$E(x) = F^1E(x) \supset F^2E(x) \supset \cdots \supset F^{k_x}E(x) \supset F^{k_x+1}E(x) = 0$$

of linear subspaces in $E(x)$ satisfying the following property:

(RP1) σ^E preserve the flags, i.e., $\sigma_x^E(F^iE(x)) = F^iE(\sigma_X(x))$.

We define

$$r_j = \dim(F^jE(x)) - \dim(F^{j+1}E(x)).$$

The integer k is called the length of the flag and the sequence $(r_1, \dots, r_{k_x})$ is called the type of the flag.

A *real parabolic structure* on (E, σ^E) over S is a real quasi-parabolic structure on (E, σ^E) over S as above, together with a sequence of real numbers $0 \leq \alpha_1^x < \cdots < \alpha_{k_x}^x < 1$, which are called weights corresponding to the subspaces $(F^1E(x), F^2E(x), \dots, F^{k_x}E(x))$, with the following property:

(RP2) The weights over x and $\sigma_X(x)$ are same.

We set

$$d_xE = \sum_{j=1}^{k} r_j\alpha_j,$$

where $r_j = \dim(F^jE(x)) - \dim(F^{j+1}E(x))$.

The parabolic degree, denoted by p deg(E), is defined by

$$\text{p}\deg(E) = \deg(E) + \sum_{x\in S} d_xE, \tag{3.1}$$

where $\deg(E)$ denotes the topological degree of E, and we define the parabolic slope by

$$\text{p}\mu(E) = \frac{\text{p}\deg(E)}{\text{rank}(E)}. \tag{3.2}$$

DEFINITION 3.2

Given two real parabolic bundles (E_1, σ^{E_1}) and (E_2, σ^{E_2}) over (X, σ_X), a *real parabolic morphism* is a homomorphism $\psi : (E_1, \sigma^{E_1}) \to (E_2, \sigma^{E_2})$ of real vector bundles which respects the real parabolic structures, i.e., for each real parabolic point x with the real parabolic structures on E_l at x for $l = 1, 2$ given by

$$E_l(x) = F^1E_l(x) \supset F^2E_l(x) \supset \cdots \supset F^{k_x}E_l(x) \supset 0,$$

$$0 \leq \alpha_1^l < \alpha_2^l < \cdots < \alpha_{k_x}^l < 1,$$

we require that $\psi(x)$ satisfies

$$\alpha_i^1 > \alpha_j^2 \Longrightarrow \psi(x)(F^i E_1(x)) \subseteq F^{j+1} E_2(x). \tag{3.3}$$

An isomorphism $\psi : (E_1, \sigma^{E_1}) \to (E_2, \sigma^{E_2})$ is said to be an isomorphism of real parabolic bundles if ψ and ψ^{-1} are real parabolic morphisms.

Remark 3.3. We can replace condition (3.3) by the following equivalent condition on $\psi(x)$. Given the weight α_i^1, let α_j^2 be the smallest weight such that $\alpha_i^1 \le \alpha_j^2$, then we require

$$\psi(x)(F^i E(x)) \subseteq F^j E_2(x). \tag{3.4}$$

Lemma 3.4. *If $\psi : E_1 \to E_2$ and $\phi : E_2 \to E_3$ are morphisms of real parabolic bundles, then $\phi \circ \psi$ is a real parabolic morphism.*

Proof. First note that $\phi \circ \psi$ is a morphism of real holomorphic vector bundles. Suppose $x \in X$ is a real parabolic point. We use the notation $\{F_j^n, \alpha_j^n\}$ for the weighted flag in E_n at x for $n = 1, 2, 3$. Given the weight α_i^1, let α_j^2 be the smallest weight with $\alpha_i^1 \le \alpha_j^2$. Then by condition (3.4), $\psi(x)(F_i^1) \subseteq F_j^2$. Also, if α_k^3 is the smallest weight with $\alpha_j^2 \le \alpha_k^3$, then $\phi(x)(F_j^2) \subset F_k^3$. Thus we see that $(\phi \circ \psi)(x)(F_i^1) \subseteq F_k^3$. On the other hand, let $\alpha_{k'}^3$ be the smallest weight with $\alpha_i^1 \le \alpha_{k'}^3$. Since $\alpha_i^1 \le \alpha_{k'}^3$ we see that $\alpha_{k'}^3 \le \alpha_k^3$. Thus $F_k^3 \subseteq F_{k'}^3$ and hence $(\phi \circ \psi)(x)(F_i^1) \subseteq F_{k'}^3$. □

We denote by $\mathbf{RP}(X)$ the category whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S and morphisms are real parabolic morphisms.

Remark 3.5. Given a short exact sequence of real holomorphic bundles over (X, σ_X),

$$0 \to E_1 \xrightarrow{\iota} E_2 \xrightarrow{\pi} E_3 \to 0,$$

it is easy to see that a real parabolic structure on E_2 determines a unique real parabolic structure on E_1 and E_3. Conversely, real parabolic structures on E_1 and E_3 determine a real parabolic structure on E_2. We call E_1 with this canonical real parabolic structure, a *real parabolic subbundle* of E_2 and E_3 a *real parabolic quotient* (cf. [1, 2, 5]).

Indeed, assume that we have a real parabolic structure on E_2. Then at each real parabolic point $x \in X$, we have the flag

$$E_2(x) = F_2^1(x) \supset F_2^2(x) \supset \cdots \supset F_2^{r_2}(x) \supset 0$$

with the following weights

$$0 \le a_{2_x}^1 < a_{2_x}^2 < \cdots < a_{2_x}^{r_2} < 1.$$

We define the real parabolic structure on E_1 as follows:

Let $H_i = \iota^{-1}(F_2^i(x))$. Then we obtain a sequence of subspaces

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{r_2}.$$

To obtain a strictly increasing sequence of subspaces, we choose a subset $\{i_1, \ldots, i_{r_1}\} \subset \{1, \ldots, r_2\}$ such that

$$H_1 = \cdots = H_{i_1} \supset H_{i_1+1} = \cdots = H_{i_2} \supset \cdots \supset H_{i_{r_1-1}+1} = \cdots = H_{i_{r_1}}.$$

We set $F_1^j(x) = H_{i_j}$ and $a_{1_x}^j = a_{2_x}^{i_j}$ for $j = 1, \ldots, r_1$. This gives at each real parabolic point $x \in X$ the following flag for $E_1(x)$:

$$E_1(x) = F_1^1(x) \supset F_1^2(x) \supset \cdots \supset F_1^{r_1}(x) \supset 0$$

with the following weights

$$0 \leq a_{1_x}^1 < a_{1_x}^2 < \cdots < a_{1_x}^{r_1} < 1.$$

Since E_1 is a real subbundle of E_2, we have the following commutative diagram:

$$\begin{array}{ccc} E_1(x) & \xrightarrow{\iota} & E_2(x) \\ \downarrow{\scriptstyle \sigma_x^{E_1}} & & \downarrow{\scriptstyle \sigma_x^{E_2}} \\ E_1(\sigma_X(x)) & \xrightarrow[\iota]{} & E_2(\sigma_X(x)) \end{array}.$$

From the commutativity of the above diagram, it easy to see that $\sigma_x^{E_1}(F_1^i(x)) = F_1^i(\sigma(x))$ and $a_{1_x}^i = a_{1_{\sigma(x)}}^i$ for $i = 1, \ldots, r_1$. This shows that E_1 with the above weighted flag structure over real parabolic points is a real parabolic vector bundle on (X, σ) with real parabolic structure over S.

DEFINITION 3.6

A real parabolic vector bundle (E, σ^E) is said to be *real parabolic semistable* (respectively, *real parabolic stable*) if for every proper real subbundle F of the real vector bundle E, we have

$$\mathrm{p}\mu(F) \leq \mathrm{p}\mu(E) \quad \text{(respectively, } \mathrm{p}\mu(F) < \mathrm{p}\mu(E)\text{)},$$

We say that a real parabolic vector bundle (E, σ^E) is real parabolic polystable if E is a direct sum of real parabolic stable bundles with the same slope $\mathrm{p}\mu(E)$.

4. Construction of the covering

Let (X, σ_X) be a real curve, and $S \subset X$ be a finite subset of X such that $\sigma_X(S) = S$. Consider the divisor $D = \sum_{x \in S} x$ on X. Let L be the line bundle corresponding to the divisor D on X.

We will show that the line bundle L is real holomorphic line bundle over (X, σ_X). It is enough to give a real structure on the $\mathcal{O}_X$-module $\mathcal{L} = \mathcal{O}_X(D)$. For $U \subset X$ open, define a map

$$\sigma_U^{\mathcal{L}} : \Gamma(U, \mathcal{O}_X(D)) \to \Gamma(\sigma_X(U), \mathcal{O}_X(D))$$

by $s \mapsto \overline{s \circ \sigma_X}$. Then $\sigma_U^{\mathcal{L}}$ is well-defined, since $\sigma_X(S) = S$ and $\mathrm{ord}_x(s) = \mathrm{ord}_{\sigma_X(x)}(\overline{s \circ \sigma_X})$. Since σ_X is an involution, $\sigma_{\sigma(U)}^{\mathcal{L}} \circ \sigma_U^{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}(U)}$. Thus, we get an $\mathcal{O}_X$-module homomorphism

$$\sigma^{\mathcal{L}} : \mathcal{O}_X(D) \to \mathcal{O}_X(D)^{\sigma}$$

such that $(\sigma^{\mathcal{L}})^{\sigma} \circ \sigma^{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}}$. We denote by σ^L the corresponding anti-holomorphic involution on L.

Let $N \geq 2$ be a positive integer. Consider the holomorphic map

$$\phi \,:\, L \to L^{\otimes N}$$

given by $v \mapsto v^{\otimes N}$

Let $s \in H^0(X, L^{\otimes N}) - \{0\}$ be a real holomorphic section such that $\mathrm{div}(s) = ND$. Let $Y = \phi^{-1}(s(X))$. Then Y is a compact Riemann surface. Let

$$p \,:\, Y \to X$$

be the restriction of the natural projection $\pi : L \to X$. Note that

$$Y = \{y \in L \mid y^{\otimes N} \in \mathrm{Image}(s)\}.$$

For $y \in Y$, we have $y^{\otimes N} = s(x)$ for some $x \in X$. From this, we have

$$\sigma^{L^N}(y^{\otimes N}) = (\sigma^L(y))^{\otimes N} = \sigma^{L^N}(s(x)).$$

Since s is a real holomorphic section, we have $\sigma^{L^N}(s(x)) = s(\sigma_X(x))$. It follows that $\sigma^L(y) \in Y$. Let

$$\sigma_Y : Y \to Y$$

be the restriction of the anti-holomorphic involution $\sigma^L : L \to L$. Therefore, Y is a real curve and we have the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\sigma_Y} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\sigma_X]{} & X \end{array}, \qquad (4.1)$$

where $\sigma_Y : Y \to Y$ is an anti-holomorphic involution.

Consider the action of the multiplicative group $\mathbb{C}^*$ on the total space of L. The action of the subgroup

$$\mu_N := \{c \in \mathbb{C} \mid c^N = 1\}$$

preserves the real curve Y. Clearly, the action of μ_N satisfies the following: for $y \in Y$ and $c \in \mu_N$, we have $\sigma_Y(c \cdot y) = c^{-1}\sigma_Y(y)$. Also, note that for each $x \in X \setminus S$, the cardinality of $p^{-1}(x)$ is N. It follows that $p : Y \to X$ is a Galois covering with Galois group $\Gamma = \mu_N$.

We summarize the above discussion in the following:

Lemma 4.1. *Let (X, σ_X) be a real curve, and let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. For any positive integer $N \geq 2$, there exists an N-fold cyclic cover $p : Y \to X$ which is precisely ramified over each point of S, such that the following diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\sigma_Y} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\sigma_X]{} & X \end{array} \qquad (4.2)$$

commutes, where σ_Y is an anti-holomorphic involution on Y. Moreover, the action of Γ on Y has the following property:

$$\sigma_Y(gy) = g^{-1}\sigma_Y(y) \qquad \textit{for all } g \in \Gamma \textit{ and } y \in Y.$$

Equivariant real vector bundles. Let (Y, σ_Y) be a real curve. Let G be a finite group acting holomorphically and effectively on Y with the property that $\sigma_Y(gy) = g^{-1}\sigma_Y(y)$ for all $g \in G$. Let $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$.

DEFINITION 4.2

Let Y be a Riemann surface, and let G be a subgroup of $\mathrm{Aut}(Y)$. An *admissible chart* for G at a point $y \in Y$ is a chart $\phi_y : U_y \to \mathbb{D}$ on Y centred at y such that

- U_y is G_y-invariant, where G_y denotes the isotropy subgroup of G at y; and
- $g(U_y) \cap U_y = \emptyset$ for all $g \in G \setminus G_y$.

We call U_y an *admissible neighbourhood* of y.

Remark 4.3. Let Y be a Riemann surface, and let G be a properly discontinuous subgroup of $\mathrm{Aut}(Y)$. Then the isotropy group G_y is finite and cyclic for each $y \in Y$, and has a unique generator g_y such that for every admissible chart $\phi_y : U_y \to \mathbb{D}$ at y, we have

$$\phi_y(g_y(x)) = \exp(2\pi\sqrt{-1}/m_y)\phi_y(x) \qquad \text{for all } x \in U_y,$$

where m_y is the cardinality of G_y. We call g_y the *isotropy generator* at y.

DEFINITION 4.4

A *G-equivariant real vector bundle* on (Y, σ_Y) consists of the following data: a real holomorphic vector bundle (W, σ^W) on (Y, σ_Y), and a lift of the natural action of G on Y to W such that

(a) the bundle projection $\pi : W \to Y$ is G-equivariant;
(b) if $y \in Y$ and $g \in G$, the map $W(y) \to W(g \cdot y)$, given by $v \mapsto g \cdot v$ is linear isomorphism;
(c) the following diagram;

$$\begin{array}{ccc} G \times W & \longrightarrow & W \\ {\scriptstyle (\mathrm{inv}, \sigma^W)} \downarrow & & \downarrow {\scriptstyle \sigma^W} \\ G \times W & \longrightarrow & W \end{array}$$

commutes, where $\mathrm{inv} : G \to G$ is an inverse map $g \mapsto g^{-1}$.

Let W_1 and W_2 be two G-equivariant real vector bundles on (Y, σ_Y). A real homomorphism $\phi : W_1 \to W_2$ of real vector bundles is called a morphism of G-equivariant real vector bundles if ϕ is a G-equivariant map.

DEFINITION 4.5

A G-equivariant holomorphic vector bundle E over a Riemann surface Y is said to be *G-semistable* (respectively, *G-stable*) if for every proper G-invariant vector subbundle $\mathcal{F}$ of E, we have

$$\mu(\mathcal{F}) \leq \mu(E) \quad (\text{respectively,} \quad \mu(\mathcal{F}) < \mu(E)).$$

DEFINITION 4.6

A G-equivariant real holomorphic vector bundle (E, σ^E) over a real curve is said to be *G-real semistable* (respectively, *G-real stable*) if for every proper G-invariant real vector subbundle $\mathcal{F}$ of E, we have

$$\mu(\mathcal{F}) \leq \mu(E) \quad (\text{respectively,} \quad \mu(\mathcal{F}) < \mu(E)).$$

A G-equivariant real holomorphic vector bundle E over a real curve Y is said to be *G-real polystable* if $E = \oplus_{i=1}^{n} E_i$, where E_i is G-invariant real subbundle of E which is G-real stable satisfying $\mu(E_i) = \mu(E)$ for $i = 1, \ldots, n$.

5. Correspondence

Let (X, σ_X) be a real curve, and let $S \subset X$ be a non-empty finite subset of X such that $\sigma_X(S) = S$. Let $N \geq 2$ be a positive integer. Let $p : Y \to X$ be an N-fold cyclic ramified covering as in the Lemma 4.1.

PROPOSITION 5.1

Let W be a Γ-equivariant real vector bundle over Y. Then the vector bundle $p_*^{\Gamma} W$ on X is a real parabolic vector bundle on X with real parabolic structure over S.

Proof. Let W be a Γ-equivariant real vector bundle over Y. Then the invariant direct image $p_*^{\Gamma} E$ defines a holomorphic vector bundle over X. The real structure on W induces a real structure on $p_*^{\Gamma} W$. Therefore, $p_*^{\Gamma} W$ is a real vector bundle over X.

Let $y \in Y$ be a ramified point of p over $x \in S$, and let ξ_y be the isotropy generator of $\Gamma = \Gamma_y$ at y (see Remark 4.3). Note that the fibre $W(y)$ is canonically identified with the fibre $p_*^{\Gamma} W(x)$. Note that $W(y)$ is a Γ-module. For the generator ξ_y of Γ, the distinct eigen-values of the operator $\xi : W(y) \to W(y)$ will be $\omega^{k_1}, \ldots, \omega^{k_{r_x}}$ with their multiplicities $n_1, \ldots, n_{r_x}$ respectively, where $0 \leq k_1 < k_2 < \cdots < k_{r_x} < N$ and $\omega = \exp(2\pi\sqrt{-1}/N)$. Let V_i be the ω^{k_i}-eigenspace of ξ and define

$$F_y^i = V_i \oplus \cdots \oplus V_{r_x}$$

with the associated weight $a_i = k_i/N$ for $i = 1, \ldots, r_x$. Then

$$p_*^{\Gamma} W(x) \simeq W(y) = F_y^1 \supset F_y^2 \supset \cdots \supset F_y^{r_x} \supset F_y^{r_x+1} = 0$$

is a flag with weights $0 \leq a_1 < a_2 < \cdots < a_n < 1$. The multiplicity of the weight a_i is $n_i = \dim(F_y^i / F_y^{i+1})$. Since the following diagram

$$\begin{array}{ccc} W(y) & \xrightarrow{\ \xi_y\ } & W(y) \\ {\scriptstyle \sigma_y^W}\downarrow & & \downarrow{\scriptstyle \sigma_y^W} \\ W(\sigma_Y(y)) & \xrightarrow[\ \xi_{\sigma_Y(y)}^{-1}\ ]{} & W(\sigma_Y(y)) \end{array}$$

commutes by Definition 4.4(c), we have

$$\sigma_y^W(V_i) = \omega^{-k_i}\text{-eigenspace for } \xi_{\sigma_Y(y)}^{-1} \text{ in } W(\sigma_Y(y)).$$

Let v' be a eigenvector of $\xi_{\sigma_Y(y)}^{-1}$ for the eigenvalue ω^{-k_i}. Then we have $\xi_{\sigma_Y(y)}(v') = \omega^{k_i} v'$. This shows that

$$\sigma_y^W(V_i) = \omega^{k_i}\text{-eigenspace for } \xi_{\sigma_Y(y)} \text{ in } W(\sigma_Y(y)).$$

Therefore, we obtain a real parabolic structure on $p_*^\Gamma W$ over S. □

PROPOSITION 5.2

Let (E, σ^E) be a real parabolic vector bundle on X with parabolic structure over S, all of whose weights are integral multiples of $1/N$. Then there exists a Γ-equivariant real vector bundle W on Y such that $p_*^\Gamma W$ is isomorphic to E as a real parabolic vector bundle.

Proof. Let E be a real parabolic vector bundle over X, i.e., for each $x \in S$,

$$E(x) = F_x^1 \supset F_x^2 \supset \cdots \supset F_x^{r_x} \supset 0$$

$$0 \le a_x^1 < a_x^2 < \cdots < a_x^{r_x} < 1,$$

where $a_x^i = k_i^x/N, 0 \le k_i^x < N$, satisfying the following:

$$\sigma_x^E(F_x^i) = F_{\sigma_X(x)}^i \qquad \text{and} \qquad a_x^i = a_{\sigma_X(x)}^i \quad \text{for all } i = 1, \ldots, r_x. \tag{5.1}$$

Consider weights $0 \le \alpha_x^1 \le \cdots \le \alpha_x^n < N$ according to their multiplicities, where $\alpha_x^i = k_i^x/N$ for all i. Define a map $\Delta_x : \mathbb{C}^* \to \mathrm{GL}(n, \mathbb{C})$ by

$$\Delta_x(z) = \begin{pmatrix} z^{k_1^x} & & 0 \\ & \ddots & \\ 0 & & z^{k_n^x} \end{pmatrix}.$$

Let y be a ramified point over x. Choose admissible neighbourhoods U_y of y and $U_{\sigma_Y(y)}$ of $\sigma_Y(y)$ in Y such that $\sigma_Y(U_y) = U_{\sigma_Y(y)}$ (in case if $x = \sigma(x)$ then we can choose U_y so that $\sigma_Y(U_y) = U_y$). We may assume that U_y's are pairwise disjoint and p^*E is trivial over U_y for all $y \in p^{-1}(S)$. Identify U_y with its isomorphic image via ϕ_y (see Definition 4.2) for all $y \in p^{-1}(S)$.

Let $U = Y \setminus p^{-1}(S)$ and $V = \cup_{y \in p^{-1}(S)} U_y$. Consider a vector bundle E_1 over U defined by $p^*E|_U$. Then E_1 is naturally a Γ-equivariant real vector bundle over U. For $y \in p^{-1}(S)$, let $\phi_y : p^*E|_{U_y} \xrightarrow{\simeq} U_y \times \mathbb{C}^n$ be an isomorphism. Let $\{e_1, \ldots, e_n\}$ be a flag basis for $E(x) \overset{\text{can}}{\simeq} p^*E(y)$, where $x = p(y)$. Via the isomorphism ϕ_y this gives a basis of $\mathbb{C}^n$ which we again denote by $\{e_1, \ldots, e_n\}$. Now consider the trivial vector bundle $E_2 = V \times \mathbb{C}^n$ on V. Define a map

$$\delta : (U \cap V) \times \mathbb{C}^n \to (U \cap V) \times \mathbb{C}^n$$

by

$$\delta(z, v) = (z, \Delta_x(z)v), \; z \in U_y \cap U.$$

Let W be the vector bundle obtained by glueing E_1 and E_2 using δ.

On E_2 take the Γ-action (defined by the action of the generator ξ_y),

$$\tau_1 : V \times \mathbb{C}^n \to V \times \mathbb{C}^n$$

given by

$$\tau_1(z, v) = (\omega z, \Delta_x(\omega)v), \quad z \in U_y.$$

Let

$$\sigma^{V\times\mathbb{C}^n} : V \times \mathbb{C}^n \to V \times \mathbb{C}^n$$

be the anti-holomorphic involution induced from p^*E. The action of Γ on $E_1|_{U\cap V} = (U \cap V) \times \mathbb{C}^n$ is given by

$$\tau_{12} : (U \cap V) \times \mathbb{C}^n \to (U \cap V) \times \mathbb{C}^n, \ (z, v) \mapsto (\omega z, v)$$

and the vector bundle $E_2|_{U\cap V} = (U \cap V) \times \mathbb{C}^n$ has the following Γ-action:

$$\tau_{21} : (U \cap V) \times \mathbb{C}^n \to (U \cap V) \times \mathbb{C}^n$$

defined by

$$\tau_{21}(z, v) = (\omega z, \Delta_x(\omega)v), \ z \in U_y \cap U.$$

Then the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times \mathbb{C}^n & \xrightarrow{\tau_{12}} & (U \cap V) \times \mathbb{C}^n \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ (U \cap V) \times \mathbb{C}^n & \xrightarrow[\tau_{21}]{} & (U \cap V) \times \mathbb{C}^n \end{array} .$$

For, let $(z, v) \in (U \cap V) \times \mathbb{C}^n$. If $z \in U_y \cap U$, then we have

$$\begin{aligned} \delta \circ \tau_{12}(z, v) &= \delta(\omega z, v) \\ &= (\omega z, \Delta_x(\omega z)v) \end{aligned}$$

and

$$\begin{aligned} \tau_{21} \circ \delta(z, v) &= \tau_{21}(z, \Delta_x(z)v) \\ &= (\omega z, \Delta_x(\omega)\Delta_x(z)v) \end{aligned}$$

This implies that $\delta \circ \tau_{12}(z, v) = \tau_{21} \circ \delta(z, v)$. Therefore, $\delta \circ \tau_{12} = \tau_{21} \circ \delta$. This gives the Γ-action on W.

Similarly, the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times \mathbb{C}^n & \xrightarrow{\sigma_{12}} & (U \cap V) \times \mathbb{C}^n \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ (U \cap V) \times \mathbb{C}^n & \xrightarrow[\sigma_{21}]{} & (U \cap V) \times \mathbb{C}^n \end{array} ,$$

where $\sigma_{12} = \sigma_{21} = \sigma^{V\times\mathbb{C}^n}|_{(U\cap V)\times\mathbb{C}^n}$. For, let $(z, v) \in (U \cap V) \times \mathbb{C}^n$. If $z \in U_y \cap U$, we have

$$\begin{aligned} \delta \circ \sigma_{12}(z, v) &= \delta(\bar{z}, \sigma^{V\times\mathbb{C}^n}(v)) \\ &= (\bar{z}, \Delta_{\sigma(x)}(\bar{z})\sigma^{V\times\mathbb{C}^n}(v)) \end{aligned}$$

and

$$\begin{aligned}\sigma_{21} \circ \delta(z, v) &= \sigma_{21}(z, \Delta_x(z)v) \\ &= (\bar{z}, \sigma^{V\times\mathbb{C}^n}(\Delta_x(z)v))\,.\end{aligned}$$

From (5.1), it follows that $\delta \circ \sigma_{12} = \sigma_{21} \circ \delta$.

This gives the real structure on W compatible with the Γ-action. It is easy to see that $p_*^{\Gamma} W$ and E are isomorphic as real parabolic vector bundles. □

Let $\mathbf{RP}(X, N)$ denote the full sub-category of $\mathbf{RP}(X)$ whose objects are real parabolic vector bundles on (X, σ_X) with parabolic structure over S, all of whose weights are integral multiples of $1/N$. Let $\mathbf{RE}_{\Gamma}(Y)$ denote the category whose objects are all Γ-equivariant real vector bundles on $(Y, \ \sigma_Y)$ and morphisms are morphisms of Γ-equivariant real vector bundles.

Theorem 5.3. *There is a canonical functor* $\Psi : \mathbf{RP}(X, N) \to \mathbf{RE}_{\mathbf{\Gamma}}(Y)$ *which is an equivalence of categories.*

Proof. We first define a functor $\Psi : \mathbf{RE}_{\mathbf{\Gamma}}(Y) \to \mathbf{RP}(X, N)$ as follows: Let W be a Γ-equivariant real vector bundle on Y of rank n. Then $p_*^{\Gamma} W$ is a locally free sheaf of rank n on X. The real structure on W induces a real structure on $p_*^{\Gamma} W$. We define $\Psi(W)$ to be the corresponding real vector bundle on X. By Proposition 5.1, we obtain a real parabolic structure on $\Psi(W)$ with the real parabolic structure over S having all the weights integral multiples of $1/N$. This implies that $\Psi(W)$ is an object of $\mathbf{RP}(X, N)$. Let $\phi : W_1 \to W_2$ be a morphism in $\mathbf{RE}_{\Gamma}(Y)$. Then $\Psi(\phi) : \Psi(W_1) \to \Psi(W_2)$ is defined as follows: For $U \subseteq X$ open and a section $s \in \Gamma(U, \Psi(W_1))$, we define

$$\Psi(\phi)_U(s) := \phi_{p^{-1}(U)}(s).$$

Since ϕ is Γ-equivariant, $\phi_{p^{-1}(U)}(s) \in \Gamma(p^{-1}(U), W_2)^{\Gamma}$. Clearly, $\Psi(\phi)$ is a morphism of real vector bundles on X. Therefore, we obtain a canonical functor $\Psi : \mathbf{RE}_{\mathbf{\Gamma}}(Y) \to \mathbf{RP}(X, N)$ which is fully faithful (cf. [2]). To show that Ψ is an equivalence of categories, we only need to check that Ψ is essentially surjective, which follows from Proposition 5.2. □

PROPOSITION 5.4

A real parabolic vector bundle (E, σ^E) *in* $\mathbf{RP}(X, N)$ *is real parabolic semistable if and only if the corresponding* Γ*-equivariant real vector bundle* W *on* Y *is semistable in the usual sense.*

Proof. Let E be a real parabolic semistable vector bundle over X. Let W be the corresponding Γ-equivariant real vector bundle over Y. Note that

$$\operatorname{p deg}(E) = \frac{\deg(W)}{N}, \tag{5.2}$$

where N is the order of the Galois group of the covering $p : Y \to X$ (see p. 165 of [3]). Let

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{l-1} \subset W_l = W$$

be the Harder–Narasimhan filtration of W. Since $\sigma^W : W \to W^\sigma$ is an isomorphism, we have the following filtration:

$$0 = \sigma^W(W_0) \subset \sigma^W(W_1) \subset \sigma^W(W_2) \subset \cdots \subset \sigma^W(W_{l-1}) \subset \sigma^W(W_l) = W^\sigma. \tag{5.3}$$

We also have another filtration

$$0 = {W_0}^\sigma \subset {W_1}^\sigma \subset {W_2}^\sigma \subset \cdots \subset {W_{l-1}}^\sigma \subset {W_l}^\sigma = W^\sigma. \tag{5.4}$$

Since filtrations in (5.3) and (5.4) of W^σ satisfy the conditions of the Harder–Narasimhan filtration, by the uniqueness of the Harder–Narasimhan filtration we conclude that

$$\sigma^W(W_i) = {W_i}^\sigma$$

for all $i = 1, \dots l$. Thus, W_1 is a real semistable subbundle of W. Similarly, by the uniqueness of W_1 it follows that W_1 is left invariant under the action of Γ on W. Therefore, W_1 is a Γ-equivariant real semistable subbundle of W. For every subsheaf $\mathcal{F}$ of W, we have $\mu(\mathcal{F}) \leq \mu(W_1)$. By the correspondence, $p_*^\Gamma W_1$ is a real parabolic subbundle of E. Since E is a real parabolic semistable, using (5.2), we have

$$\mu(W_1) \leq \mu(W).$$

Since W_1 is a maximal semistable subbundle of W, we have $\mu(W_1) = \mu(W)$. This proves that W is semistable.

Conversely, assume that W is semistable. Let F be the proper real parabolic subbundle of E. Let V be the corresponding Γ-equivariant real vector bundle over Y. By the construction, it follows that V is a Γ-equivariant real subbundle of W. Since W is semistable, we have $\mu(V) \leq \mu(W)$. Using (5.2), we have $\mathrm{p}\mu(F) \leq \mathrm{p}\mu(E)$. This shows that E is real parabolic semistable. □

Remark 5.5. Using (5.2), it can be easily seen that a real parabolic vector bundle E is real parabolic stable if and only if the corresponding Γ-equivariant real vector bundle W on Y is Γ-real stable. Consequently, a real parabolic vector bundle E is real parabolic polystable if and only if the corresponding Γ-equivariant real vector bundle W on Y is Γ-real polystable.

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References

[1] Boden Hans U, Representations of orbifold groups and parabolic bundles, *Comment. Math. Helv.* **66(3)** (1991) 389–447

[2] Mehta V B and Seshadri C S, Moduli of vector bundles on curves with parabolic structure, *Math. Ann.* **248** (1980) 205–239

[3] Seshadri C S, Moduli of π–vector bundles over an algebraic curve, in: Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1970) (Rome Edizioni Cremonese) pp. 139–260

[4] Seshadri C S, Moduli of vector bundles on curves with parabolic structures, *Bull. Amer. Math. Soc.* **83** (1977) 124–126

[5] Seshadri C S, Fibrés vectoriels sur les courbes algébriques, Astérisque (1982) (Société Mathématique de France: Paris) vol. 96