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Alexander duals of multipermutohedron ideals

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Abstract. An Alexander dual of a multipermutohedron ideal has many combinatorial properties. The standard monomials of an Artinian quotient of such a dual correspond bijectively to some λ-parking functions, and many interesting properties of these Artinian quotients are obtained by Postnikov and Shapiro (*Trans. Am. Math. Soc.* **356** (2004) 3109–3142). Using the multigraded Hilbert series of an Artinian quotient of an Alexander dual of multipermutohedron ideals, we obtained a simple proof of Steck determinant formula for enumeration of λ-parking functions. A combinatorial formula for all the multigraded Betti numbers of an Alexander dual of multipermutohedron ideals are also obtained.

Keywords. Multipermutohedron; Alexander dual; Hilbert series; parking functions.

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1. Introduction

The notion of Alexander duality for a squarefree monomial ideals has been extended to all monomial ideals by Miller [4]. Let $\mathbb{N}$ be the set of nonnegative integers and $R = k[x_1, x_2, \ldots, x_n]$ be the standard polynomial ring over a field k. For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$, let $\mathbf{x}^{\mathbf{b}}$ be the monomial $\prod_{i=1}^{n} x_i^{b_i} \in R$. Consider a monomial ideal I in the polynomial ring R. Choose $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ so that all the minimal generators of I divide the monomial $\mathbf{x}^{\mathbf{a}}$. The Alexander dual $I^{[\mathbf{a}]}$ of I with respect to $\mathbf{a}$ (Definition 2.1) is a monomial ideal in the polynomial ring R. Many properties of the Alexander dual $I^{[\mathbf{a}]}$ of I are given in Chapter 5 in the book of Miller and Sturmfels [5].

In this paper, we study Alexander duals of multipermutohedron ideals. As permutohedron and multipermutohedron are polytopes having very rich combinatorial properties, it is not surprising that the associated multipermutohedron ideals as well as its Alexander duals also have many interesting combinatorial properties. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ with $0 \leq u_1 < u_2 < \cdots < u_n$. For any permutation σ of $\mathbf{u}$, $\sigma\mathbf{u} = (\sigma u_1, \ldots, \sigma u_n) \in \mathbb{N}^n$. The convex hull of $n!$ points $\pi\mathbf{u} \in \mathbb{R}^n$ is an $(n-1)$-dimensional polytope in $\mathbb{R}^n$ called a *permutohedron* $P(\mathbf{u})$. The associated ideal

$I(\mathbf{u}) = \langle \mathbf{x}^{\pi \mathbf{u}} \rangle$ in the polynomial ring $R = k[x_1, \ldots, x_n]$, generated by the monomial vertex labels $\mathbf{x}^{\pi \mathbf{u}}$ of $P(\mathbf{u})$ is called a *permutohedron ideal*. The Alexander dual of the permutohedron ideal $I(1, 2, \ldots, n)$ with respect to $\mathbf{n} = (n, n, \ldots, n)$ is given by $I(1, 2, \ldots, n)^{[\mathbf{n}]} = \langle \left(\prod_{i \in A} x_i\right)^{n-|A|+1} : \emptyset \neq A \subset [n] \rangle$. The quotient $R/I(1, 2, \ldots, n)^{[\mathbf{n}]}$ is an Artinian k-algebra and $\dim_k(R/I(1, 2, \ldots, n)^{[\mathbf{n}]}) = (n+1)^{n-1}$. In other words, the number of standard monomials in the Artinian quotient $R/I(1, 2, \ldots, n)^{[\mathbf{n}]}$ is precisely $(n+1)^{n-1}$, which is the number of labeled trees on $(n+1)$ vertices. Thus the monomial ideal $I(1, 2, \ldots, n)^{[\mathbf{n}]}$ is called a *tree ideal*. The vertices of the first barycentric subdivision of an $(n-1)$-simplex can be naturally labeled with the minimal generators of the tree ideal $I(1, 2, \ldots, n)^{[\mathbf{n}]}$ and the free resolution of tree ideal supported by the *first barycentric subdivision* $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex Δ_{n-1} is minimal.

If we consider $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ with $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n$ and $u_i = u_j$ for some $i \neq j$, then also the polytope $P(\mathbf{u})$ and the ideal $I(\mathbf{u})$ are well-defined. In this case, we call $P(\mathbf{u})$ a *multipermutohedron* and $I(\mathbf{u})$ a *multipermutohedron ideal*. For any integer $c \geq 1$, we consider $\mathbf{u_n} + \mathbf{c} - \mathbf{1} = (u_n + c - 1, u_n + c - 1, \ldots, u_n + c - 1)$ and the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. If $u_1 \geq 1$, then the quotient $R' = R/I(\mathbf{u})^{[\mathbf{u_n-c+1}]}$ is an Artinian k-algebra and the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ is again given by

$$I(\mathbf{u})^{[\mathbf{u_n+c-1}]} = \left\langle \left(\prod_{i \in A} x_i\right)^{u_n - u_{|A|} + c} : \emptyset \neq A \subset [n] \right\rangle,$$

except the generating set need not be minimal. In this case also, one would like to know, what is the dimension $\dim_k(R')$ or equivalently, the number of standard monomials in $R' = R/I(\mathbf{u})^{[\mathbf{u_n-c+1}]}$? A solution of this problem is known and it lies in counting *generalized parking functions*. The dimension $\dim_k(R')$ equals the number of λ-parking functions, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$; $\lambda_i = u_n - u_i + c$. Using a free resolution of R' and its multigraded Hilbert series, we give a simple proof of the Steck determinant formula for counting λ-parking functions (Theorem 2.8). We have obtained combinatorial formula for all the multigraded Betti numbers of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ of the multipermutohedron ideal (Theorem 3.2). As an application, we characterized minimality of the cellular free resolution supported by a labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ spanned by the barycenters of some of the faces of an $n-1$-simplex Δ_{n-1} (Theorem 3.7).

Some of the results in this paper are implicit in the work of Postnikov and Shapiro [7]. They considered certain classes of monomial ideals; namely *monotone monomial ideals*, *order monomial ideals*, and obtained a formula for the number of standard monomials in their Artinian quotient algebras. They also computed the (coarse) Hilbert series of these quotient algebras and obtained an extension of a formula for enumerating λ-parking functions due to Pitman and Stanley [6]. We have used multigraded (or fine) Hilbert series of the quotient algebra R' and recovered Steck determinant formula for counting λ-parking functions. Postnikov and Shapiro also gave combinatorial formula for multigraded Betti numbers of strictly monotone monomial ideals. The monomial ideal $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ is strictly monotone if $u_1 < u_2 < \cdots < u_n$. We have obtained multigraded Betti numbers of $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ in all the cases.

2. Alexander duals of monomial ideals

Let n be a positive integer and $[n] = \{1, 2, \ldots, n\}$. Let Δ be a simplicial complex on the vertex set $[n]$. The *Alexander dual* Δ^* of Δ is a simplicial complex on $[n]$ given by

$$\Delta^* = \{A \subseteq [n] : [n] - A \notin \Delta\}.$$

For any subset $A \subseteq [n]$, $\mathbf{x}_A = \prod_{i \in A} x_i$ is a squarefree monomial in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ over a field k. The *Stanley–Reisner ideal* I_Δ of the simplicial complex Δ is defined to be the squarefree monomial ideal

$$I_\Delta = \langle \mathbf{x}_A : A \text{ is a minimal nonface of } \Delta \rangle$$

in R. Now the *Alexander dual of the squarefree monomial ideal* I_Δ is defined to be the Stanley–Reisner ideal I_{Δ^*} of the Alexander dual Δ^*.

The notion of Alexander duals of a squarefree monomial ideals has been extended to monomial ideals by Miller [4]. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$. Then $\mathbf{x}^{\mathbf{b}}$ denotes the monomial $\prod_{i=1}^n x_i^{b_i}$ and $\mathbf{m}^{\mathbf{b}}$ denotes the monomial ideal $\langle x_i^{b_i} : b_i > 0 \rangle$ in the standard polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ over a field k. Consider a monomial ideal I in the polynomial ring R. Then I has a unique minimal set of monomial generators and all the (monomial) primary components of I are unique. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathbb{N}^n$ such that $\{\mathbf{x}^{\mathbf{b}} : \mathbf{b} \in \mathcal{A}\}$ be the set of minimal generators of I and $\{\mathbf{m}^{\mathbf{b}} : \mathbf{b} \in \mathcal{B}\}$ be the set of (monomial) primary components of I. Thus, we have

$$I = \langle \mathbf{x}^{\mathbf{b}} : \mathbf{b} \in \mathcal{A} \rangle = \bigcap \{\mathbf{m}^{\mathbf{b}} : \mathbf{b} \in \mathcal{B}\}.$$

Choose $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ such that $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$ for all $\mathbf{b} \in \mathcal{A}$. In other words, all the minimal generator $\mathbf{x}^{\mathbf{b}}$ of I divide $\mathbf{x}^{\mathbf{a}}$. Whenever $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$, we set $\mathbf{a} \ominus \mathbf{b} \in \mathbb{N}^n$ by defining its ith coordinate

$$(\mathbf{a} \ominus \mathbf{b})_i = \begin{cases} a_i + 1 - b_i, & \text{if } b_i > 0, \\ 0, & \text{if } b_i = 0. \end{cases}$$

DEFINITION 2.1

The *Alexander dual* $I^{[\mathbf{a}]}$ of the monomial ideal I with respect to $\mathbf{a}$ is defined to be the monomial ideal

$$I^{[\mathbf{a}]} = \bigcap \{\mathbf{m}^{\mathbf{a} \ominus \mathbf{b}} : \mathbf{b} \in \mathcal{A}\}.$$

Equivalently, $I^{[\mathbf{a}]} = \langle \mathbf{x}^{\mathbf{a} \ominus \mathbf{b}} : \mathbf{b} \in \mathcal{B} \rangle$.

Remark 2.2.

(1) The Alexander dual is indeed a duality in the sense that $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$. Also, the Alexander dual $(I_\Delta)^{[\mathbf{1}]}$ of a Stanley–Reisner ideal I_Δ with respect to $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$ is precisely I_{Δ^*}. Therefore, the notion of Alexander duality of monomial ideals introduced by Miller turns out to be an appropriate generalization.

(2) Let $\mathbf{a}_I$ be the exponent on the LCM of all minimal generators of the monomial ideal I. Then we define the (*tight*) *Alexander dual* $I^* = I^{[\mathbf{a}_I]}$. The only inadequacy of this notion is that $(I^*)^*$ need not equal I, unlike $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$.

We now define multipermutohedron and associated multipermutohedron ideals. Let n be a positive integer and $\mathbf{m} = (m_1, m_2, \dots, m_l) \in \mathbb{N}^l$ such that each $m_i \geq 1$ and $n = \sum_{i=1}^{l} m_i$. Set $s_0 = 0$ and $s_j = \sum_{i=1}^{j} m_i$. Let $\mathbf{u}(\mathbf{m}) = (u_1, u_2, \dots, u_n) \in \mathbb{N}^n$ such that the first m_1 coordinates be equal to u_1 and next m_2 coordinates be equal to u_{m_1+1} and so on. In other words,

$$u_1 = \cdots = u_{s_1} < u_{s_1+1} = \cdots = u_{s_2} < u_{s_2+1} = \cdots < u_{s_{l-1}+1} = \cdots = u_n.$$

Let $e_1, e_2, \dots, e_n$ be the standard basis vectors in $\mathbb{R}^n$. For $0 \leq i < j$, set $E(i, j) = \sum_{\alpha=i+1}^{j} e_\alpha$. Then, $\mathbf{u}(\mathbf{m}) = \sum_{i=1}^{l} u_{s_i} E(s_{i-1}, s_i)$ and $u_{s_i} < u_{s_{i+1}}$ for $1 \leq i < l$. For a permutation π of $\mathbf{u}(\mathbf{m})$, let $\pi\mathbf{u}(\mathbf{m}) = (\pi u_1, \dots, \pi u_n) \in \mathbb{R}^n$. The convex hull of all the points $\pi\mathbf{u}(\mathbf{m})$; π a permutation of $\mathbf{u}(\mathbf{m})$, is an $(n-1)$-dimensional polytope $P(\mathbf{u}(\mathbf{m}))$ called *a multipermutohedron*. Each vertex $\pi\mathbf{u}(\mathbf{m})$ of the multipermutohedron $P(\mathbf{u}(\mathbf{m}))$ is naturally labeled with the monomial $\mathbf{x}^{\pi\mathbf{u}(\mathbf{m})}$ making it a labeled polyhedral cell complex. The associated monomial ideal $I(\mathbf{u}(\mathbf{m})) = \langle \mathbf{x}^{\pi\mathbf{u}(\mathbf{m})} : \pi \text{ a permutation of } \mathbf{u}(\mathbf{m}) \rangle$ is called a *multipermutohedron ideal*. The cellular free complex associated to the multipermutohedron is always a free resolution of the multipermutohedron ideal. A computation of the Betti numbers of multipermutohedron ideals and characterization of minimality of the cellular resolution are carried out in [3].

The (tight) Alexander dual $I(\mathbf{u}(\mathbf{m}))^* = I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}]}$, where $\mathbf{u_n} = (u_n, u_n, \dots, u_n)$. In this paper, more generally, the Alexander duals $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n+c-1}]}$ of the multipermutohedron ideal $I(\mathbf{u}(\mathbf{m}))$ with respect to $\mathbf{u_n} + \mathbf{c} - \mathbf{1} = (u_n + c - 1, \dots, u_n + c - 1)$ for $c \geq 1$ are considered.

Lemma 2.3. *The minimal generators of the Alexander dual* $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n+c-1}]}$ *of the multipermutohedron ideal is given by*

$$I(\mathbf{u})^{[\mathbf{u_n+c-1}]} = \left\langle \left(\prod_{j \in A} x_j \right)^{u_n - u_{|A|} + c} : A \subseteq [n],\ |A| = s_i + 1 \text{ for } 0 \leq i < l \text{ and } u_{|A|} \geq 1 \right\rangle.$$

Therefore, the quotient $R/(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ *is an Artinian k-algebra if and only if* $u_1 \geq 1$.

Proof. Suppose all the minimal generators of a monomial ideal I in R divides $\mathbf{x}^{\mathbf{a}}$. Then, for any $\mathbf{b} = (b_1, \dots, b_n)$ with $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$, the monomial $\mathbf{x}^{\mathbf{b}} \notin I$ if and only if $\mathbf{x}^{\mathbf{a}-\mathbf{b}} \in I^{[\mathbf{a}]}$ (see Proposition 5.23 of [5]). Clearly, $\mathbf{x}^{\mathbf{a}-\mathbf{b}}$ is a minimal generator of $I^{[\mathbf{a}]}$ precisely when $\mathbf{b}$ is maximal. If $u_{s_i} \geq 1$, then taking $\mathbf{b} = (u_{s_i} - 1)E(0, s_{i-1} + 1) + (u_n + c - 1)E(s_{i-1} + 1, n)$ or its permutation, we see that $\mathbf{x}^{\mathbf{u_n+c-1-b}}$ is a minimal generator of $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n+c-1}]}$. □

Remark 2.4. It follows from Lemma 2.3 that Alexander dual of a multipermutohedron ideal is a sum of special multipermutohedron ideals. In fact, the tree ideal $I(1, 2, \dots, n)^{[\mathbf{n}]}$ is a sum of multipermutohedron ideals

$$I(0, \dots, 0, n) + I(0, \dots, 0, n-1, n-1) + \cdots + I(0, 2, 2, \dots, 2) + I(1, 1, \dots, 1).$$

This gives another motivation for studying multipermutohedron ideals.

If $u_1 \geq 1$, then the multigraded Hilbert series of the Artinian k-algebra $R/(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ is the sum of finitely many standard monomial in $R' = R/I(\mathbf{u})^{[\mathbf{u_n-c+1}]}$. In order to describe the multigraded Hilbert series of R', we need the notion of λ-*parking functions*. For more on parking functions or λ-parking functions, we refer to [6,7,9].

DEFINITION 2.5

A sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is said to be a *parking function of length n* if its nondecreasing rearrangement $q_1 \leq q_2 \leq \cdots \leq q_n$ satisfy $q_i \leq i, \ \forall i$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, a sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is said to be a λ-*parking function of length n* if its nondecreasing rearrangement $q_1 \leq q_2 \leq \cdots \leq q_n$ satisfy $q_i \leq \lambda_{n-i+1}, \ \forall i$.

A λ-parking function for $\lambda = (n, n-1, n-2, \ldots, 1)$ is a parking function.

Lemma 2.6. *A monomial* $\mathbf{x^p} = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ *is a standard monomial in the Artinian k-algebra* $R' = R/(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ *if and only if* $\mathbf{p}+\mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ *is a* λ-*parking function for* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (u_n - u_1 + c, u_n - u_2 + c, \ldots, u_n - u_n + c)$. *Thus, the multigraded Hilbert series of* R' *is given by* $H(R', \mathbf{x}) = \sum_{\mathbf{p}\in\Lambda_n} \mathbf{x^{p-1}}$, *where* Λ_n *is the set of all* λ-*parking functions of length n. Also,* $\dim_k(R') = H(R', \mathbf{1}) = |\Lambda_n|$.

Proof. Suppose $\mathbf{p}+\mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ is not a λ-parking function. Thus there is a nondecreasing rearrangement $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ of $\mathbf{p}+\mathbf{1}$ such that $q_j > \lambda_{n-j+1} = u_n - u_{n-j+1} + c$ for some j. Equivalently, there are at least $n-j+1$ indices $i_1, i_2, \ldots, i_{n-j+1}$ such that $p_{i_r} \geq u_n - u_{n-j+1} + c$ for $1 \leq r \leq n-j+1$. This condition holds if and only if $\mathbf{x^p}$ is divisible by $(\prod_{i\in A} x_i)^{u_n - u_{n-j+1}+c}$ with $A = \{i_1, i_2, \ldots, i_{n-j+1}\}$. Hence, $\mathbf{x^p} \in I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. This shows that $\mathbf{p}+\mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ is not a λ-parking function $\Leftrightarrow$ $\mathbf{x^p}$ is not a standard monomial in R'. Since the multigraded Hilbert series is the sum of all standard monomials, the second and third parts of the lemma follows. □

We now proceed to give a proof of Steck determinant [8] formula for counting λ-parking functions. Consider the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1}. A vertex of $\mathbf{Bd}(\Delta_{n-1})$ corresponds to a nonempty subset $A \subseteq [n]$ and hence it is naturally labeled with the monomial $(\prod_{\alpha\in A} x_\alpha)^{u_n - u_{|A|}+c}$ in R. Also, an $(i-1)$-dimensional face of $\mathbf{Bd}(\Delta_{n-1})$ corresponds to a tuple $(A_1, A_2, \ldots, A_i)$ of nonempty subsets of $[n]$ with $\emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_i$ and the monomial label on this $i-1$-face is

$$\prod_{j=1}^{i} \left(\prod_{\alpha\in A_j - A_{j-1}} x_\alpha \right)^{u_n - u_{|A_j|}+c}.$$

Thus, $\mathbf{Bd}(\Delta_{n-1})$ is a labeled simplicial complex. To each labeled simplicial complex or a labeled polyhedral cell complex $\mathbf{X}$, one can associate a free complex of R-modules [1,2].

Let $\mathcal{F}_{i-1}(\mathbf{X})$ (or simply $\mathcal{F}_{i-1}$) be the set of $i-1$-faces of $\mathbf{X}$. Then the associated free complex $\mathbb{F}_*(\mathbf{X})$ is given by

$$\mathbb{F}_*(\mathbf{X}) : \cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0, \tag{2.1}$$

where $F_i = \bigoplus_{\sigma \in \mathcal{F}_{i-1}(\mathbf{X})} R[-\nu(\sigma)]$, monomial $\mathbf{x}^{\nu(\sigma)}$ is the label on the face σ and δ_i is a differential. If $\mathbb{F}_*(\mathbf{X})$ is a resolution of $R/I(\mathbf{X})$, where $I(\mathbf{X})$ is the ideal in R generated by the vertex labels of $\mathbf{X}$, then the multigraded Hilbert series of $R/I(\mathbf{X})$ is given by

$$\begin{aligned} H(R/I(\mathbf{X}), \mathbf{x}) &= \sum_{i=0}^{\dim(\mathbf{X})+1} (-1)^i H(F_i, \mathbf{x}) \\ &= \sum_{i=0}^{\dim(\mathbf{X})+1} (-1)^i \sum_{\sigma \in \mathcal{F}_{i-1}} \frac{\mathbf{x}^{\nu(\sigma)}}{(1-x_1)(1-x_2)\cdots(1-x_n)}. \end{aligned}$$

The free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ associated to the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ is in fact a free resolution of the quotient $R' = R/(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ (see [7]). This resolution is usually nonminimal, but it can be used to calculate the multigraded Hilbert series $H(R', \mathbf{x})$ of the quotient R'. We have

$$\begin{aligned} H(R', \mathbf{x}) &= \frac{1}{\prod_{j=1}^{n}(1-x_j)} \sum_{i=0}^{n} (-1)^i \sum_{(A_1,\ldots,A_i) \in \mathcal{F}_{i-1}} \\ &\quad \times \left[\prod_{j=1}^{i} \left(\prod_{\alpha \in A_j - A_{j-1}} x_\alpha \right)^{u_n - u_{|A_j|} + c} \right]. \end{aligned} \tag{2.2}$$

PROPOSITION 2.7

$$|\Lambda_n| = (n!) \sum_{i=0}^{n} (-1)^{n-i} \left\{ \sum_{0=t_0<t_1<\cdots<t_{i-1}<t_i=n} \left(\prod_{j=1}^{i} \frac{(\lambda_{t_j})^{t_j - t_{j-1}}}{(t_j - t_{j-1})!} \right) \right\}.$$

Proof. From Lemma 2.6, we have

$$|\Lambda_n| = H(R', \mathbf{1}) = \lim_{\substack{x_1 \to 1, \\ \ldots, \\ x_n \to 1}} H(R', \mathbf{x}) = \lim_{\substack{x_1 \to 1, \\ \ldots, \\ x_n \to 1}} \frac{Q(\mathbf{x})}{\prod_{j=1}^{n}(1-x_j)},$$

where the polynomial $Q(\mathbf{x})$, in view of eq. (2.2), is

$$Q(\mathbf{x}) = \sum_{i=0}^{n} (-1)^i \sum_{(A_1,\ldots,A_i) \in \mathcal{F}_{i-1}(\mathbf{Bd}(\Delta_{n-1}))} \left[\prod_{j=1}^{i} \left(\prod_{\alpha \in A_j - A_{j-1}} x_\alpha \right)^{u_n - u_{|A_j|} + c} \right].$$

Now applying L'Hospital's rule, we see that

$$|\Lambda_n| = \frac{1}{(-1)^n} \left.\frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \ldots \partial x_n}\right|_{\mathbf{x}=\mathbf{1}}.$$

In the partial derivative $\frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \ldots \partial x_n}$, the term corresponding to the tuple $(A_1, \ldots, A_i)$ survives only if $|A_i| = n$. Putting $|A_j| = t_j, \lambda_j = u_n - u_j + c$, and observing that the number of $i-1$-faces $(A_1, \ldots, A_i)$ with $|A_j| = t_j$ is precisely $\frac{n!}{\prod_{j=1}^{i}(t_j - t_{j-1})!}$, we get the desired result. □

Theorem 2.8 [8]. *Let* $\mu_{ij} = \frac{(\lambda_{n-i+1})^{j-i+1}}{(j-i+1)!}$ *if* $1 \le i \le j+1$ *and* $\mu_{ij} = 0$ *if* $j+1 < i \le n$. *Then*

$$|\Lambda_n| = (n!) \det[\mu_{ij}]_{n \times n}.$$

Proof. Let $\mathbf{v}_r = \sum_{j=0}^{r} \frac{(\lambda_{n-j})^{r-j}}{(r-j)!} e_{j+1}$ for $1 \le r \le n$ and $e_{n+1} = 0$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The column vector $\mathbf{v}_r$ is the r-th column of the $n \times n$ matrix $[\mu_{ij}]$. Thus

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n = (\det[\mu_{ij}]) e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

It is a straightforward verification that the exterior product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n$ equals

$$\sum_{i=0}^{n} (-1)^{n-i} \left\{ \sum_{0=t_0<t_1<\cdots<t_{i-1}<t_i=n} \left(\prod_{j=1}^{i} \frac{(\lambda_{t_j})^{t_j - t_{j-1}}}{(t_j - t_{j-1})!} \right) \right\} e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

Since exterior product is distributive and $e_i \wedge e_i = 0$, terms in the product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n$ are obtained by choosing one term from each vector $\mathbf{v}_r$ so that their product give rise to a multiple of $e_1 \wedge e_2 \wedge \cdots \wedge e_n$. For $0 \le i \le n$ and a tuple $(t_1, t_2, \ldots, t_i)$ with $0 = t_0 < t_1 \cdots < t_{i-1} < t_i = n$, we choose a term $\mathbf{f}_r$ from the vector $\mathbf{v}_r$ $(1 \le r \le n)$ as follows:

$$\mathbf{f}_r = \begin{cases} \frac{(\lambda_{t_j})^{t_j - t_{j-1}}}{(t_j - t_{j-1})!} e_{n-t_j+1}, & \text{if } r = n - t_{j-1}, \\ e_{r+1}, & \text{if } r \ne n - t_{j-1}. \end{cases}$$

Then $\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \cdots \wedge \mathbf{f}_n$ is clearly equal to

$$\left(\prod_{j=1}^{i} \frac{(\lambda_{t_j})^{t_j - t_{j-1}}}{(t_j - t_{j-1})!} \right) \left(\prod_{j=1}^{i} (-1)^{t_j - t_{j-1} - 1} \right) e_1 \wedge e_2 \wedge \cdots \wedge e_n. \tag{2.3}$$

As $\prod_{j=1}^{i} (-1)^{t_j - t_{j-1} - 1} = (-1)^{n-i}$ and the product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n$ is obtained by summing quantity (2.3) over all the possible values of i and $(t_1, t_2, \ldots, t_i)$, using Proposition 2.7, we get

$$(n!) \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_n = |\Lambda_n| e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

This completes the proof. □

3. Multigraded Betti numbers

Postnikov and Shapiro studied the monomial ideal $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ in [7], without referring it as an Alexander dual. They explicitly constructed a finite free resolution of the so-called *monotone monomial ideals* and this free resolution is minimal if the monomial ideal is *strictly monotone*. In particular, the ideal $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is strictly monotone if $\mathbf{m} = (1, 1, \ldots, 1)$, or equivalently $u_1 < u_2 < \cdots < u_n$, and in this case, the minimal resolution of $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is the cellular resolution supported by the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex Δ_{n-1} with the vertex label $(\prod_{i\in A} x_i)^{u_n-u_{|A|}+c}$ on the vertex corresponding to $\emptyset \neq A \subseteq [n]$ (see Corollary 6.4 and Corollary 12.2 of [7]). The minimal resolution of the monomial ideal $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ and their Betti numbers for the case $\mathbf{m} \neq (1, 1, \ldots, 1)$ are not discussed in [7].

The multigraded Betti numbers of the multipermutohedron ideal are obtained in [3] and by realizing the monomial ideal $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ as an Alexander dual of a multipermutohedron ideal, its Betti numbers are calculated on similar lines.

For a monomial ideal I in the polynomial ring $k[x_1, x_2, \ldots, x_n]$ and $\mathbf{b} \in \mathbb{N}^n$, we consider the *upper Koszul simplicial complex* $K^{\mathbf{b}}(I) = \{\text{squarefree vectors } \tau : \mathbf{x}^{\mathbf{b}-\tau} \in I\}$ and the *lower Koszul simplicial complex* $K_{\mathbf{b}}(I) = \{\text{squarefree vectors } \tau : \mathbf{x}^{\mathbf{b}'+\tau} \notin I\}$ of I in degree $\mathbf{b}$, where $\mathbf{b}' = \max\{\mathbf{b}-\mathbf{1}, \mathbf{0}\} = (b'_1, b'_2, \ldots, b'_n)$ with $b'_i = \max\{b_i - 1, 0\}$, $\forall i$. The multigraded Betti numbers of I in degree $\mathbf{b}$ are given by

$$\begin{aligned}\beta_{i,\mathbf{b}}(I) &= \dim_k \tilde{H}_{i-1}(K^{\mathbf{b}}(I); k) \quad \text{and} \quad \beta_{i-1,\mathbf{b}}(I)\\ &= \dim_k \tilde{H}^{|\mathrm{Supp}(\mathbf{b})|-i-1}(K_{\mathbf{b}}(I); k); \quad i \geq 1,\end{aligned}$$

where the support $\mathrm{Supp}(\mathbf{b}) = \{i : b_i > 0\}$ (see Theorem 5.11 of [5]). We will be primarily using lower Koszul simplicial complexes in computing multigraded Betti numbers of the Alexander dual of multipermutohedron ideal $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. The minimal generators of $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are of the form $(\prod_{j\in A} x_j)^{u_n-u_{|A|}+c}$, where $A \subseteq [n]$, $|A| = s_i + 1$ for $0 \leq i < l$ and $u_{|A|} \geq 1$. Thus

$$\beta_{0,\mathbf{b}}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = 1$$

for $\mathbf{b} = (u_n - u_{s_i+1} + c)E(0, s_i + 1)$ or its permutation, where $0 \leq i < l$ if $u_1 \geq 1$ or $0 < i < l$ if $u_1 = 0$. Therefore,

$$\beta_0(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \begin{cases} \sum_{i=0}^{l-1} \binom{n}{s_i+1}, & \text{if } u_1 \geq 1,\\ \sum_{i=1}^{l-1} \binom{n}{s_i+1}, & \text{if } u_1 = 0.\end{cases}$$

For computing higher Betti numbers, we need the following topological result: Consider a disjoint family $\{\Delta_{n_j}\}$ of simplexes and let $\Gamma = \Delta_{n_1}^{(i_1)} * \Delta_{n_2}^{(i_2)} * \cdots * \Delta_{n_t}^{(i_t)}$ be a join of i_j-skeleton of n_j-simplex Δ_{n_j} for $1 \leq j \leq t$. Then $\dim_k \tilde{H}_j(\Gamma; k) = \left[\prod_{\alpha=1}^{t} \binom{n_\alpha}{i_\alpha+1}\right] \delta_{\dim(\Gamma), j}$, where $\delta_{i,j}$ be the Kronecker delta and $\dim(\Gamma) = \sum_{j=1}^{t} i_j + (t-1)$ (see Lemma 3.3 of [3]).

Let $p, q \in \mathbb{N}$ and $p \leq q$. Then $[p, q]$ denotes an integral interval $\{r \in \mathbb{N} : p \leq r \leq q\}$. We also write $(p, q]$ for $[p+1, q]$. In order to describe multigraded Betti numbers of the Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$, we need the following notion.

DEFINITION 3.1

Let $J = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ with $0 = j_0 < j_1 < j_2 < \cdots < j_t \le n$. Then J is said to be a *dual* $\mathbf{m}$-*isolated* if $J \cap (s_{j-1}, s_j]$ is either empty or singleton for $1 \le j \le l$. Thus for each α, there is a unique i_α with $s_{i_\alpha - 1} + 1 \le j_\alpha \le s_{i_\alpha}$. In other words, J contains at most one point from each of the integral intervals $(s_{j-1}, s_j]$ $(1 \le j \le l)$, which is the reason for the name $\mathbf{m}$-*isolated*. For $\mathbf{u}(\mathbf{m}) = (u_1, \ldots, u_n)$, set $\mathbf{b}(J) = \sum_{\alpha=1}^{t} \lambda_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$, $\lambda_i = u_n - u_i + c$ and set *dual weight* $\mathrm{dwt}(J) = \mathrm{dwt}(\mathbf{b}(J)) = \left[\sum_{\alpha=1}^{t} (j_\alpha - s_{i_\alpha - 1})\right] - 1$. Also, the size of the support $|\mathrm{Supp}(\mathbf{b}(J))| = j_t$. The set of all dual $\mathbf{m}$-isolated subsets of $[n]$ is denoted by $\mathcal{I}^*_{\mathbf{m}}$. If $J \subseteq [n]$ is a dual $\mathbf{m}$-isolated subset with $\mathrm{dwt}(J) = i$, we write $J \in \mathcal{I}^*_{\mathbf{m}}(\langle i \rangle)$.

Definition of $\mathbf{m}$-isolated subsets is given in [3] and it is used to describe multigraded Betti numbers of the multipermutohedron ideals, but the notion of dual $\mathbf{m}$-isolated subsets is somewhat different. In the following theorem, multigraded Betti numbers of the Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are computed using the notion of dual $\mathbf{m}$-isolated subsets.

Theorem 3.2. *For* $\mathbf{b} \in \mathbb{N}^n$ *and* $i \ge 1$, *let* $\beta_{i-1,\mathbf{b}}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ *be an* $i-1$*-th multigraded Betti number of* $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ *in the degree* $\mathbf{b}$. *If* $u_1 \ge 1$, *then the multigraded Betti numbers* $\beta_{i-1,\mathbf{b}}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ *are given as follows*:

(1) *For* $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m}}$,

$$\beta_{i-1,\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \left[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\right] \delta_{i-1,\mathrm{dwt}(J)},$$

where $J \cap (s_{i_\alpha-1}, s_{i_\alpha}] = \{j_\alpha\}$. *If* π *is a permutation of* $\mathbf{b}(J)$, *then*

$$\beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \beta_{i-1,\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}).$$

(2) *If* $\mathbf{b} \ne \pi\mathbf{b}(J)$ *for any* $J \in \mathcal{I}^*_{\mathbf{m}}(\langle i-1 \rangle)$ *and any permutation* π *of* $\mathbf{b}(J)$, *then*

$$\beta_{i-1,\mathbf{b}}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = 0.$$

Proof. The lower Koszul complex $K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ of the dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of the multipermutohedron ideal $I(\mathbf{u}(\mathbf{m}))$ is claimed to be the join of skeletons of simplices

$$\Delta_{j_1-1}^{(s_{i_1-1}-1)} * \Delta_{j_2-j_1-1}^{(s_{i_2-1}-j_1-1)} * \cdots * \Delta_{j_\alpha-j_{\alpha-1}-1}^{(s_{i_\alpha-1}-j_{\alpha-1}-1)} * \cdots * \Delta_{j_t-j_{t-1}-1}^{(s_{i_t-1}-j_{t-1}-1)},$$

where simplex $\Delta_{j_\alpha - j_{\alpha-1}-1}$ is spanned by vertices $\{e_\nu : j_{\alpha-1} + 1 \le \nu \le j_\alpha\}$. This claim is proved by a straightforward verification. Consider the vector

$$\mathbf{v} = E(0, s_{i_1-1}) + E(j_1, s_{i_2-1}) + E(j_2, s_{i_3-1}) + \cdots + E(j_{t-1}, s_{i_t-1}).$$

Then $\mathbf{v}$ is the vector

$$\mathbf{v} = (1, \ldots, 1, 0, \ldots, 0, \vdots 1, \ldots, 1, 0, \ldots, 0, \vdots \ldots\ldots \vdots 1, \ldots, 1, 0, \ldots, 0, \vdots 0, \ldots, 0)$$

consisting of exactly t strands of 1's followed by 0's together with $n - j_t$ zeros at the end. The length of the α-th strand is $j_\alpha - j_{\alpha-1}$ and precisely first $s_{i_\alpha - 1} - j_{\alpha-1}$ entries of the α-th strand are 1's followed by 0's. Now, set $\mathbf{b}'(J) = \sum_{\alpha=1}^{t}(\lambda_{j_\alpha} - 1)E(j_{\alpha-1}, j_\alpha)$. Clearly, $\mathbf{b}'(J) + \mathbf{v}$ cannot be bigger than or equal to an exponent of any minimal generator of $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. Thus $\mathbf{v} \in K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. Let π_j be a permutation of the j-th strand of the vector $\mathbf{v}$ and let π be the product of the (disjoint) permutations π_j for $1 \le j \le t$. Then we also have $\pi\mathbf{v} \in K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. If $\mathbf{v}'$ is another vector obtained from $\mathbf{v}$ by replacing at least one of the 0 by 1, then $\mathbf{b}'(J) + \pi\mathbf{v}'$ becomes bigger than or equal to an exponent of some minimal generator of $K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. This proves the claim.

The dimension $\dim(K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})) = \sum_{\alpha=1}^{t}(s_{i_{\alpha-1}} - j_{\alpha-1} - 1) + (t-1) = j_t - \sum_{\alpha=1}^{t}(j_\alpha - s_{i_{\alpha-1}}) - 1$. The multigraded Betti number

$$\beta_{i-1,\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \dim_k(\tilde{H}^{j_t-i-1}(K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}); k)) \quad \text{for } i \ge 1.$$

Clearly, $i - 1 = dwt(\mathbf{b}(J)) \Leftrightarrow j_t - i - 1 = \dim(K_{\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}))$. Thus from the above result on homology groups of the join of skeletons of simplexes, the first part of (1) follows. Since minimal generators of $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are invariant under a permutation we have $\beta_{i,\pi\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \beta_{i,\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$, for a permutation π of $\mathbf{b}(J)$.

Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ such that $\mathbf{b} \ne \pi\mathbf{b}(J)$ for any permutation π. Changing $\mathbf{b}$ by a permutation, we may assume that $b_1 \ge b_2 \ge \cdots \ge b_n$. The nonzero Betti numbers of a monomial ideal exist in a multidegree $\mathbf{b}$ only if the monomial $\mathbf{x}^{\mathbf{b}}$ is a LCM of some set of minimal generators of the monomial ideal. Therefore, $\mathbf{b} = \sum_{\alpha=1}^{t}\lambda_{j_\alpha}E(j_{\alpha-1}, j_\alpha)$ for some $J' = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m}}^{*}$. But by the given condition, $J' \notin \mathcal{I}_{\mathbf{m}}^{*}(\langle i-1\rangle)$. Thus $\beta_{i-1,\mathbf{b}}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = 0$. This proves (2). □

Remark 3.3. Theorem 3.2 looks quite similar to Theorem 3.5 in [3]. This is not surprising as there is a general duality for Betti numbers of a monomial ideal I and its Alexander dual $I^{[\mathbf{a}]}$. In fact,

$$\beta_{n-i,\mathbf{b}}(S/I) = \beta_{i,\mathbf{a}+\mathbf{1}-\mathbf{b}}(I^{[\mathbf{a}]})$$

for $b = (b_1, \ldots, b_n)$ with $1 \le b_i \le a_i$, $\forall i$ (Theorem 5.48 of [5]). However, this duality does not give all the multigraded Betti numbers of the dual $I^{[\mathbf{a}]}$.

COROLLARY 3.4

Let $\beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ be the $i-1$-th Betti number of the Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. Suppose $u_1 \ge 1$ and for $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m}}^{*}(\langle i-1\rangle)$, we set

$$\beta_{i-1}^{J} = \prod_{\alpha=1}^{t}\left[\binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\binom{j_{\alpha+1}}{j_\alpha}\right],$$

where $J \cap (s_{i_\alpha - 1}, s_{i_\alpha}] = \{j_\alpha\}$ and $j_{t+1} = n$. Then $\beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \sum_{J\in\mathcal{I}_{\mathbf{m}}^{*}(\langle i-1\rangle)}\beta_{i-1}^{J}$.

Proof. Let $\mathrm{Per}(\mathbf{b}(J))$ be the set of all permutations of $\mathbf{b}(J)$. Then, in view of Theorem 3.2, we have

$$\begin{aligned}\beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) &= \sum_{\mathbf{b}\in\mathbb{N}^n} \beta_{i-1,\mathbf{b}}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) \\ &= \sum_{J\in\mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)} \left[\sum_{\pi\in\mathrm{Per}(\mathbf{b}(J))} \beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})\right] \\ &= \sum_{J\in\mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)} \beta^J_{i-1},\end{aligned}$$

where $\beta^J_{i-1} = \sum_{\pi\in\mathrm{Per}(\mathbf{b}(J))} \beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. For $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)$ with $J \cap (s_{i_\alpha -1}, s_{i_\alpha}] = \{j_\alpha\}$, we have

$$\beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha -1} - j_{\alpha-1}},$$

for all $\pi \in \mathrm{Per}(\mathbf{b}(J))$. The number of permutations π of $\mathbf{b}(J)$ is

$$|\mathrm{Per}(\mathbf{b}(J))| = \frac{n!}{\prod_{\alpha=1}^{t}(j_{\alpha+1} - j_\alpha)!} = \prod_{\alpha=1}^{t}\binom{j_{\alpha+1}}{j_\alpha}.$$

Therefore,

$$\beta^J_{i-1} = \left[\prod_{\alpha=1}^{t}\binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha -1} - j_{\alpha-1}}\right]\left[\prod_{\alpha=1}^{t}\binom{j_{\alpha+1}}{j_\alpha}\right].$$

□

Remark 3.5. Let $\tilde{\mathcal{I}}^*_{\mathbf{m}}(\langle i-1\rangle) = \{J \in \mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle) : J \cap (s_0, s_1] = \emptyset\}$. Then for $u_1 = 0$, the formula for Betti numbers of the Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ as in Theorem 3.2 or Corollary 3.4 remain valid by just replacing $\mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)$ with $\tilde{\mathcal{I}}^*_{\mathbf{m}}(\langle i-1\rangle)$.

We have already obtained formula for the zeroth Betti number of the Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. For the first Betti number, we need to determine all dual **m**-isolated subsets $J \in \mathcal{I}^*_{\mathbf{m}}(\langle 1\rangle)$ of dual weight 1. Let $J_\alpha = \{s_\alpha + 2\}$ for $0 \le \alpha < l$ with $s_{\alpha+1} - s_\alpha \ge 2$ and $J_{\nu,\omega} = \{s_\nu + 1, s_\omega + 1\}$ with $0 \le \nu < \omega < l$. Then $J_\alpha, J_{\nu,\omega} \in \mathcal{I}^*_{\mathbf{m}}(\langle 1\rangle)$ and

$$\beta_1^{J_\alpha} = (s_\alpha + 1)\binom{n}{s_\alpha + 2} \text{ while } \beta_1^{J_{\nu,\omega}} = \binom{n}{s_\omega + 1}\binom{s_\omega + 1}{s_\nu + 1}.$$

Thus

$$\begin{aligned}\beta_1(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) &= \sum_{\substack{0\le\alpha<l,\\ s_{\alpha+1}-s_\alpha\ge 2}} \beta_1^{J_\alpha} + \sum_{0\le\nu<\omega<l} \beta_1^{J_{\nu,\omega}} \\ &= \sum_{\substack{0\le\alpha<l,\\ s_{\alpha+1}-s_\alpha\ge 2}} (s_\alpha + 1)\binom{n}{s_\alpha + 2} \\ &\quad + \sum_{0\le\nu<\omega<l} \binom{n}{s_\omega + 1}\binom{s_\omega + 1}{s_\nu + 1}, \qquad (3.1)\end{aligned}$$

provided $u_1 \geq 1$. On the other hand, if $u_1 = 0$, then

$$\beta_1(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \sum_{\substack{0<\alpha<l,\\ s_{\alpha+1}-s_\alpha\geq 2}} (s_\alpha+1)\binom{n}{s_\alpha+2} + \sum_{0<\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1}.$$

We have seen that the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex Δ_{n-1} supports a free resolution of the quotient $R/I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of the Alexander dual of the multipermutohedron ideal. Now consider a polyhedral cell complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ obtained by modifying the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ as follows: First assume that $u_1 \geq 1$. In this case, the vertices of the polyhedral cell complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ are precisely the barycenters corresponding to the subsets $A \subseteq [n]$ with $|A| = s_i + 1$ for $0 \leq i < l$ and the edges correspond to the chain of subsets $A \subset B$ of $[n]$ with $|A| = s_\nu + 1$ and $|B| = s_\omega + 1$ for $0 \leq \nu < \omega < l$, or the subsets C of $[n]$ of the form $C = A \cup B$ with $|C| = s_\alpha + 2$, $|A| = |B| = s_\alpha + 1$, and $s_{\alpha+1} - s_\alpha \geq 2$. Higher dimensional faces of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ are spanned by the vertices and edges so that the polyhedral cell complex gives a subdivision of the $n-1$-simplex Δ_{n-1}. Thus the dimension of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is $n-1$. Now assume that $u_1 = 0$. In this case, the polyhedral cell complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is obtained as in the earlier case, but now we delete all the faces containing the vertices of $n-1$-simplex Δ_{n-1}, i.e. barycenters corresponding to the subsets $A \subseteq [n]$ with $|A| = 1$. The dimension of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ in the case $u_1 = 0$ can be any number from 0 to $n-1$ depending on $\mathbf{m}$.

Let $f_i(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1}))$ be the number of i-dimensional faces of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$. Clearly, the number of vertices $f_0(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1}))$ equals the zeroth Betti number $\beta_0(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. We now proceed to count the number of edges of the polyhedral cell complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$. Firstly, we consider the case $u_1 \geq 1$. For any two vertices of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ corresponding to the subsets $A, B \subseteq [n]$ with $|A| = s_\nu + 1$ and $|B| = s_\omega + 1$ for $0 \leq \nu < \omega < l$, there is an edge between these two vertices if and only if $A \subseteq B$. Also if $|A| = |B| = s_\alpha + 1$, then there is an edge between these vertices if $|A \cup B| = s_\alpha + 2$ and $s_{\alpha+1} - s_\alpha \geq 2$. Now counting these subsets, we obtain a combinatorial formula

$$f_1(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) = \sum_{0\leq\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1} + \sum_{\substack{0\leq\alpha<l,\\ s_{\alpha+1}-s_\alpha\geq 2}} \frac{(s_\alpha+1)(s_\alpha+2)}{2}\binom{n}{s_\alpha+2}. \tag{3.2}$$

If $u_1 = 0$, then deleting all the edges containing the vertices of $n-1$-simplex Δ_{n-1}, the combinatorial formula takes the form

$$f_1(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) = \sum_{0<\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1} + \sum_{\substack{0<\alpha<l,\\ s_{\alpha+1}-s_\alpha\geq 2}} \frac{(s_\alpha+1)(s_\alpha+2)}{2}\binom{n}{s_\alpha+2}.$$

A combinatorial formula for the higher dimensional faces of the polyhedral cell complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ are quite cumbersome. But if $\mathbf{m} = (m_1, 1, \ldots, 1)$, then $f_i(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1}))$ can be easily calculated.

Theorem 3.6. *Let* $\mathbf{m} = (m_1, 1, \ldots, 1)$. *Then*

$$f_{i-1}(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) = \beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) \qquad \forall i \geq 1.$$

Proof. Firstly we consider the case $u_1 \geq 1$. We know that $(i-1)$-faces of the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ correspond to a chain of nonempty subsets of $[n]$ of length i. Since the barycenters corresponding to the subsets $A \subset [n]$ with $1 < |A| \leq m_1$ are missing, an $(i-1)$-face of the polyhedral complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ corresponds to a chain

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_t$$

of subsets of $[n]$ such that either all A_i's represent vertices of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ or $1 < |A_1| \leq m_1 < |A_2|$. In the former case, $t = i$, while in the latter case, $t = i - |A_1| + 1$ as it represents the $(i-1)$-face spanned by vertices of Δ_{n-1}, corresponding to singleton subsets of A_1, and the barycenters $A_2, \ldots, A_t$. Let $|A_i| = j_i$. Then $J = \{j_1, j_2, \ldots, j_t\}$ is a dual $\mathbf{m}$-isolated subset with $dwt(J) = j_1 + (t-1) - 1 = i - 1$, and every $(i-1)$-face of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ arises in this way. In this case, all the faces of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ are simplicial and thus $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is a $(n-1)$-dimensional simplicial complex.

Let f^J_{i-1} be the number of $(i-1)$-faces of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ associated to a dual $\mathbf{m}$-isolated subset $J \in \mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)$ with dual weight $i-1$. Then $f^J_{i-1} = \prod_{\alpha=1}^{t} \binom{j_{\alpha+1}}{j_\alpha}$, where $j_{t+1} = n$. For $\mathbf{m} = (m_1, 1, \ldots, 1)$, using Corollary 3.4, $\beta^J_{i-1} = \prod_{\alpha=1}^{t} \binom{j_{\alpha+1}}{j_\alpha}$, because either $j_{\alpha+1} - j_\alpha - 1 = 0$ or $s_{i_\alpha - 1} - j_\alpha = 0$. Thus

$$\begin{aligned} f_{i-1}(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) &= \sum_{J \in \mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)} f^J_{i-1} \\ &= \sum_{J \in \mathcal{I}^*_{\mathbf{m}}(\langle i-1\rangle)} \beta^J_{i-1} = \beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}). \end{aligned}$$

If $u_1 = 0$, then vertices of the $n-1$-simplex Δ_{n-1} are no longer vertices of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$. Thus an $(i-1)$-face of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ corresponds to a chain

$$A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_t$$

of subsets of $[n]$ with $t = i$ and $|A_1| > m_1$. Clearly, maximal such chain has length $n - m_1$ and hence the dimension of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is $n - m_1 - 1$. In this case, we have

$$\begin{aligned} f_{i-1}(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) &= \sum_{J \in \tilde{\mathcal{I}}^*_{\mathbf{m}}(\langle i-1\rangle)} f^J_{i-1} \\ &= \sum_{J \in \tilde{\mathcal{I}}^*_{\mathbf{m}}(\langle i-1\rangle)} \beta^J_{i-1} = \beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}). \end{aligned}$$

This completes the proof. □

A vertex of $\mathbf{Bd^m}(\Delta_{n-1})$ corresponds to a nonempty subset $A \subseteq [n]$, and it is naturally labeled with the monomial $(\prod_{i\in A} x_i)^{u_n - u_{|A|}+c}$. Thus $\mathbf{Bd^m}(\Delta_{n-1})$ is a labeled polyhedral cell complex. The free complex associated to the labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ gives a cellular free resolution of the ideal $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. We now investigate minimality of the cellular resolution supported by $\mathbf{Bd^m}(\Delta_{n-1})$.

Theorem 3.7. *The cellular resolution supported by* $\mathbf{Bd^m}(\Delta_{n-1})$ *is the minimal resolution of* $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ *if and only if* $m_\alpha = 1$ *for* $2 \leq \alpha \leq l$.

Proof. Suppose the free resolution supported by the labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ minimally resolves $I(\mathbf{u}(\mathbf{m}))$. Then $\beta_1(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = f_1(\mathbf{Bd^m}(\Delta_{n-1}))$, the number of edges of $\mathbf{Bd^m}(\Delta_{n-1})$. Using equations (3.1) and (3.2), and similar equations for the case $u_1 = 0$, we see that there are at most one α with $s_{\alpha+1} - s_\alpha \geq 2$; namely $\alpha = 0$ if $u_1 \geq 1$ and no such α if $u_1 = 0$. Thus in either case, $m_{\alpha+1} = s_{\alpha+1} - s_\alpha = 1$ for $\alpha \geq 1$. This proves the direct part.

Conversely, let $m_\alpha = 1$ for $\alpha \geq 2$. Then the cellular free resolution supported by the labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ is minimal, because $\beta_{i-1}(I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = f_{i-1}(\mathbf{Bd^m}(\Delta_{n-1}))\ \forall i \geq 1$, in view of Theorem 3.6. □

Remark 3.8. In [3], it is proved that the cellular resolution supported by the multipermutohedron $P(\mathbf{u}(\mathbf{m}))$ is the minimal resolution of the multipermutohedron ideal $I(\mathbf{u}(\mathbf{m}))$ if and only if $m_\alpha = 1$ for $2 \leq \alpha \leq l$. In spite of the identical resemblance, in view of Remark 3.3, Theorem 3.7 about the minimal resolution of Alexander dual $I(\mathbf{u}(\mathbf{m}))^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ can not simply be deduced from the minimal resolution of $I(\mathbf{u}(\mathbf{m}))$.

At the end, we give some examples of cellular free resolutions of the Alexander duals of multipermutohedron ideals for $n = 3$.

(1) Let $\mathbf{u}(\mathbf{m}) = (a, a, b)$, $0 < a < b$. Consider $\mathbf{b} + \mathbf{c} - \mathbf{1} = (b + c - 1, b + c - 1, b + c - 1)$, $c \geq 1$. Then the Alexander dual $I(a, a, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]} = \langle x^{b-a+c}, y^{b-a+c}, z^{b-a+c}, (xyz)^c\rangle$ is an ideal in the polynomial ring $R = k[x, y, z]$. The polyhedral cell complex $\mathbf{Bd^m}(\Delta_2)$ is obtained by subdividing a 2-simplex into three triangular regions by choosing the fourth vertex as the centroid and joining it with the three vertices of the 2-simplex. Thus $\mathbf{Bd^m}(\Delta_2)$ is a simplicial complex with four vertices, six edges and two triangular faces. Clearly, the labels on the vertices of the 2-simplex are $x^{b-a+c}, y^{b-a+c}, z^{b-a+c}$ while the label on the centroid is $(xyz)^c$. The dual $\mathbf{m}$-isolated subsets are $J_0 = \{1\}$, $\bar{J}_0 = \{3\}$, $J_1 = \{2\}$, $\bar{J}_1 = \{1, 3\}$ and $J_2 = \{2, 3\}$, where J_i (or $\bar{J}_i$) has dual weight i. Using Corollary 3.4, we have $\beta_0^{J_0} = 3$, $\beta_0^{\bar{J}_0} = 1$, $\beta_1^{J_1} = 3$, $\beta_1^{\bar{J}_1} = 3$ and $\beta_2^{J_2} = 3$. Thus the Betti numbers of $I(a, a, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]}$ are $\beta_0 = 4$, $\beta_1 = 6$ and $\beta_2 = 3$. Thus the free complex associated with the labeled simplicial complex $\mathbf{Bd^m}(\Delta_2)$ is the minimal resolution of the Alexander dual $I(a, a, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]}$, which is also indicated by Theorem 3.7.

(2) Let $\mathbf{u}(\mathbf{m}) = (a, b, b)$, $0 < a < b$. The Alexander dual is given by $I(a, a, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]} = \langle x^{b-a+c}, y^{b-a+c}, z^{b-a+c}, (xy)^c, (xz)^c, (yz)^c\rangle$. The polyhedral cell complex $\mathbf{Bd^m}(\Delta_2)$ is obtained by subdividing a 2-simplex into four triangular regions by choosing three more vertices as the barycenters of the three edges of the 2-simplex and joining the barycenters with each other. Thus $\mathbf{Bd^m}(\Delta_2)$ is a simplicial complex having six vertices, nine edges and four triangular faces. Clearly, labels on the vertices of the 2-simplex are $x^{b-a+c}, y^{b-a+c}, z^{b-a+c}$, while the labels on the barycenters of the three edges are

$(xy)^c$, $(xz)^c$, $(yz)^c$. The dual **m**-isolated subsets in this case are $J_0 = \{1\}$, $\bar{J}_0 = \{2\}$, $J_1 = \{1, 2\}$, $\bar{J}_1 = \{3\}$ and $J_2 = \{1, 3\}$, where J_i (or $\bar{J}_i$) has dual weight i. We have $\beta_0^{J_0} = 3$, $\beta_0^{\bar{J}_0} = 3$, $\beta_1^{J_1} = 6$, $\beta_1^{\bar{J}_1} = 2$ and $\beta_2^{J_2} = 3$. Thus the Betti numbers of $I(a, a, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]}$ are $\beta_0 = 6$, $\beta_1 = 8$ and $\beta_2 = 3$. Since the number of edges of $\mathbf{Bd}^{\mathbf{m}}(\Delta_2)$ is $9 > 8 = \beta_1$, the free complex associated with this labeled simplicial complex is a nonminimal resolution of the Alexander dual $I(a, b, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]}$.

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