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Hyperbolicity in median graphs

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Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) = \inf\{\delta \geq 0 : \ X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we study the hyperbolicity of median graphs and we also obtain some results about general hyperbolic graphs. In particular, we prove that a median graph is hyperbolic if and only if its bigons are thin.

Keywords. Median graph; Gromov hyperbolicity; Gromov hyperbolic graph; infinite graphs; geodesics.

1. Introduction

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces (see [1, 19, 20]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 19, 20]).

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [20] and references therein), where it was demonstrated to have a practical importance. This theory was applied principally to the study of automatic groups (see [34]), which play a role in the science of computation. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [45] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [14, 15, 18, 31]). Another important application of these spaces is secure transmission of information on the internet (see [25–27]). In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [26, 27]). The hyperbolicity is also useful in the study of DNA data (see [9]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [5–7, 9–11, 13, 17, 25–30, 32, 33, 35–38, 41–44, 46, 47].

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring–Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [21]). The study of Gromov hyperbolicity of the quasi-hyperbolic and the Poincaré metrics was the subject of [2, 8, 22, 23, 38–40, 43, 44]. In particular, in [38, 43, 44, 46] it is proved that the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our study on hyperbolic graphs we use the notations of [19]. Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L denotes the length of a curve. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

As a normalization, we consider graphs such that the length of every edge is 1 (the case of graphs with all edge lengths is equal to $k \in (0, \infty)$ can be reduced to this one). In order to consider a graph G as a geodesic metric space, we must identify any edge $[u, v] \in E(G)$ with the real interval [0, 1]; therefore, any point in the interior of any edge is a point of G. Hence, if we consider the edge $[u, v]$ as a graph with just one edge, then it is isometric to [0, 1]. A connected graph G is naturally equipped with a distance defined on its points, induced by taking the shortest paths in G. Then, we see G as a metric graph.

Along the paper we just consider simple (without loops and multiple edges) connected graphs whose edges have length equal to 1; these properties guarantee that the graphs are geodesic metric spaces. Note that to exclude multiple edges and loops is not an important loss of generality, since Theorems 8 and 10 of [6] reduce the problem to compute the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. If we have a triangle with two identical vertices, we call it a *bigon*; note that since this is a special case of the definition, every geodesic bigon in a δ-hyperbolic space is δ-thin.

Trivially, every bounded metric space X is $((\operatorname{diam} X)/2)$-hyperbolic. The real line $\mathbb{R}$ is 0-hyperbolic whereas the Euclidean plane $\mathbb{R}^2$ is not. In general, a normed vector space E is hyperbolic if and only if $\dim E = 1$. Every metric tree with arbitrary length edges is 0-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -c^2 < 0$ is hyperbolic (see [1, 19] for more background and further results).

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how 'tree-like' the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see [12]).

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Note that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take the supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and/or to study the hyperbolicity for a particular class of graphs.

The papers [7, 9, 11, 33, 35, 37] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, line graphs, cartesian product graphs, cubic graphs and tessellation graphs. In this paper we study the hyperbolicity of median graphs and we also obtain some results about general hyperbolic graphs. Median graphs and related median structures (median algebras and median complexes) have many nice properties and admit numerous characterizations. These structures have been investigated in several contexts by quite a number of authors for more than half a century. Median structures are still being rediscovered in various disguises. For more detailed information, see [3, 16, 24].

2. Hyperbolicity and median graphs

In any graph, for any two vertices a and b, we define the *interval* of vertices that lie on shortest paths $I(a,b) := \{v / d(a,b) = d(a,v) + d(v,b)\}$. A *median graph* is defined by the property that, for any three vertices a, b, c, the intervals $I(a,b), I(a,c), I(b,c)$ intersect in a single point.

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The hyperbolicity constants of some median graphs are known (see [33]).

PROPOSITION 2.1

The following equalities hold:

(1) *If G is any tree*, *then* $\delta(G) = 0$.
(2) *If G is the Cayley graph of the group* $\mathbb{Z}^n$, *then* $\delta(G) = 0$ *for* $n = 1$ *and* $\delta(G) = \infty$ *for* $n > 1$.
(3) *If G is the n-cube graph* $Q_n = K_2 \times \cdots \times K_2$, *then* $\delta(G) = n/2$.

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last vertex, which is equal to the first one.

Given any graph G, we denote by $\tau(G)$ the set of geodesic triangles T in G which are cycles and such that each vertex of T is either a vertex in $V(G)$ or a midpoint of some edge in $E(G)$.

In [5] we found the following result.

Theorem 2.2. *In any graph G,*

$$\delta(G) = \sup\left\{\delta(T) : T \in \tau(G)\right\}.$$

In particular, $\delta(G)$ is an integer multiple of $1/4$.

In order to prove Theorem 2.5, we need some previous result (see Theorem 2.4 below), which is interesting by itself. Theorem 2.4 is a version of Theorem 2.2 with an additional advantage: it involves just the vertices of G. Of course, we just obtain estimates of $\delta(G)$ instead of an equality.

DEFINITION 2.3

Given a graph G, let us consider the set $\mathbb{T}_2$ of geodesic triangles T in G which are cycles and such that the three vertices of the triangle T are vertices of G. We define the constant $\delta_2(G)$ as the supremum of $\delta(T)$ for every $T \in \mathbb{T}_2$.

Theorem 2.4. *For every graph G, we have*

$$\delta_2(G) \le \delta(G) \le 4\delta_2(G) + \frac{1}{2}.$$

Proof. Since the first inequality is direct, it suffices to prove the second one.

Let us consider the set $\mathbb{T}_3$ of geodesic triangles T in G such that the three vertices of the triangle T are vertices of G. We define the constant $\delta_3(G)$ as the supremum of $\delta(T)$ for every $T \in \mathbb{T}_3$.

Let us consider any fixed geodesic triangle $T = \{a, b, c\} \in \tau(G)$. Assume that a, b, c are different midpoints of edges in $E(G)$ (the other cases are similar). Then $a \in [a_0, a_1]$, $b \in [b_0, b_1]$ and $c \in [c_0, c_1]$, where $[a_0, a_1]$, $[b_0, b_1]$ and $[c_0, c_1]$ are different edges. Without loss of generality we can assume that the points $a_0, a_1, b_0, b_1, c_0, c_1$ are chosen in the appropriate order so that

$$[a_0, a_1] \cup [a_1 b_0] \cup [b_0, b_1] \cup [b_1 c_0] \cup [c_0, c_1] \cup [c_1 a_0]$$

is the cycle T. Let us consider now the geodesic hexagon $H = \{a_0, a_1, b_0, b_1, c_0, c_1\}$ with sides

$$[a_0, a_1], [a_1 b_0], [b_0, b_1], [b_1 c_0], [c_0, c_1], [c_1 a_0].$$

Since $a_0, a_1, b_0, b_1, c_0, c_1 \in V(G)$, splitting the hexagon in four triangles, we deduce that H is $4\delta_3(G)$-thin. An argument similar to the one in the proof of Lemma 2.1 of [43] gives $\delta_3(G) = \delta_2(G)$; hence, H is $4\delta_2(G)$-thin.

Let us fix $p \in T$; without loss of generality we can assume that $p \in [ab]$. If $p \notin [a_1 b_0]$, then $p \in [aa_1] \cup [b_0 b]$ and $d(p, [bc] \cup [ca]) \le d(p, \{a, b\}) < 1/2$. If $p \in [a_1 b_0]$, then there exists $q \in [b_0, b_1] \cup [b_1 c_0] \cup [c_0, c_1] \cup [c_1 a_0] \cup [a_0, a_1]$ with $d(p, q) \le 4\delta_2(G)$. Since $[b_1 c_0] \cup [c_0, c_1] \cup [c_1 a_0] \subset [bc] \cup [ca]$, we have that

$$d(p, [bc] \cup [ca]) \le d(p, q) + \frac{1}{2} \le 4\delta_2(G) + \frac{1}{2}.$$

Hence, $\delta(T) \le 4\delta_2(G) + 1/2$ for any geodesic triangle T which is a cycle, and we conclude that $\delta(G) \le 4\delta_2(G) + 1/2$. □

Theorem 2.5. *Let G be any median graph. Then G is hyperbolic if and only if the bigons of G are uniformly thin. In fact, if every geodesic bigon in G is δ-thin, then*

$$\delta_2(G) \le 3\delta, \qquad \delta(G) \le 12\delta + \frac{1}{2}\,.$$

Proof. If G is δ-hyperbolic, then every bigon of G is δ-thin, since a bigon is a triangle with two vertices equal.

Assume now that every geodesic bigon in G is δ-thin. In order to prove the upper bound of $\delta(G)$, by Theorem 2.4, it suffices to prove that $\delta_2(G) \le 3\delta$. Let us consider any geodesic triangle $T = \{x, y, z\}$ in G with $x, y, z \in V(G)$ and $p \in T$. Denote the geodesics of T by $\gamma_1 = [xy]$, $\gamma_2 = [xz]$, $\gamma_3 = [yz]$. Without loss of generality, we can assume that $p \in \gamma_1$. Since G is a median graph, we know that there exist geodesics $\gamma_1', \gamma_2', \gamma_3'$, with the same endpoints, respectively, than $\gamma_1, \gamma_2, \gamma_3$, and such that $\gamma_1' \cap \gamma_2' \cap \gamma_3' = \{m\}$. Since $\{\gamma_1, \gamma_1'\}$ is a geodesic bigon, we know that there exists $p_1 \in \gamma_1'$ with $d(p, p_1) \le \delta$. Without loss of generality we can assume that p_1 belongs to the geodesic $g_1 \subseteq \gamma_1'$ joining x and m. Let us consider the geodesic $g_2 \subseteq \gamma_2'$ joining x and m. Since $\{g_1, g_2\}$ is a geodesic bigon, we know that there exists $p_2 \in g_2$ with $d(p_1, p_2) \le \delta$. Since $\{\gamma_2', \gamma_2\}$ is a geodesic bigon, we know that there exists $q \in \gamma_2$ with $d(q, p_2) \le \delta$. Then $d(p, \gamma_2 \cup \gamma_3) \le d(p, q) \le d(p, p_1) + d(p_1, p_2) + d(p_2, q) \le 3\delta$.

Since T is any geodesic triangle with vertices in $V(G)$, we conclude that $\delta_2(G) \le 3\delta$. Hence, Theorem 2.4 gives $\delta(G) \le 12\delta + \frac{1}{2}$. □

There are important results stating that the hyperbolicity constant of a graph $T(G)$ (obtained from an original graph G via some transformation T), is bounded in terms of the hyperbolicity constant of G (see p. 88 of [19] and [11]). Theorem 2.11 below states a similar result for another transformation. In order to prove it we need to introduce a useful concept.

Given a graph G, we say that a family of subgraphs $\{G_n\}_n$ of G is a *tree-decomposition* of G if $\cup_n G_n = G$, $G_n \cap G_m$ is either a vertex or the empty set for each $n \neq m$, and if the graph R defined as follows is a tree: for each n let us define a point v_n (v_n is an abstract point, it is not contained in G); we have $V(R) := \{v_n\}_n$ and $[v_n, v_m] \in E(R)$ if and only if $G_n \cap G_m \neq \emptyset$.

A tree-decomposition of G always exists, as we will show now (although it can be trivial: if the graph is two-connected then tree-decomposition has just one element). We say that a vertex v of a graph G is a *tree-vertex* if $G \setminus \{v\}$ is not connected. Given any edge in G, let us consider the maximal two-connected subgraph containing it. It is clear that the set of these maximal two-connected subgraphs $\{G_n\}_n$ is a tree-decomposition of G; we call it the *canonical tree-decomposition* of G.

Note that every G_n in any tree-decomposition of G is an isometric subgraph of G.

We will need the following result, which allows to obtain global information about the hyperbolicity of a graph from local information (see [6]).

Lemma 2.6. *Let G be any graph and let $\{G_n\}_n$ be any tree-decomposition of G. Then* $\delta(G) = \sup_n \delta(G_n)$.

Proof. Note that, since $\{G_n\}_n$ is a tree-decomposition of G, if g is a cycle in G, then there exists n such that $g \subseteq G_n$. By Theorem 2.2, we have

$$\begin{aligned}\delta(G) &= \sup\{\delta(T) : T \text{ is a geodesic triangle that is a cycle}\}\\ &= \sup_n \{\sup\{\delta(T) : T \text{ is a geodesic triangle that is a cycle in } G_n\}\}\\ &= \sup_n \delta(G_n).\end{aligned}$$

□

One can check that the following property about median graph and tree-decompositions holds.

PROPOSITION 2.7

The following statements are equivalents for any graph G:

(1) *G is a median graph.*
(2) *G_n is a median graph for every n for some tree-decomposition $\{G_n\}_n$ of G.*
(3) *G_n is a median graph for every n for every tree-decomposition $\{G_n\}_n$ of G.*

A subgraph Γ of G is said to be *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$. The following results appear in [42].

Lemma 2.8. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$.*

Lemma 2.9. *The complete graphs with n vertices K_n verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ for every $n \ge 4$. The wheel graphs with n vertices W_n verify $\delta(W_n) = 3/2$ if and only if $7 \le n \le 10$.*

In [32] the following result appear which characterize the hyperbolic graphs G with hyperbolicity constant verifying $\delta(G) < 1$.

Theorem 2.10. *If G is any graph with $\delta(G) < 1$, then we have either $\delta(G) = 0$ or $\delta(G) = 3/4$. Furthermore*:

(1) *$\delta(G) = 0$ if and only if G is a tree.*
(2) *$\delta(G) = 3/4$ if and only if G is not a tree and every cycle in G has length 3.*

Theorem 2.11. *Given any graph G, r subgraphs $G_1^0, \dots, G_r^0$ of G isomorphic, respectively, to the complete graphs $K_{c_1}, \dots, K_{c_r}$ (such that $c_j \ge 2$ for $j = 1, \dots, r$, and $G_i^0 \cap G_j^0$ is either the empty set or some vertex of G if $i \ne j$) and natural numbers $d_1 > c_1, \dots, d_r > c_r$, we denote by G^* the graph obtained from G by replacing each G_j^0 by a graph $G_j \supset G_j^0$ isomorphic to K_{d_j}, for $j = 1, \dots, r$.*

(1) *If G is not a tree, then*

$$\delta(G) \le \delta(G^*) \le \delta(G) + \frac{1}{2}$$

and both inequalities are sharp.

(2) *If G is a tree and $d_1 = \cdots = d_r = 3$, then $\delta(G^*) = 3/4$; if G is a tree and $\max\{d_1, \dots, d_r\} \geq 4$, then $\delta(G^*) = 1$.*

Proof. We deal first with the case in which G is not a tree. Since G is an isometric subgraph of G^*, Lemma 2.8 gives $\delta(G) \leq \delta(G^*)$.

In order to prove the upper bound for $\delta(G^*)$, let us consider any geodesic triangle T in G^* with sides $\gamma_1, \gamma_2, \gamma_3$. By Theorem 2.2 we can assume that T is a cycle and that each vertex of T is either a vertex in $V(G^*)$ or a midpoint of some edge in $E(G^*)$. If the cycle T is contained in some G_m, then $\delta(T) \leq \delta(K_{d_m}) \leq 1$ by Lemma 2.9. If the cycle T is contained in G, then $\delta(T) \leq \delta(G)$.

Assume now that the cycle T is contained neither in G nor in some G_m. Without loss of generality we can assume that for each j with $T \cap G_j \neq \varnothing$, the set $T \cap G_j$ is not isometric to a subset of G_j^0 with the same endpoints, since otherwise we can replace $T \cap G_j$ by its isometric subset of G_j^0 with the same endpoints. Then, since the sides of T are geodesics, if T intersects some $G_m \setminus G_m^0$, then at least a vertex of the geodesic triangle T belongs to this $G_m \setminus G_m^0$.

Since G is an isometric subgraph of G^*, $\gamma_i^0 := \gamma_i \cap G$ is a (connected) geodesic in G for $i = 1, 2, 3$. We are going to construct a geodesic triangle T_0 in G containing $\gamma_1^0, \gamma_2^0, \gamma_3^0$. Since T is a cycle, if some endpoint x_i of γ_i^0 is not an endpoint of γ_i, then there exists an edge $e_{ij} \in E(G)$ connecting x_i with some endpoint x_j of γ_j^0. Let us define w' as the midpoint of the edge $e_{ij} = [x_i, x_j]$. Let us denote by T' the connected component of $T \setminus G$ which joins x_j and x_i. Then $T' \subseteq G_s \setminus G_s^0$ for some $1 \leq s \leq r$.

Assume first that exactly one vertex w of the geodesic triangle T belongs to this $G_s \setminus G_s^0$.

We claim that if there exists a vertex $y_j \in V(G_s) \setminus V(G_s^0)$ with $[x_j, y_j] \subset \gamma_j \cap G_s$, then $g_j^0 := \gamma_j^0 \cup [x_j w']$ is a geodesic in G.

In fact, we now prove that if $\gamma_j^0 = [z_j x_j]$, then $d_{G^*}(z_j, x_j) \leq d_{G^*}(z_j, x_i)$. Seeking for a contradiction, assume that $d_{G^*}(z_j, x_j) > d_{G^*}(z_j, x_i)$. Then

$$\begin{aligned} d_{G^*}(z_j, y_j) &\leq d_{G^*}(z_j, x_i) + d_{G^*}(x_i, y_j) < d_{G^*}(z_j, x_j) + 1 \\ &= L(\gamma_j^0) + L([x_j, y_j]), \end{aligned}$$

and this implies that γ_j is not a geodesic. This is the contradiction we were looking for. Therefore, $d_{G^*}(z_j, x_j) \leq d_{G^*}(z_j, x_i)$. Hence, g_j^0 is a geodesic in G.

Assume now that $L(T') = 3$. Then there exist vertices $y_i, y_j \in V(G_s) \setminus V(G_s^0)$ with $y_i \neq y_j$, $[x_i, y_i] \subset \gamma_i \cap G_s$, $[x_j, y_j] \subset \gamma_j \cap G_s$ and the vertex $w = \gamma_i \cap \gamma_j$ of T is the midpoint of $[y_i, y_j]$. The claim gives that $g_i^0 := \gamma_i^0 \cup [x_i w']$ and $g_j^0 := \gamma_j^0 \cup [x_j w']$ are geodesics in G.

Assume now that $L(T') = 2$. If $w \in V(G^*)$, then let us define $g_i^0 := \gamma_i^0 \cup [x_i w']$ and $g_j^0 := \gamma_j^0 \cup [x_j w']$; the claim gives that g_i^0 and g_j^0 are geodesics in G. If w is a midpoint of some edge in $E(G_s)$, then without loss of generality we can assume that it is the midpoint of $[x_i, a]$, with $a \in V(G_s)$; let us define $g_i^0 := \gamma_i^0$ and $g_j^0 := \gamma_j^0 \cup [x_j, x_i]$; we have that g_i^0 and g_j^0 are geodesics in G (note that $\gamma_j^0 \cup [x_j, a] \cup [aw]$ and $g_j^0 \cup [x_i w] = \gamma_j^0 \cup [x_j, x_i] \cup [x_i w]$ have the same endpoints and the same length).

If there are two vertices of T in T', assume first that $L(T') = 2$. Denote by $a \in V(G_s)$ the midpoint of T'. Since each vertex of T is either a vertex in $V(G^*)$ or a midpoint of some edge in $E(G^*)$, we have just two cases. If the vertices of T in T' are the midpoints of

$[x_i, a]$ and $[x_j, a]$, let us define the geodesics in G: $g_i^0 := \gamma_i^0$, $g_j^0 := \gamma_j^0$ and $g_k^0 := [x_j, x_i]$. Otherwise, without loss of generality we can assume that the vertices of T in T' are a and the midpoint of $[x_i, a]$; let u be the other vertex of T. Note that we have either $d_{G^*}(u, x_j) = d_{G^*}(u, x_i)$ or $d_{G^*}(u, x_j) = d_{G^*}(u, x_i) - 1$. If $d_{G^*}(u, x_j) = d_{G^*}(u, x_i)$, then it is clear that $g_i^0 := \gamma_i^0 \cup [x_i w']$ and $g_j^0 := \gamma_j^0 \cup [x_j w']$ are geodesics in G. If $d_{G^*}(u, x_j) = d_{G^*}(u, x_i) - 1$, then $g_i^0 := \gamma_i^0$ and $g_j^0 := \gamma_j^0 \cup [x_j, x_i]$ are geodesics in G.

Assume now that $L(T') = 3$; then the closure of T' is $\overline{T'} = [x_j, y_j] \cup [y_j, y_i] \cup [y_i, x_i]$, with $y_i, y_j \in V(G_s) \setminus V(G_s^0)$ and $y_i \neq y_j$. We have the next three cases.

If $[x_i, y_i] \subset \gamma_i \cap G_s$ and $[x_j, y_j] \subset \gamma_j \cap G_s$, then the claim gives that $g_i^0 := \gamma_i^0 \cup [x_i w']$ and $g_j^0 := \gamma_j^0 \cup [x_j w']$ are geodesics in G.

If the vertices of T in T' are the midpoints of $[x_i, y_i]$ and $[x_j, y_j]$, then $g_i^0 := \gamma_i^0$, $g_j^0 := \gamma_j^0$ and $g_k^0 := [x_j, x_i]$ are geodesics in G.

If one of the vertices of T in T' is in $[y_j, y_i]$ and the other is the midpoint of $[x_i, y_i]$, then the claim gives that $g_i^0 := \gamma_i^0$, $g_j^0 := \gamma_j^0 \cup [x_j w']$ and $g_k^0 := [w' x_i]$ are geodesics in G.

Iterating this process at most three times, we obtain a geodesic triangle T_0 in G with sides $\gamma_1^1, \gamma_2^1, \gamma_3^1$, containing $\gamma_1^0, \gamma_2^0, \gamma_3^0$, respectively.

Let $p \in T$ be any fixed point. Without loss of generality we can assume that $p \in \gamma_1$. It is not difficult to check the following two facts in any case: if $p \in G$, then $d_{G^*}(p, \gamma_2 \cup \gamma_3) \leq d_G(p, \gamma_2^1 \cup \gamma_3^1) + 1/2 \leq \delta(G) + 1/2$; if $p \notin G$, then $d_{G^*}(p, \gamma_2 \cup \gamma_3) \leq 5/4$.

Since T is arbitrary, we deduce that $\delta(G^*) \leq \max\{\delta(G) + 1/2, 5/4\}$.

In order to finish the proof, if G is not a tree, then Theorem 2.10 gives that $\delta(G) \geq 3/4$, and we have $\delta(G) + 1/2 \geq 5/4$.

The lower bound for $\delta(G^*)$ is attained in any complete graph $G = K_c$ with $c \geq 4$ by replacing K_c by $G^* = K_d$ with $d > c$. The upper bound is attained in any cycle graph $G = C_{2n}$ by replacing two opposite edges of G by two graphs isomorphic to K_3.

Assume now that G is a tree. In this case, we have $c_j = 2$ for $j = 1, \ldots, r$. Then Lemma 2.6 gives $\delta(G^*) = \max_{1 \leq j \leq r} \leq \delta(K_{d_j})$. Hence, Lemma 2.9 gives that if $d_1 = \cdots = d_r = 3$, then $\delta(G^*) = 3/4$, and if $\max\{d_1, \ldots, d_r\} \geq 4$, then $\delta(G^*) = 1$. $\square$

3. Hyperbolicity constant and large degree

In this section we compute the hyperbolicity constant of any graph of order n with a vertex with degree $n - 1$ (see Theorem 3.4 below). This result is useful in the study of the hyperbolicity constant of a large class of graphs. We need some definitions.

Given $(G_1, v_1), (G_2, v_2), \ldots, (G_m, v_m)$, where G_j are finite graphs and $v_j \in V(G_j)$, we define the $(v_1, v_2, \ldots, v_m)$-union of $G_1, G_2, \ldots, G_m$, as the graph obtained by pasting the graphs $G_1, G_2, \ldots, G_m$, by identifying the vertices $v_1, v_2, \ldots, v_m$ in a single vertex of the new graph. As an example, if the graphs $G_1, G_2, \ldots, G_m$ are isomorphic to a complete graph with two vertices, then the $(v_1, v_2, \ldots, v_m)$-union of $G_1, G_2, \ldots, G_m$ is isomorphic to a star graph with $m + 1$ vertices.

Define $\mathcal{D}_0$ as the set of graphs isomorphic to some star graph.

Define $\mathcal{D}_{3/4}$ as the set of graphs isomorphic to some $(v_1, v_2, \ldots, v_m)$-union of $G_1, G_2, \ldots, G_m$, where each G_j is isomorphic to a complete graph with either two or three vertices.

Define $\mathcal{D}_1$ as the set of graphs isomorphic to some $(v_1, v_2, \ldots, v_m)$-union of $G_1, G_2, \ldots, G_m$, where each G_j is a graph with order n_j verifying the following properties: $\deg v_j = n_j - 1$, there exists a cycle isomorphic to C_4, and for every cycle γ with $L(\gamma) \geq 5$ we have $\deg_\gamma w \geq 3$ for every vertex $w \in \gamma$.

Denote by W_n the wheel graph with n vertices, for $7 \leq n \leq 10$. Denote by CW_n the cycle subgraph of W_n containing just the $n-1$ vertices with degree 3. Let us fix a vertex p in $V(CW_n)$, and denote by x and y the two points in CW_n at distance $3/2$ from p. Denote by z the antipodal point of p in CW_n, i.e., the point verifying $d_{CW_n}(z, p) = (n-1)/2$. Since $d_{W_n}(z, x) = d_{W_n}(z, y) = (n-4)/2 \leq 3$, it is possible to choose geodesics $[xy]$, $[xz]$, $[yz]$ in W_n such that $[xy] \cup [xz] \cup [yz]$ is equal to the cycle CW_n.

Let E_7 be the set of edges joining vertices in $V(CW_7)$,

$$E_7 := \{[z, w]/w \in V(CW_7), w \in [xy] \setminus \{p\}\}.$$

For each $8 \leq n \leq 10$, let $z_x, z_y \in V(CW_n)$ be the vertices verifying $z_x \in [xz]$, $z_y \in [yz]$ and $d_{W_n}(p, z_x) = d_{W_n}(p, z_y) = 3$ (then $d_{W_n}(z, z_x) = d_{W_n}(z, z_y) = (n-7)/2$). Let E_n be the set of edges joining vertices in $V(CW_n)$,

$$\begin{aligned} E_n := &\{[z_x, w]/w \in V(CW_7) \cap ([xy] \cup [yz] \setminus \{p\})\} \\ &\cup \{[z_y, w]/w \in V(CW_7) \cap ([xy] \cup [xz] \setminus \{p\})\}. \end{aligned}$$

For each $7 \leq n \leq 10$, let $\mathcal{E}_n$ be the set of graphs G verifying $V(G) = V(W_n)$ and $E(W_n) \subseteq E(G) \subseteq E(W_n) \cup E_n$. Note that the geodesics $[xy]$, $[xz]$, $[yz] \subset CW_n$ in W_n are also geodesics in every $G \in \mathcal{E}_n$; in fact, one can check that the set E_n is *maximal* in the following sense: if we add to $G \in \mathcal{E}_n$ some edge $e \notin E(W_n) \cup E_n$, then it is not possible to join x, y, z by geodesics contained in CW_n.

Define the class $\mathcal{D}_{3/2}$ as the set of graphs containing as an isometric subgraph some graph isomorphic to one graph in $\mathcal{E}_7 \cup \mathcal{E}_8 \cup \mathcal{E}_9 \cup \mathcal{E}_{10}$.

The definition of $\mathcal{E}_n$ gives that $\delta(G) \geq 3/2$ for every $G \in \mathcal{E}_7 \cup \mathcal{E}_8 \cup \mathcal{E}_9 \cup \mathcal{E}_{10}$. Hence, Lemma 2.8 gives $\delta(G) \geq 3/2$ for every $G \in \mathcal{D}_{3/2}$.

The following result appear in [42].

Lemma 3.1. *In any graph G the inequality*

$$\delta(G) \leq \frac{1}{2} \operatorname{diam} G$$

holds, *and furthermore*, *it is sharp.*

The following result appear in [4] which characterize the hyperbolic graphs G with hyperbolicity constant verifying $\delta(G) = 1$.

If C is any path or cycle in a graph G and $w \in V(C)$, we denote by $\deg_C w$ the degree of the vertex w in the subgraph induced by $V(C)$.

Theorem 3.2. *Let G be a graph. Then* $\delta(G) = 1$ *if and only if the following conditions hold*:

(1) *There exists a cycle isomorphic to* C_4.
(2) *For every cycle* γ *with* $L(\gamma) \geq 5$ *we have* $\deg_\gamma w \geq 3$ *for every vertex* $w \in \gamma$.

PROPOSITION 3.3

Let G be any graph of order n with a vertex v_0 *verifying* $\deg v_0 = n - 1$. *Then*

(1) $G \in \mathcal{D}_0$ *if and only if*

$$\delta(G) = 0\,.$$

(2) $G \in \mathcal{D}_{3/4}$ *if and only if*

$$\delta(G) = \frac{3}{4}\,.$$

(3) $G \in \mathcal{D}_1$ *if and only if*

$$\delta(G) = 1\,.$$

Proof. Items (1), (2) and (3) are a consequence of Theorems 2.10 and 3.2 and Lemma 2.6. □

Theorem 3.4. *Let G be any graph of order n with a vertex* v_0 *verifying* $\deg v_0 = n - 1$. *Then*

(1) $G \in \mathcal{D}_{3/2}$ *if and only if*

$$\delta(G) = \frac{3}{2}\,.$$

(2) $G \notin \mathcal{D}_0 \cup \mathcal{D}_{3/4} \cup \mathcal{D}_1 \cup \mathcal{D}_{3/2}$ *if and only if*

$$\delta(G) = \frac{5}{4}\,.$$

Proof. Since $\deg v_0 = n - 1$, it follows that $\operatorname{diam} V(G) \le 2$, and Lemma 3.1 gives the inequality

$$\delta(G) \le \frac{1}{2} \operatorname{diam} G \le \frac{1}{2}\big(\operatorname{diam} V(G) + 1\big) \le \frac{3}{2}\,.$$

As we have seen, $\delta(G) \ge 3/2$ for every $G \in \mathcal{D}_{3/2}$; hence, $\delta(G) = 3/2$ for every $G \in \mathcal{D}_{3/2}$. By Theorems 2.2 and 2.10 we have either $\delta(G) = 0$, $\delta(G) = 3/4$, $\delta(G) = 1$, $\delta(G) = 5/4$ or $\delta(G) = 3/2$; hence, in order to finish the proof it suffices to show that if $\delta(G) = 3/2$, then $G \in \mathcal{D}_{3/2}$. Assume that $\delta(G) = 3/2$. By Theorem 2.2 there exist a geodesic triangle $T = \{x, y, z\}$ which is a cycle and $p \in [xy]$ such that $d_G(p, [xz] \cup [zy]) = 3/2$ and each point x, y, z is either a vertex in $V(G)$ or a midpoint of some edge in $E(G)$. It is clear that $d_G(p, x), d_G(p, y) \ge 3/2$; thus, $d_G(x, y) \ge 3 = \operatorname{diam} V(G) + 1 \ge \operatorname{diam} G$ and we conclude that $d_G(x, y) = 3 = \operatorname{diam} G$, $\operatorname{diam} V(G) = 2$ and $d_G(p, x) = d_G(p, y) = 3/2$. Therefore, $6 \le L(T) \le 9$.

Since $\operatorname{diam} V(G) = 2$ and $d_G(x, y) = 3$, x and y are midpoints of some edges in $E(G)$. If $v_0 \in [xy]$, then $p = v_0$ and $d_G(p, w) = 1$ for every $w \in V(G) \setminus \{p\}$, which is a contradiction with $d_G(p, [xz] \cup [zy]) = 3/2$ since $L([xz] \cup [zy]) \ge 3$; hence, $v_0 \notin [xy]$. If $v_0 \in [xz] \cup [zy]$, then $3/2 = d_G(p, [xz] \cup [zy]) \le d_G(p, v_0) = 1$, is a contradiction; hence, $v_0 \notin [xz] \cup [zy]$. Therefore, $v_0 \notin T$.

Let us consider the subgraph G_0 obtained by the union of the cycle T with the edges $[v_0, w]$ for every $w \in V(T)$ (G_0 is isomorphic to a wheel graph with $L(T) + 1$ vertices, where $7 \leq L(T) + 1 \leq 10$). Note that since $v_0 \in G_0$, the subgraph G_0^* induced by the vertices of G_0 ($V(G_0) = V(T) \cup \{v_0\}$) is an isometric subgraph of G. Note also that $E(G_0^*) \setminus E(G_0)$ is larger if z is the antipodal point of p in T. Then the maximality of $E_{L(T)+1}$ gives that G_0^* is isomorphic to one graph in $\mathcal{E}_{L(T)+1}$. Hence, $G \in \mathcal{D}_{3/2}$ and the proof is complete. □

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