

## A general vanishing theorem

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**Abstract.** Let  $E$  be a vector bundle and  $L$  be a line bundle over a smooth projective variety  $X$ . In this article, we give a condition for the vanishing of Dolbeault cohomology groups of the form  $H^{p,q}(X, S^\alpha E \otimes \wedge^\beta E \otimes L)$  when  $S^{\alpha+\beta} E \otimes L$  is ample. This condition is shown to be invariant under the interchange of  $p$  and  $q$ . The optimality of this condition is discussed for some parameter values.

**Keywords.** Vector bundle; symmetric product; exterior product; Borel–le Potier spectral sequence; vanishing theorems.

### 1. Introduction

Throughout this paper  $X$  will denote a smooth projective variety of dimension  $n$  over the field of complex numbers,  $E$  a vector bundle of rank  $e$ , and  $L$  a line bundle on  $X$ .

For any non-negative integers  $\alpha, \beta$  we denote by  $S^\alpha E$ ,  $\wedge^\beta E$  the symmetric product and the exterior product of  $E$ .  $H^{p,q}(X, S^\alpha E \otimes \wedge^\beta E \otimes L)$  will denote the Dolbeault cohomology group

$$H^q(X, S^\alpha E \otimes \wedge^\beta E \otimes L \otimes \Omega_X^p),$$

where  $\Omega_X^p$  is the bundle of exterior differential forms of degree  $p$  on  $X$ .

We start with some definitions.

#### DEFINITION 1.1

The function  $\delta : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$  is the one which satisfies

$$\delta(x) = m \iff \binom{m}{2} \leq x < \binom{m+1}{2}.$$

The last two inequalities imply

$$\delta(x) = \left\lceil \frac{\sqrt{8x+1} + 1}{2} \right\rceil,$$

where the symbol  $[ \ ]$  denotes the integral part, i.e.,

$$\begin{aligned}\delta(0) &= 1, \delta(1) = \delta(2) = 2, \delta(3) = \delta(4) = \delta(5) = 3, \\ \delta(6) &= \delta(7) = \delta(8) = \delta(9) = 4, \dots\end{aligned}$$

**Theorem 1.2.** *Let  $\alpha, \beta \in \mathbb{N}$ . If  $S^{\alpha+\beta} E \otimes L$  is ample, then*

$$H^{p,q}(X, S^\alpha E \otimes \wedge^\beta E \otimes L) = 0$$

for  $q + p - n > (r_0 + \alpha)(e + \alpha - \beta) - \alpha(\alpha + 1)$ , where  $r_0 = \min\{\beta, \delta(n - p), \delta(n - q)\}$ .

**COROLLARY 1.3**

*Let  $\beta$  be a positive integer. If  $S^\beta E \otimes L$  is ample, then*

$$H^{p,q}(X, \wedge^\beta E \otimes L) = 0,$$

for  $q + p - n > r_0(e - \beta)$ , where  $r_0 = \min\{\beta, \delta(n - p), \delta(n - q)\}$ .

This Corollary improves the result of Manivel (Theorem 1, p. 91 of [11]).

**COROLLARY 1.4**

*Assume  $S^\alpha E \otimes L$  is ample. Then*

$$H^{p,q}(X, S^\alpha E \otimes L) = 0,$$

for  $q + p - n > \alpha(e - 1)$ .

This article is the final version of several attempts [10, 14]. The results of these were used by Chaput in [3] and by Laytimi–Nagaraj in [7].

In [13], Manivel studied the vanishing of Dolbeault cohomology of a product of vector bundles tensored with certain power of their determinant. The presence of the latest allowed to deal with the problem by a more direct method.

## 2. The Schur functor version of the theorem

Our main result is a consequence of a Schur functor version of the theorem, but before giving this version, we need to recall some definitions and results:

We start by some preparation on partitions and Schur functors (for a definition see, [5]).

A partition  $u = (u_1, u_2, \dots, u_r)$  is a sequence of non increasing positive integers  $u_i$ . Its length is  $r$  and its weight is  $|u| = \sum_{i=1}^r u_i$ . For  $i > r$  we put  $u_i = 0$ . The zero-partition is the one where all  $u_i$  are zero.

For any partition  $u$  the corresponding Schur functor is denoted by  $\mathcal{S}_u$ .

Let  $V$  be a vector space of dimension  $d$ . To each partition  $u$  corresponds an irreducible  $Gl(V)$ -module  $\mathcal{S}_u(V)$  which vanishes iff  $u_{d+1} > 0$ . For example,  $\mathcal{S}_{(k)}V = S^k V$ . By functoriality the definition of Schur functors carries over to vector bundles  $E$  on  $X$ .

By abuse of language we say that  $\mathcal{S}_u$  has a certain property, if  $u$  has this property. For example, we will say  $S^k$  has weight  $k$ .

DEFINITION 2.1

The Young diagram  $Y(u)$  of a partition  $u$  is given by

$$Y(u) = \{(i, j) \in \mathbb{N}^2 \mid j \leq u_i\}.$$

The transposed partition  $\tilde{u}$  is defined by

$$Y(\tilde{u}) = \{(i, j) \in \mathbb{N}^2 \mid (j, i) \in Y(u)\}.$$

We use the notation  $\wedge_u = \mathcal{S}_{\tilde{u}}$ .

DEFINITION 2.2

The rank of a partition  $u$  is

$$\text{rk}(u) = \max\{\rho \mid (\rho, \rho) \in Y(u)\}.$$

If  $\text{rk}(u) = 1$ , then  $u$  is called a hook.

*Notation 2.3.* If  $u$  is a hook with  $u_1 = \alpha + 1$  and  $|u| = k$ , we write

$$\mathcal{S}_u = \Gamma_k^\alpha.$$

In particular,  $\Gamma_k^0 = \wedge^k$  and  $\Gamma_k^{k-1} = S^k$ . Recall that

$$S^\alpha E \otimes \wedge^\beta E = \Gamma_{\alpha+\beta}^\alpha E \oplus \Gamma_{\alpha+\beta}^{\alpha-1} E.$$

DEFINITION 2.4

For partitions  $u, v$  of the same weight, the dominance partial ordering is defined by

$$u \geq v, \quad \text{iff} \quad \sum_{i=1}^j u_i \geq \sum_{i=1}^j v_i \quad \text{for all } j.$$

This partial ordering can be extended to a pre-ordering of the set of all non-zero partitions of arbitrary weight  $u, v$  with  $|u| = n, |v| = m$ , by comparing as above the partitions of the same weight  $mu$  and  $nv$ , where the multiplication

$$mu = m(u_1, u_2, \dots, u_r) = (mu_1, \dots, mu_r) \quad \forall m \in \mathbb{N}.$$

More precisely,  $u \geq v$  iff  $mu \geq nv$ . We write  $u \simeq v$  iff  $u \geq v$  and  $v \leq u$ .

For convenience we will write  $\mathcal{S}_u \geq \mathcal{S}_v$  instead of  $u \geq v$ . For example,  $\wedge^r \succ \wedge^{r+1}$  and  $S^\alpha \simeq S^1$  for any  $\alpha \in \mathbb{N}$ .

*Lemma 2.5 (Dominance lemma) (Theorem 3.7 of [8]).* For any partition  $u$  and  $v$ , if  $u \geq v$ , then  $\mathcal{S}_u E$  ample  $\implies \mathcal{S}_v E$  ample.

For example, if  $\wedge^2 E$  is ample, then  $\wedge^3 E$  is ample.

Now we give the Schur presentation of the main theorem under which the main theorem will be shown. With notation 2.3 we have as follows.

**Theorem 2.6.** *Let  $k \in \mathbb{N}$ . If  $S^k E \otimes L$  is ample, then*

$$H^{p,q}(X, \Gamma_k^\alpha E \otimes L) = 0,$$

for  $q + p - n > (r_0 + \alpha)(e - k + 2\alpha) - \alpha(\alpha + 1)$ , where  $r_0 = \min\{k - \alpha, \delta(n - p), \delta(n - q)\}$ .

**PROPOSITION 2.7**

*Theorem 2.6 is equivalent to Theorem 1.2.*

*Proof.* Since

$$S^\alpha E \otimes \wedge^{k-\alpha} E = \Gamma_k^\alpha E \oplus \Gamma_k^{\alpha-1} E,$$

we have to only show that for  $1 \leq \alpha \leq k - 1$  the conditions of Theorem 1.2 imply the vanishing of  $H^{p,q}(X, \Gamma_k^{\alpha-1} E)$ , but this is clear since the function  $(r_0 + \alpha)(e - k + 2\alpha) - \alpha(\alpha + 1)$  is increasing in  $\alpha$ . □

**3. Some technical lemmas**

We start with some properties of the function  $\delta$  defined in 1.1.

*Lemma 3.1.* *For  $\mu \in \mathbb{N}$ ,  $x \in \mathbb{N}$  such that  $(x + \mu\delta(x), x - \mu\delta(x)) \in \mathbb{N} \times \mathbb{N}$ . We have*

- (1)  $\delta(x + \delta(x)) = \delta(x) + 1$
- (2)  $\delta(x + \mu\delta(x)) \leq \delta(x) + \mu$
- (3)  $\delta(x - \mu\delta(x)) \leq \delta(x) - \mu$ .

*Proof.* The first assertion and the case  $\mu = 1$  in (2) and (3) are obvious. For both the remaining assertions we use induction on  $\mu$ .

For (2)  $\delta(x + \mu\delta(x)) = \delta(x + \delta(x) + (\mu - 1)\delta(x))$ , since  $\delta(x) \leq \delta(x + \delta(x)) = \delta(x) + 1$ , we have  $\delta(x + \delta(x) + (\mu - 1)\delta(x)) \leq \delta(x + \delta(x) + (\mu - 1)\delta(x + \delta(x)))$ .

Now induction hypothesis gives  $\delta(x + \delta(x) + (\mu - 1)\delta(x + \delta(x))) \leq \delta(x + \delta(x)) + \mu - 1 = \delta(x) + \mu$ .

For (3),  $\delta(x - \mu\delta(x)) = \delta(x - \delta(x) - (\mu - 1)\delta(x))$ ,  $\delta(x - \delta(x) - (\mu - 1)\delta(x)) \leq \delta(x - \delta(x) - (\mu - 1)\delta(x - \delta(x)))$ .

Induction hypothesis gives  $\delta(x - \delta(x) + (\mu - 1)\delta(x - \delta(x))) \leq \delta(x - \delta(x)) - (\mu - 1)$ .

Now since it is true for  $\mu = 1$ , we get  $\delta(x - \delta(x)) - (\mu - 1) \leq \delta(x) - \mu$ . □

**DEFINITION 3.2**

Let  $\phi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \times \mathbb{N}$  be the following injection  $\phi(x, \alpha) = (\phi_1(x, \alpha), \phi_2(x, \alpha), \phi_3(x, \alpha))$ , where

$$\begin{aligned} \phi_1(x, \alpha) &= \delta(x) + \alpha, \\ \phi_2(x, \alpha) &= x - \binom{\delta(x)}{2}, \\ \phi_3(x, \alpha) &= \alpha. \end{aligned}$$

We define an order on the pairs  $(x, \alpha) \in \mathbb{N} \times \mathbb{N}$  by the lexicographic order on  $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$  induced by  $\phi$ , we denote this order by

$$(x', \alpha') \leq_\phi (x, \alpha).$$

The set  $\mathbb{N} \times \mathbb{N}$  endowed with the above order will be denoted by

$$\{\mathbb{N} \times \mathbb{N}, \leq_\phi\} := \mathfrak{U}. \quad (3.1)$$

*Lemma 3.3.* For  $\mu \in \mathbb{Z} - \{0\}$  and  $(x + \mu\delta(x), \alpha - \mu) \in \mathbb{N} \times \mathbb{N}$ , then

$$(x + \mu\delta(x), \alpha - \mu) \leq_\phi (x, \alpha),$$

where the order  $\leq_\phi$  is given in Definition 3.2.

*Proof.* By Lemma 3.1,  $\phi_1(x + \mu\delta(x), \alpha - \mu) \leq \alpha + \delta(x)$ . If  $\delta(x + \mu\delta(x)) = \mu + \delta(x)$ , then

$$\phi_2(x + \mu\delta(x), \alpha - \mu) = x - \binom{\delta(x)}{2} - \binom{\mu}{2} \leq x - \binom{\delta(x)}{2}.$$

If  $\binom{\mu}{2} = 0$ , which means  $\mu = 1$ , then

$$\phi_3(x + \mu\delta(x), \alpha - \mu) = \alpha - 1 < \alpha.$$

□

We need to use these following results.

*Lemma 3.4.* Let  $E$  be an ample vector bundle and  $G$  an arbitrary vector bundle on a projective variety  $X$ . Then for sufficiently large enough  $n$ ,  $S^n E \otimes G$  is ample.

*Lemma 3.5* [2]. Let  $L$  be a line bundle on a projective variety  $X$  and  $d$  be a positive integer. Then there exist a projective variety  $Y$ , a finite surjective morphism  $f : Y \rightarrow X$ , and a line bundle  $M$  on  $Y$ , such that  $f^*L \simeq M^d$ .

*Lemma 3.6.* Let  $p, q, n, f_1, \dots, f_r$  be fixed positive integers and  $\alpha^1, \dots, \alpha^r$  be fixed non-zero partitions. If  $H^{p,q}(X, \otimes_{i=1}^r \mathcal{S}_{\alpha^i} F_i) = 0$  for all smooth projective varieties  $X$  of dimension  $n$  and all ample vector bundles  $F_1, \dots, F_r$  of ranks  $f_1, \dots, f_r$  on  $X$ , then this vanishing statement remains true if one of the  $F_i$  is ample and the others are nef.

*Proof.* We can reorder  $F_i$  such that  $F_1$  is ample. Let  $E = F_1$  and  $\alpha = \alpha^1$ . Let  $N$  be a sufficiently large number such that  $S^N E \otimes \det E^*$  is ample (for the existence of such  $N$ , see Lemma 3.4) and let  $a = \sum_{i=2}^m |\alpha^i|$ . By Lemma 3.5 we can find a finite surjective morphism  $f : Y \rightarrow X$ , and a line bundle  $M$  on  $Y$ , such that  $f^*(\det E) = M^{Na}$ . Then  $E_a = f^*E \otimes (M^*)^a$  is ample since  $S^N E_a$  is. We have

$$f^*(\mathcal{S}_\alpha E \otimes_{i=2}^m \mathcal{S}_{\alpha^i} F_i) = \mathcal{S}_\alpha E_a \otimes_{i=2}^m \mathcal{S}_{\alpha^i} F'_i,$$

where  $F'_i = M^{|\alpha^i|} \otimes f^*F_i$  for  $i = 2, \dots, m$ . Thus all  $F'_i$  are ample. To complete the proof, we use Lemma 10 of [12] which says that, for any vector bundle  $\mathcal{F}$  on  $X$  and any finite surjective morphism  $f : Y \rightarrow X$ , the vanishing of  $H^{p,q}(Y, f^*\mathcal{F})$  implies the vanishing of  $H^{p,q}(X, \mathcal{F})$ . □

*Lemma 3.7.* Fix  $n, p, q, k, \alpha \in \mathbb{N}$  and  $t \in \mathbb{Z}$ . Assume that

$$H^{p,q}(X, \Gamma_k^\alpha E)$$

vanishes for all smooth projective varieties  $X$  of dimension  $n$  and all ample vector bundles  $E$  of rank  $e = k + t$  on  $X$ . Let  $\alpha < k' < k$ . Then  $H^{p,q}(X, \Gamma_k^\alpha E')$  vanishes for all ample vector bundles  $E'$  of rank  $e' = k' + t$  on  $X$ .

*Proof.* For a given  $E'$ , put  $E = E' \oplus L^{\oplus(k-k')}$ , where  $L$  is any ample line bundle. Since  $\Gamma_k^\alpha E' \otimes L^{k-k'}$  is a direct summand of  $\Gamma_k^\alpha E$ , we have

$$H^{p,q}(X, \Gamma_k^\alpha E' \otimes L^{k-k'}) = 0$$

for ample vector bundle  $E'$  of rank  $e'$  and ample line bundle  $L$ . By Lemma 3.6, this vanishing result remains true, when  $L$  is replaced by the trivial line bundle.  $\square$

**COROLLARY 3.8**

*Assume that there is an integer  $k_0$  such that*

$$H^{p,q}(X, \Gamma_k^\alpha E) = 0 \quad \text{if } k > k_0,$$

*for any projective smooth variety  $X$  of dimension  $n$  and any ample vector bundle  $E$  of rank  $e$ , under the condition  $C(n, p, q, \alpha, e - k)$ . Then under this same condition the vanishing remains true for all  $k$ .*

The Bloch–Gieseker lemma can be used in other ways to generalize vanishing theorems. In particular, one has the following.

*Lemma 3.9.* Fix  $n, p, q, e \in \mathbb{N}$  and partitions  $u, v$  of the same weight. Assume that  $H^{p,q}(X, \mathcal{S}_u E)$  vanishes for all projective varieties  $X$  of dimension  $n$  and all vector bundles  $E$  of rank  $e$  for which  $\mathcal{S}_v E$  is ample. Let  $L$  be a line bundle and  $F$  a vector bundle of rank  $e$ . Then  $H^{p,q}(X, \mathcal{S}_u F \otimes L) = 0$ , if  $\mathcal{S}_v F \otimes L$  is ample.

*Proof.* Let us denote  $|u| = |v| = d$ . By Lemma 3.5 we can find a finite surjective morphism  $f : Y \rightarrow X$ , and a line bundle  $M$  on  $Y$ , such that  $f^* L = M^d$ . Then

$$\mathcal{S}_v(f^* F \otimes M) = f^*(\mathcal{S}_v F \otimes L) \text{ is ample.} \tag{3.2}$$

Due to the analogous equation (3.2) for  $\mathcal{S}_u$  one has by assumption

$$H^{p,q}(Y, f^*(\mathcal{S}_u F \otimes L)) = 0,$$

and the vanishing of  $H^{p,q}(X, \mathcal{S}_u F \otimes L)$  follows by using Lemma 10 of [12].  $\square$

The lemma applies, for example, if  $\mathcal{S}_v F$  is nef and  $L$  is ample.

**COROLLARY 3.10**

*To generalize vanishing of type  $H^{p,q}(X, \mathcal{S}_u F \otimes L)$ , from  $L = \mathcal{O}_X$  to arbitrary  $L$ , it suffices to use Lemma 3.9.*

We need to recall

*Lemma 3.11 (Lemma 1.3 of [6]).* Let  $X$  be a projective variety and  $E, F$  be vector bundles on  $X$ . If  $E$  is ample and  $F$  is nef, then  $E \otimes F$  is ample.

#### 4. The Borel–Le Potier spectral sequence

To prepare the proof, we need a lemma and some properties of the Borel–Le Potier spectral sequence, which has been made a standard tool in the derivation of vanishing theorems [4].

Let  $E$  be a vector bundle over a smooth projective variety  $X$ ,  $\dim(X) = n$ . Let  $Y = G_r(E)$  be the corresponding Grassmann bundle and  $Q$  be the canonical quotient bundle over  $Y$ .

*Lemma 4.1.* *Let  $l, r$  be positive integer and  $k = lr$ , if  $\wedge^r E$  is ample. Then for  $P + q > n + r(e - r)$ ,*

$$H^{P,q}(G_r(E), \det Q^l) = 0.$$

*Proof.* Since  $\det Q = \mathcal{O}_{\mathbb{P}(\wedge^r E)}(1)|_{G_r(E)}$ , thus  $\wedge^r E$  ample implies that  $\det Q$  is ample. One concludes by using the Nakano–Akizuki–Kodaira vanishing theorem [1].  $\square$

#### DEFINITION 4.2

Let  $\pi : Y \rightarrow X$  be a morphism of projective manifolds,  $P$  a positive integer and  $\mathcal{F}$  a vector bundle over  $Y$ . The Borel–Le Potier spectral sequence  ${}^P E$  given by the data  $\pi, P, \mathcal{F}$  is the spectral sequence which converge to  $H^{P,q}(Y, \mathcal{F})$ . It is obtained from the filtration on  $\Omega_Y^P \otimes \mathcal{F}$  which is induced by the filtration

$$F^P(\Omega_Y^P) = \pi^* \Omega_X^P \wedge \Omega_Y^{P-P}$$

on the bundle  $\Omega_Y^P$  of exterior differential forms of degree  $P$ .

The graded bundle which corresponds to the filtration on  $\Omega_Y^P$  is given by

$$F^P(\Omega_Y^P)/F^{P+1}(\Omega_Y^P) = \pi^* \Omega_X^P \otimes \Omega_{Y/X}^{P-P},$$

where  $\Omega_{Y/X}^{P-P}$  is the bundle of relative differential forms of degree  $P - p$ . Thus the  $E_1$  terms of  ${}^P E$  have the form

$${}^P E_1^{p,q-p} = H^q(Y, \pi^* \Omega_X^p \otimes \Omega_{Y/X}^{P-p} \otimes \mathcal{F}).$$

These  $E_1$  terms can be calculated as limit groups of the Leray spectral sequence associated to the projection  $\pi$ ,

$${}^{p,P} E_{2,L}^{q-j,j} = H^{p,q-j}(X, R^j \pi_* (\Omega_{Y/X}^{P-p} \otimes \mathcal{F})).$$

Now we consider the Borel–Le Potier spectral sequence which converge to  $H^{P,q}(G_r(E), \det Q^l)$ .

#### PROPOSITION 4.3

*Let  $\pi : G_r(E) = Y \rightarrow X$ , the  $E_1$  terms of the Borel–Le Potier spectral sequence given by  $\pi, P, \det Q^l$  have the form*

$${}^P E_1^{p,q-p} = \bigoplus_{u \in \sigma(P-p,r)} H^q(G_r(E), S_u Q^* \otimes \det Q^l \otimes \wedge_u S \otimes \pi^* \Omega_X^p).$$

Here  $S$  is the tautological sub-bundle of  $\pi^*E$  over  $Y$  and  $\sigma(p, r)$  is the set of partitions of weight  $p$  and length at most  $r$ .

*Proof.* One has  $\Omega_{Y/X} = Q^* \otimes S$ . Thus

$$\Omega_{Y/X}^{P-p} = \bigoplus_{u \in \sigma(P-p, r)} S_u Q^* \otimes \wedge_u S.$$

□

Obviously Leray spectral sequence degenerates at the  $E_{2,L}$  level.

Using Corollary 1 of Bott formula in page 94 of [11], Manivel computes the  $E_1$  terms under some condition on  $P$  (Proposition 3, page 96 of [11]). He states his result under the supplementary condition  $e \geq k$ , which is not necessary for the calculation.

**PROPOSITION 4.4 [11]**

Assume  $P \geq n + (l - 1)\binom{r+1}{2} - l(r - 1)$ , and  $k = lr$ . Let

$$\alpha(p) = \frac{(l - 1)(r + 1)}{2} - \frac{P - p}{r},$$

$$j(p) = (l - 1)\binom{r}{2} - (r - 1)\alpha(p).$$

Then the  $E_1$  terms of the spectral sequence have the form

$${}^P E_1^{p,q} = \begin{cases} H^{P,q-j(p)}(X, \Gamma^{\alpha,k} E) & \text{for } (n - p, \alpha(p)) \in \mathfrak{U} \\ 0 & \text{otherwise,} \end{cases}$$

where the set  $\mathfrak{U}$  is defined in (3.1).

Note that the connecting morphisms of Borel–Le Potier spectral sequence

$$d_m : {}^P E_m^{p,q-p} \longrightarrow {}^P E_m^{p+m,q-p+1-m}$$

all vanish, unless  $m$  is a multiple of  $r$  since under  $d_m$  the integer  $\alpha$  goes to the integer  $\alpha + \frac{m}{r}$ .

**5. Proof of the main theorem**

Before giving the proof of the main theorem, we will first explain the case  $r_0 = \beta$  in the main theorem, which corresponds to Corollary 5.2 bellow.

We need to recall these results.

**Theorem 5.1 [9].** *Let  $E_i$  be vector bundles, with  $\text{rank}(E_i) = e_i$ , over a smooth projective variety  $X$  of dimension  $n$ , and let  $L$  be a line bundle on  $X$ . If  $\otimes_{i=1}^m \Lambda^{r_i} E_i \otimes L$  is ample, then*

$$H^{p,q}(X, \otimes_{i=1}^m \Lambda^{r_i} E_i \otimes L) = 0 \text{ for } p + q - n > \sum_{i=1}^m r_i(e_i - r_i).$$



COROLLARY 5.2

Let  $E$  be a vector bundle of rank  $e$ , and let  $L$  be a line bundle on a smooth projective variety  $X$  of dimension  $n$ . If  $S^{\alpha+\beta}E \otimes L$  is ample, then

$$H^{p,q}(X, S^\alpha E \otimes \Lambda^\beta E \otimes L) = 0 \text{ for } q + p - n > \alpha(e - 1) + \beta(e - \beta).$$

*Proof.* We will apply Theorem 5.1 to the vector bundle  $\underbrace{E \otimes E \otimes \cdots \otimes E}_{\alpha \text{ times}} \otimes \Lambda^\beta E \otimes L$ ,

which  $S^\alpha E \otimes \Lambda^\beta E \otimes L$  is a direct summand.

Let us first show this equivalence of ampleness

$$S^\alpha E \otimes F \simeq \underbrace{E \otimes E \otimes \cdots \otimes E}_{\alpha \text{ times}} \otimes F \tag{5.1}$$

for any vector bundles  $F$ .

Indeed for the first direction, note that  $S^\alpha E \otimes L$  is direct summand of  $\underbrace{E \otimes E \otimes \cdots \otimes E}_{\alpha \text{ times}} \otimes F$ .

For the second direction, Littlewood–Richardson rules give

$$\underbrace{E \otimes E \otimes \cdots \otimes E}_{\alpha \text{ times}} = S^\alpha E \oplus \sum_{|\lambda|=\alpha} S_\lambda E.$$

We clearly have  $\alpha > \lambda$  in the dominance partial order and use Remark 2.5 to conclude.

Now by Littlewood–Richardson rules,

$$S^\alpha E \otimes \Lambda^\beta E = \oplus S_\nu E \text{ with } |\nu| = \alpha + \beta,$$

satisfying  $S_\nu \prec S^{\alpha+\beta}$ . Thus the ampleness of  $S^{\alpha+\beta}E \otimes L$  implies the ampleness of  $S^\alpha E \otimes \Lambda^\beta E \otimes L$  by Remark 2.5. We use the equivalence of ampleness (5.1) to conclude.  $\square$

Due to Remark 3.10 one can prove our main theorem without  $L$ .

We prove Theorem 1.2 by induction on  $(n - p, \alpha) \in \mathfrak{U}$ , where the set  $\mathfrak{U}$  is given in Definition 3.2.

Assume that the result is true for all pairs  $(p', \alpha')$  such that

$$(n - p', \alpha') \leq_\phi (n - p, \alpha),$$

with respect to the order introduced in Definition 3.2.

Choose  $r = \delta(n - p)$ . Let  $l$  be arbitrary if  $n = p$ , otherwise let  $l \geq \frac{r\alpha+n-p}{r-1}$ . Choose  $P$  such that  $\alpha(p) = \alpha$ , and consider the Borel–Le Potier spectral sequence. Then for  $k = lr$ ,

$${}^P E_1^{p,q+j(p)-p} = H^{p,q}(X, \Gamma^{\alpha,k} E).$$

When  $m$  is a multiple of  $r$ ,  $m = \mu r$ , the morphisms  $d_m$  connect  ${}^P E_1^{p,q+j(p)-p}$  with  ${}^P E_1^{p',q'+j(p')-p'}$  where for the terms on the right of  ${}^P E_1^{p,q+j(p)-p}$ ,

$$p' = p + \mu r, \quad q' = q + \mu(r - 1) + 1, \tag{5.2}$$

and

$$p' = p - \mu r, \quad q' = q - \mu(r - 1) - 1 \tag{5.3}$$

for the terms on the left. Here  $\mu$  is any positive integer.

*Lemma 5.3.* For any integers  $p'$  and  $q'$  of the form (5.2) or (5.3),

$${}^P E_1^{p', q'+j(p')-p'} = 0,$$

when  $q > Q(n - p, \alpha)$ , where

$$Q(n - p, \alpha) = n - p + (\delta(n - p) + \alpha)(e - k + 2\alpha) - \alpha(\alpha + 1).$$

*Proof.* The assertion is trivially true for  $\alpha(p') < 0$  or  $e - k + \alpha(p') < 0$ , such that we may assume  $e - k + 2\alpha(p') \geq 0$ .

We need to prove that the assertion  $q > Q(n - p, \alpha)$  implies the assertion  $q' > Q(n - p', \alpha')$ .

The terms on the right of  ${}^P E_1^{p, q+j(p)-p}$  have  $\alpha' = \alpha + \mu$ , and the same parameters as in (5.2), a straight calculation yields

$$\begin{aligned} & Q(n - p, \alpha) - Q(n - p', \alpha') + \mu(\delta(n - p) - 1) + 1 \\ &= (e - k + 2(\alpha + \mu)(\delta(n - p) - \mu - \delta(n - p - \mu\delta(n - p))) + \mu^2 + 1. \end{aligned} \tag{5.4}$$

The terms on the left of  ${}^P E_1^{p, q+j(p)-p}$  have  $\alpha' = \alpha - \mu$ , and the parameters in (5.3), the calculation yields

$$\begin{aligned} & Q(n - p, \alpha) - Q(n - p', \alpha') - \mu(\delta(n - p) - 1) - 1 \\ &= (e - k + 2(\alpha - \mu)(\delta(n - p) + \mu - \delta(n - p + \mu\delta(n - p))) + \mu^2 - 1. \end{aligned} \tag{5.5}$$

By Lemma 3.1, both terms of (5.4) and (5.5) are non negative and positive if  $\mu \neq 1$ . Thus  $q > Q(n - p, \alpha)$  implies  $q' > Q(n - p', \alpha(p'))$ .

By Lemma 3.3,  $(n - p', \alpha(p')) \leq_\phi (n - p, \alpha)$  such that the groups  ${}^P E_1^{p', q'+j(p')-p'}$  vanish by induction hypothesis. Thus all co-bordant morphisms of  ${}^P E_1^{p, q+j(p)-p}$  vanish. This implies that  ${}^P E_1^{p, q+j(p)-p}$  is a sub-factor of  $H^{P, q+j(p)-p}(Y, F)$ , where  $F = \det(Q)^l$ .

Recall that  $P = p + (l - 1)\binom{r+1}{2} - \alpha r$  and  $\dim Y = n + r(e - r)$ . Thus the condition  $q > Q(n - p, \alpha)$  is equivalent to

$$P + q + j(p) - \dim Y > \alpha(e - k + \alpha).$$

When the right-hand side is non-negative,  $H^{P, q+j(p)-p}(Y, F) = 0$  by the Nakano–Kodaira–Akizuki vanishing theorem. Thereby

$$H^{P, q}(X, \Gamma_k^\alpha E) = 0 \text{ for } q > Q(n - p, \alpha).$$

Remember that this proof was under the condition  $k = rl$  (see Proposition 4.4), but this condition can be removed by Corollary 3.8.

To get  $r_0 = \delta(n - q)$  in our theorem, we interchange the role of  $p$  and  $q$  at every stage of the proof, in particular we use  $r = \delta(n - q)$ .

### 6. Optimality

#### PROPOSITION 6.1

Let  $G = \text{Gr}_{(r,d)}$  be the Grassmannian of all co-dimensional  $r$  subspaces of a vector space  $V$  of dimension  $d = f + r$ . Let  $Q$  be the universal sub-bundle of rank  $r$  on  $G$ ,  $\dim G = n = fr$ .

Then, for  $q = n - f$ ,  $\alpha = f - 1$ ,

$$H^q(G, S^\alpha Q \otimes Q \otimes \det Q \otimes K_X) \neq 0.$$

*Proof.* Since  $S^{\alpha+1} Q$  is a direct summand of  $S^\alpha Q \otimes Q$ , it is enough to show that

$$H^q(G, S^{\alpha+1} Q \otimes \det Q \otimes K_X) \neq 0.$$

For the universal sub-bundle  $S$  on  $G$ , we have  $K_G = ((\det Q)^*)^{\otimes d} = \det S^{\otimes d}$ . Thus since  $\alpha = f - 1$ ,

$$H^q(G, S^f Q \otimes \det Q \otimes K_X) = H^q(G, S^f Q \otimes \det S^{\otimes(d-1)}).$$

Now by Bott formula (see corollary 1, page 94 of [11])

$$H^q(G, S^f Q \otimes \det S^{\otimes(d-1)}) = \delta_{q, i(a,b)-c(d)} \mathcal{S}_{\psi(a,b)} V,$$

where

$$a = (f, \underbrace{0, \dots, 0}_{r-1 \text{ times}}), \quad b = (\underbrace{d-1, \dots, d-1}_{d-r \text{ times}}).$$

For any sequence  $v = (v_1, v_2, \dots)$ ,

$$i(v) = \text{card}\{(i, j) / i < j, v_i < v_j\},$$

where

$$\psi(v) = (v - c(d))^\geq + c(d),$$

$$c(d) = (1, 2, \dots, d),$$

and  $(v)^\geq$  is the partition obtained by ordering the terms of  $v$  in non increasing order.

$$(a, b) = (f, \underbrace{0, \dots, 0}_{r-1 \text{ times}}), \quad (\underbrace{d-1, \dots, d-1}_{d-r \text{ times}}),$$

$$((a, b) - c(d)) = (f - 1, -2, -3, \dots, -r, f - 2, f - 3, \dots, 0, -1),$$

we get  $i((a, b) - c(d)) = f(r - 1) = n - f$ , and

$$\psi(a, b) = \underbrace{(f, f, \dots, f)}_{d \text{ times}}.$$

Thus  $\mathcal{S}_{\psi((a,b))} V = (\det V)^{\otimes f}$ .

Note that the non-vanishing example of the above proposition happens for the limit condition

$$q + p - n = (r_0 + \alpha)(e + \alpha - \beta) - \alpha(\alpha + 1),$$

where  $r_0 = \min\{\beta, \delta(n - p), \delta(n - q)\}$ . □

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