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Determination of star bodies from p-centroid bodies

LUJUN GUO and GANGSONG LENG

Department of Mathematics, Shanghai University, Shanghai 200444, China
E-mail: lujunguo0301@163.com; gleng@staff.shu.edu.cn

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Abstract. In this paper, we prove that an origin-symmetric star body is uniquely determined by its p-centroid body. Furthermore, using spherical harmonics, we establish a result for non-symmetric star bodies. As an application, we show that there is a unique member of $\Gamma_p\langle K\rangle$ characterized by having larger volume than any other member, for all real $p \geq 1$ that are not even natural numbers, where $\Gamma_p\langle K\rangle$ denotes the p-centroid equivalence class of the star body K.

Keywords. p-centroid body; star body; spherical integral transformation; p-cosine transformation.

1. Introduction

The centroid body ΓK of a convex body $K \in \mathbb{R}^d$ is a classical notion from geometry (see for e.g. [1, 2, 4–6, 9–11, 15, 17–19, 21–23, 27]) that has attracted much attention in recent years. Centroid bodies were first defined and investigated by Petty [24], but the concept had previously appeared in the work of Dupin, in connection with problems about floating bodies (see Section 7.4 of [26]). If K is an origin symmetric convex body, it turns out that ΓK is bounded by the locus of the centroids of all the halves of K obtained by cutting K with hyperplanes through the origin.

For a star-shaped body $K \subset \mathbb{R}^d$ (star-shaped with respect to the origin) and each p such that $1 \leq p \leq \infty$, let $\Gamma_p K$ be the p-centroid body of K, that is, the convex body whose support function is given by

$$h_{\Gamma_p K}(x) = \left(\frac{1}{V(K)} \int_K |x \cdot y|^p \mathrm{d}y\right)^{\frac{1}{p}}, \tag{1.1}$$

where the integration is with respect to the Lebesgue measure.

For $p = 1$, the set $\Gamma_1 K$ is known in the literature as the centroid body ΓK of K.

For $p = 2$, $\Gamma_2 K$ is homothetic to the Legendré ellipsoid of K, which arises in classical mechanics in connection with the moments of inertia of K (see [16, 22]).

Since $\Gamma_\infty K$ is interpreted as the limit of (1.1), as $p \to \infty$, therefore $\Gamma_\infty K = \mathrm{conv}(K \cup (-K))$, where 'conv' stands for the convex hull. This body was investigated by Fáry and Rédei [3] in the framework of affine inequalities related to the geometry of numbers.

An immediate consequence of the definition of the p-centroid body of K is that for any linear transformation $\phi \in GL(n)$,

$$\Gamma_p \phi K = \phi \Gamma_p K.$$

Let $\Gamma_p \langle K \rangle$ denote the p-centroid equivalence class of K defined by

$$\Gamma_p \langle K \rangle = \{L \in \mathscr{S}^d : \Gamma_p K = \Gamma_p L\},$$

where $\mathscr{S}^d$ denotes the set of star bodies.

Let K, L be two origin symmetric star bodies in $\mathbb{R}^d$ such that $\Gamma_p K \subset \Gamma_p L$, what is the relation between $V(K)$ and $V(L)$?

For $p = 1$, Lutwak [19] proved that, under this assumption, if L is a polar projection body, then $V(K) \leq V(L)$. He also showed that if K is not a polar projection body, then there exists a star body L such that $\Gamma K \subset \Gamma L$, but $V(K) > V(L)$.

In [7], Grinberg and Zhang proved the following general result: For $p \geq 1$, let K and L be two origin symmetric star bodies in $\mathbb{R}^d$ such that

$$\Gamma_p K \subset \Gamma_p L.$$

If the space $(\mathbb{R}^d, \|\cdot\|_L)$ embeds in L_p, then $V(K) \leq V(L)$. On the other hand, they also showed that if $(\mathbb{R}^d, \|\cdot\|_K)$ does not embed in L_p, then there is a body L so that $\Gamma_p K \subset \Gamma_p L$, but $V(K) > V(L)$.

For $p < 1$, the function $h_{\Gamma_p K}(\cdot)$ in (1.1) is not necessarily convex, therefore it is not a support function, but the definition of the polar p-centroid body still makes sense, even though these bodies may not be convex. For $p > -1$, $p \neq 0$, Yaskin and Yaskina [28] defined the polar p-centroid body of a star body K by the formula:

$$\|\xi\|_{\Gamma_p^* K} = \left(\frac{1}{V(K)} \int_K |x \cdot \xi|^p \mathrm{d}x\right)^{\frac{1}{p}}, \qquad \xi \in \mathbb{R}^d.$$

For $p = 0$, they gave the definition as follows:

$$\|\xi\|_{\Gamma_0^* K} = \exp\left(\frac{1}{V(K)} \int_K \ln |x \cdot \xi| \mathrm{d}x\right), \qquad \xi \in \mathbb{R}^d.$$

They proved the following extension of the results by Lutwak [19] and Grinberg and Zhang [7]. For $p > -1$, let K and L be two origin-symmetric star bodies in $\mathbb{R}^d$ such that $\Gamma_p^* L \subset \Gamma_p^* K$. If $(\mathbb{R}^d, \|\cdot\|_L)$ embeds in L_p, then $V(K) \leq V(L)$. However, if $(\mathbb{R}^d, \|\cdot\|_K)$ does not embed in L_p, they constructed counter examples to the later result.

Inspired by the works of Lutwak [19], Grinberg and Zhang [7] and Yaskin and Yaskina [28], we will study the following questions in our paper.

If K, L are two star bodies in $\mathbb{R}^d$ such that $\Gamma_p K = \Gamma_p L$, what can be said about K and L? What can be said about a star body in $\Gamma_p \langle K \rangle$ with maximal volume?

The main results of this paper are as follows:

Theorem 1.1. *Let K, L be two origin symmetric star bodies in $\mathbb{R}^d$. If $\Gamma_p K = \Gamma_p L$ for some $p \geq 1$ that is not even natural number, then $K = L$.*

The case for $p = 1$ was remarked by Lutwak [19], i.e., for two origin-symmetric star bodies $K, L \in \mathbb{R}^d$, if $\Gamma K = \Gamma L$, then $K = L$.

For a star body M and $u \in S^{d-1}$, let u^+ denote the closed half space $\{x : x \in \mathbb{R}^d, x \cdot u \geq 0\}$, and $V(M \cap u^+)$ the volume of $M \cap u^+$.

Theorem 1.2. *Let K and L be two star bodies in $\mathbb{R}^d$ satisfying $V(K \cap u^+) = V(L \cap u^+)$, for all $u \in S^{d-1}$. If $\Gamma_p K = \Gamma_p L$ for some $p \geq 1$ that is not even a natural number, then $K = L$.*

In the above two theorems, if p is even, the conclusion may not be correct. For example, in Theorem 1.1, let $p = 2$, and K, L be two origin-symmetric star bodies in $\mathbb{R}^d$. Since $\Gamma_2 M$ is an ellipsoid for any star body M, there exists a linear transformation ϕ such that $\Gamma_2 K = \phi \Gamma_2 L = \Gamma_2 \phi L$, according to Theorem 1.1, and this would imply that $K = \phi L$. By the arbitrariness of K, L, this is a contradiction.

Using Theorem 1.1, we obtain the following theorem about the p-centroid equivalence class.

Theorem 1.3. *For any real $p \geq 1$ that are not even natural numbers, if $K \in \mathscr{S}^d$, then $\Gamma_p \langle K \rangle$ contains a unique member characterized by having larger volume than any other member of $\Gamma_p \langle K \rangle$, where*

$$\Gamma_p \langle K \rangle = \{L \in \mathscr{S}^d : \Gamma_p K = \Gamma_p L\},$$

and $\mathscr{S}^d$ denotes the set of star bodies.

2. Background materials

In this section, we give some background materials regarding convex bodies. We also state some known facts about spherical harmonics.

Let $\mathbb{R}^d$ denote the Euclidean d-dimensional space with corresponding Euclidean norm $|\cdot|$. Let B^d denote the origin centered unit ball in $\mathbb{R}^d$. The set S^{d-1} is the unit sphere of $\mathbb{R}^d$ and σ is its spherical Lebesgue measure. Write κ_d for $V(B^d)$, the volume of B^d, and ω_d for $\sigma(S^{d-1})$, the surface area of B^d. For $u \in S^{d-1}$, $u^+ := \{x : x \in \mathbb{R}^d, x \cdot u \geq 0\}$, where $x \cdot u$ is the scalar product.

A set K in $\mathbb{R}^n$ is star-shaped (about the origin) if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin. The radial function $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact, star-shaped (about the origin) $K \in \mathbb{R}^n$, is defined by

$$\rho_K(v) := \max\{\lambda \geq 0 : \lambda v \in K\} \quad \text{for} \quad v \in \mathbb{R}^d \setminus \{0\}.$$

If ρ_K is positive and continuous, call K a star body (about The origin). Let $V(K)$ denote the volume of K and $\mathscr{S}^d$ the set of star bodies with respect to the origin in $\mathbb{R}^d$. Let $\mathscr{S}^d_c$ be the set of star bodies which are origin symmetric. A convex body K is a compact, convex set with non-empty interior. Associated with a convex body K is its support function h_K defined for $x \in \mathbb{R}^d$ by

$$h_K(x) := \max\{x \cdot y : y \in K\}.$$

The function h_K is positively homogeneous of degree 1. We will usually be concerned with the restriction of the support function to the unit sphere S^{d-1}.

For star bodies $K, L \in \mathscr{S}^d$, $p \geq 1$ and $\varepsilon > 0$, the L_p-harmonic radial combination $K \tilde{+}_p \varepsilon \cdot L$ is the star body defined by

$$\rho(K \tilde{+}_p \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}.$$

The dual mixed volume $\tilde{V}_{-p}(K, L)$ of star bodies K, L can be defined by

$$\frac{d}{-p} \tilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \tilde{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

The definition above and the polar co-ordinate formula for volume give the following integral representation of the dual mixed volume $\tilde{V}_{-p}(K, L)$ of the star bodies K, L:

$$\tilde{V}_{-p}(K, L) = \frac{1}{d} \int_{S^{d-1}} \rho_K^{d+p}(v) \rho_L^{-p}(v) \mathrm{d}\sigma(v). \tag{2.1}$$

For star bodies K and L, the basic inequality for the dual mixed volume $\tilde{V}_{-p}$ is

$$\tilde{V}_{-p}(K, L) \geq V(K)^{(d+p)/d} V(L)^{-p/d}, \tag{2.2}$$

with equality if and only if K and L are dilates of each other. This inequality can be obtained from Hölder's inequality and the integral representation (2.1). From the definition of the dual mixed volumes $\tilde{V}_{-p}$ it follows immediately that for each star body K,

$$\tilde{V}_{-p}(K, K) = V(K). \tag{2.3}$$

An immediate consequence of the dual mixed volume inequality (2.2) and (2.3) is the following lemma [20].

Lemma 2.1. *If for star bodies K, L and $p \geq 1$ we have*

$$\tilde{V}_{-p}(K, Q)/V(K) = \tilde{V}_{-p}(L, Q)/V(L),$$

for all star bodies Q which belong to some class that contains both K and L, then $K = L$.

Let $L_2(S^{d-1})$ denote the class of all real-valued Lebesgue integral functions f on S^{d-1} with the property that $\int_{S^{d-1}} f^2(u) \mathrm{d}\sigma(u) < \infty$. If $f, g \in L_2(S^{d-1})$, the inner product $\langle f, g \rangle$ is defined by

$$\langle f, g \rangle = \int_{S^{d-1}} f(u) g(u) \mathrm{d}\sigma(u).$$

In order to prove the theorems stated in § 1, we have to introduce the following three lemmas. Since they are well-known, we omit their proofs and only provide references.

Let $\mathscr{T}$ denote the hemispherical integral transformation on S^{d-1}, i.e.

$$\mathscr{T}(f)(u) = \int_{S^{d-1}} \tau(u \cdot v) f(v) \mathrm{d}\sigma(v), \tag{2.4}$$

where $f \in L_2(S^{d-1})$ and

$$\tau(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

If f is a real-valued function on S^{d-1}, let f^+ and f^- denote the functions

$$f^+(u) = \frac{1}{2}(f(u) + f(-u)) \quad \text{and} \quad f^-(u) = \frac{1}{2}(f(u) - f(-u)),$$

respectively. So, $f = f^+ + f^-$, where f^+ is an even function and f^- is an odd function on S^{d-1}.

Lemma 2.2 [8, 12]. *If f_1, f_2 are two continuous functions on S^{d-1}, then the equality $\mathscr{T}(f_1) = \mathscr{T}(f_2)$ holds if and only if $f_1^- = f_2^-$ and $\langle f_1, 1\rangle = \langle f_2, 1\rangle$.*

For $p \geq 1$, let $\mathscr{C}_p$ denote the p-cosine transformation on S^{d-1}, i.e., for each bounded integrable function f on S^{d-1}, let $\mathscr{C}_p f$ be the function defined by

$$\mathscr{C}_p(f)(u) = \int_{S^{d-1}} |u \cdot v|^p f(v) \mathrm{d}\sigma(v),$$

for $u \in S^{d-1}$.

Lemma 2.3. *If f_1, f_2 are two bounded integrable functions on S^{d-1} and $p \geq 1$ that is not even natural number, then the equality $\mathscr{C}_p(f_1) = \mathscr{C}_p(f_2)$ holds if and only if almost everywhere $f_1^+ = f_2^+$.*

With the Funk–Hecke theorem it is easy to prove Lemma 2.3, which reveals, for properly defined classes of functions, the injectivity of our spherical integral transformation $\mathscr{C}_p$ (see e.g. [13, 14, 25]). It is obvious that if f is a continuous function on S^{d-1}, then the equality $\mathscr{C}_p(f) = 0$ holds if and only if $f^+ = 0$. Let $C_e(S^{d-1})$ denote the set of all even real-valued continuous functions on S^{d-1} and $C_e^\infty(S^{d-1})$ the subset of $C_e(S^{d-1})$ containing all infinitely differentiable functions. An immediate consequence of Lemma 2.3 and Blaschke–Levy representation (see e.g. [13, 14]) is the following:

Lemma 2.4. *If $p \geq 1$ that is not even a natural number, then the operator $\mathscr{C}_p : C_e^\infty(S^{d-1}) \to C_e^\infty(S^{d-1})$ is bijective.*

3. Proofs of the main results

Proof of Theorem 1.1. For any $f \in C_e(S^{d-1})$, there exists a star body M such that $\rho_M^{-p} = \mathscr{C}_p f$. From (2.1), it follows that

$$\tilde{V}_{-p}(K, M) = \frac{1}{d}\langle \rho_K^{d+p}, \rho_M^{-p}\rangle = \frac{1}{d}\langle \rho_K^{d+p}, \mathscr{C}_p f\rangle = \frac{1}{d}\langle f, \mathscr{C}_p \rho_K^{d+p}\rangle \tag{3.1}$$

and

$$\tilde{V}_{-p}(L, M) = \frac{1}{d}\langle \rho_L^{d+p}, \rho_M^{-p}\rangle = \frac{1}{d}\langle \rho_L^{d+p}, \mathscr{C}_p f\rangle = \frac{1}{d}\langle f, \mathscr{C}_p \rho_L^{d+p}\rangle. \tag{3.2}$$

It follows from (1.1), $h_{\Gamma_p K}(u) = h_{\Gamma_p L}(u)$ for all $u \in S^{d-1}$. The use of polar co-ordinates and the definition of the p-cosine transformation on S^{d-1} is

$$V(K)^{-1}\mathscr{C}_p \rho_K^{d+p} = V(L)^{-1}\mathscr{C}_p \rho_L^{d+p}. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we obtain

$$\tilde{V}_{-p}(K, M)/V(K) = \tilde{V}_{-p}(L, M)/V(L) \tag{3.4}$$

for those $M \in \mathscr{S}_c^d$ whose reciprocal radial function raised to the power p is in the range of the p-cosine operator $\mathscr{C}_p : C_e(S^{d-1}) \to C_e(S^{d-1})$. That (3.4) holds for all $M \in \mathscr{S}_c^d$ follows from Lemma 2.4, i.e.

$$\tilde{V}_{-p}(K, M)/V(K) = \tilde{V}_{-p}(L, M)/V(L) \tag{3.5}$$

for all $M \in \mathscr{S}_c^d$. Since $K, L \in \mathscr{S}_c^d$, (3.5) and Lemma 2.1 immediately give $K = L$. □

Proof of Theorem 1.2. By the definition of the hemispherical transformation, $V(K \cap u^+)$, for $u \in S^{d-1}$,

$$\begin{aligned} V(K \cap u^+) &= \int_K \tau(u \cdot x)\mathrm{d}x \\ &= \frac{1}{d}\int_{S^{d-1}} \tau(u \cdot v)\rho_K^d(v)\mathrm{d}\sigma(v) \\ &= \frac{1}{d}\mathscr{T}(\rho_K^d)(u). \end{aligned} \tag{3.6}$$

Similarly we get

$$V(L \cap u^+) = \frac{1}{d}\mathscr{T}(\rho_L^d)(u). \tag{3.7}$$

Let $g(v) = \rho_K^d(v) - \rho_L^d(v)$, for $v \in S^{d-1}$. Since $V(K \cap u^+) = V(L \cap u^+)$, for all $u \in S^{d-1}$, we get from (3.6) and (3.7),

$$\frac{1}{d}\mathscr{T}(g)(u) = 0.$$

Then $g^- = 0$ follows from Lemma 2.2.

Hence

$$\rho_K^d(u) - \rho_L^d(u) = \rho_K^d(-u) - \rho_L^d(-u), \qquad \forall u \in S^{d-1}. \tag{3.8}$$

It follows from the definitions of the p-centroid body and the p-cosine transformation that

$$\begin{aligned} h_{\Gamma_p K}^p(u) &= \frac{1}{(d+p)V(K)}\int_{S^{d-1}} |u \cdot v|^p \rho_K^{d+p}(v)\mathrm{d}\sigma(v) \\ &= \frac{1}{(d+p)V(K)}\mathscr{C}_p(\rho_K^{d+p})(u) \end{aligned} \tag{3.9}$$

and

$$h_{\Gamma_p L}^p(u) = \frac{1}{(d+p)V(L)}\mathscr{C}_p(\rho_L^{d+p})(u). \tag{3.10}$$

Let $f(v) = \rho_K^{d+p}(v) - \rho_L^{d+p}(v)$, for all $v \in S^{d-1}$. Since $V(K \cap u^+) = V(L \cap u^+)$, for all $u \in S^{d-1}$, we get $V(K) = V(L)$. By (3.9), (3.10) and $\Gamma_p K = \Gamma_p L$, we have $\mathscr{C}_p(f) = 0$. Then $f^+ = 0$ follows from Lemma 2.3. Hence

$$\rho_K^{d+p}(u) - \rho_L^{d+p}(u) = -(\rho_K^{d+p}(-u) - \rho_L^{d+p}(-u)), \quad \forall u \in S^{d-1}. \tag{3.11}$$

From (3.8) and (3.11), we get $\rho_K = \rho_L$. Indeed, suppose there exists some $u_o \in S^{d-1}$ with $\rho_K(u_o) \neq \rho_L(u_o)$, say $\rho_K(u_o) < \rho_L(u_o)$. By (3.11), we have $\rho_K(-u_o) > \rho_L(-u_o)$. From this and (3.8) we get $\rho_K(u_o) > \rho_L(u_o)$, a contradiction. Hence $\rho_K(u) = \rho_L(u)$ for all $u \in S^{d-1}$ and $K = L$. □

Proof of Theorem 1.3. Let us first define $\xi > 0$ by

$$\xi^{\frac{p}{d+p}} = \frac{1}{d}\int_{S^{d-1}} (\lambda V(K)^{-1}\rho_K^{d+p}(u) + \mu V(L)^{-1}\rho_L^{d+p}(u))^{\frac{d}{d+p}} \mathrm{d}\sigma(u), \tag{3.12}$$

where $K, L \in \mathscr{S}^d$ and λ, μ are positive real numbers.

The body $\lambda K \widehat{+}_p \mu L \in \mathscr{S}^d$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho^{d+p}_{\lambda K \widehat{+}_p \mu L}(\cdot) = \lambda V(K)^{-1}\rho_K^{d+p}(\cdot) + \mu V(L)^{-1}\rho_L^{d+p}(\cdot). \tag{3.13}$$

From (3.12), (3.13) and the polar co-ordinate formula for volumes, we obtain

$$\begin{aligned}\xi^{\frac{p}{d+p}} &= \frac{1}{d}\int_{S^{d-1}} (\xi^{-1}\rho^{d+p}_{\lambda K \widehat{+}_p \mu L}(u))^{\frac{d}{d+p}} \mathrm{d}\sigma(u)\\ &= \xi^{-\frac{d}{d+p}}\frac{1}{d}\int_{S^{d-1}} \rho^{d}_{\lambda K \widehat{+}_p \mu L}(u)\mathrm{d}\sigma(u).\end{aligned}$$

Therefore

$$\xi = V(\lambda K \widehat{+}_p \mu L)$$

and hence

$$V(\lambda K \widehat{+}_p \mu L)^{-1}\rho^{d+p}_{\lambda K \widehat{+}_p \mu L}(\cdot) = \lambda V(K)^{-1}\rho_K^{d+p}(\cdot) + \mu V(L)^{-1}\rho_L^{d+p}(\cdot). \tag{3.14}$$

From (3.14) and (2.1), it follows that, for $K, L, M \in \mathscr{S}^d$ and $\lambda, \mu \geq 0$,

$$\begin{aligned}&V(\lambda K \widehat{+}_p \mu L)^{-1}\tilde{V}_{-p}(\lambda K \widehat{+}_p \mu L, M)\\ &\quad = \lambda V(K)^{-1}\tilde{V}_{-p}(K, M) + \mu V(L)^{-1}\tilde{V}_{-p}(L, M).\end{aligned} \tag{3.15}$$

Applying inequality (2.2) to the dual mixed volumes on the right-hand side of (3.15) we get

$$\frac{\tilde{V}_{-p}(\lambda K \widehat{+}_p \mu L, M)}{V(\lambda K \widehat{+}_p \mu L)} V(M)^{\frac{p}{d}} \geq \lambda V(K)^{\frac{p}{d}} + \mu V(L)^{\frac{p}{d}}, \tag{3.16}$$

with equality if and only if K, L and M are dilates of each other. Now taking $\lambda K \widehat{+}_p \mu L$ for M and by using (2.3), we obtain

$$V(\lambda K \widehat{+}_p \mu L)^{\frac{p}{d}} \geq \lambda V(K)^{\frac{p}{d}} + \mu V(L)^{\frac{p}{d}}, \tag{3.17}$$

with equality if and only if K and L are dilates of each other.

For $K \in \mathscr{S}^d$, define $\hat{\nabla}_p K \in \mathscr{S}_c^d$ by

$$\hat{\nabla}_p K = \frac{1}{2} K \widehat{+}_p \frac{1}{2}(-K). \tag{3.18}$$

It follows from this denotation and (3.17) that

$$V(\hat{\nabla}_p K) \geq V(K), \tag{3.19}$$

with equality if and only if K is centered.

Since $\Gamma_p K = \Gamma_p(-K)$ can be easily obtained from (1.1), it follows, by using polar coordinates (1.1) and (3.14), that

$$\begin{aligned} & h_{\Gamma_p(\frac{1}{2}K \widehat{+}_p \frac{1}{2}(-K))}(u) \\ & = \left(\frac{1}{(d+p)V(\frac{1}{2}K \widehat{+}_p \frac{1}{2}(-K))} \int_{S^{d-1}} |u \cdot v|^p \rho^{d+p}_{\frac{1}{2}K \widehat{+}_p \frac{1}{2}(-K)}(v) \mathrm{d}\sigma(v) \right)^{\frac{1}{p}} \\ & = \left(\frac{1}{(d+p)} \int_{S^{d-1}} |u \cdot v|^p \left(\frac{1}{2} \frac{1}{V(K)} \rho_K^{d+p}(v) + \frac{1}{2} \frac{1}{V(-K)} \rho_{(-K)}^{d+p}(v) \right) \mathrm{d}\sigma(v) \right)^{\frac{1}{p}} \\ & = \left(\frac{1}{2} h^p_{\Gamma_p K}(u) + \frac{1}{2} h^p_{\Gamma_p(-K)}(u) \right)^{\frac{1}{p}} \\ & = h_{\Gamma_p K}(u), \end{aligned}$$

for $u \in S^{d-1}$. Hence

$$\Gamma_p \left(\frac{1}{2} K \widehat{+}_p \frac{1}{2}(-K) \right) = \Gamma_p K.$$

This and (3.18) show that

$$\Gamma_p \hat{\nabla}_p K = \Gamma_p K. \tag{3.20}$$

Suppose $L \in \Gamma_p\langle K \rangle$, that is $\Gamma_p L = \Gamma_p K$. It follows from (3.18) that $\hat{\nabla}_p K, \hat{\nabla}_p L \in \mathscr{S}_c^d$. From (3.20), we get

$$\Gamma_p(\hat{\nabla}_p K) = \Gamma_p K = \Gamma_p L = \Gamma_p(\hat{\nabla}_p L).$$

Hence, $\hat{\nabla}_p K, \hat{\nabla}_p L \in \Gamma_p\langle K \rangle$ and $\hat{\nabla}_p K = \hat{\nabla}_p L$ by Theorem 1.1. From (3.19), we know that $\hat{\nabla}_p L = L$ if and only if L is centered. Hence

$$V(\hat{\nabla}_p K) = V(\hat{\nabla}_p L) \geq V(L)$$

with equality if and only if L is centered.

This means that $\hat{\nabla}_p K$ is the unique member characterized by having larger volume than any other member of $\Gamma_p\langle K\rangle$. □

Remark. For any real $p \geq 1$ that are not even natural numbers, and $K \in \mathscr{S}^d \setminus \mathscr{S}_c^d$, the set $\Gamma_p\langle K\rangle$ is not a singleton. In fact, for any $K \in \mathscr{S}^d$, (3.20) shows that $\Gamma_p \hat{\nabla}_p K = \Gamma_p K$, hence $\hat{\nabla}_p K \in \Gamma_p\langle K\rangle$. From (3.19), we know that $\hat{\nabla}_p K \neq K$ when K is not centered.

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