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A class of degenerate stochastic differential equations with non-Lipschitz coefficients

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Abstract. We obtain sufficient condition for SDEs to evolve in the positive orthant. We use arguments based on comparison theorems for SDEs to achieve this. As an application we prove the existence of a unique strong solution for a class of multidimensional degenerate SDEs with non-Lipschitz diffusion coefficients.

Keywords. Degenerate SDEs; non-Lipschitz coefficients; comparison theorems.

1. Introduction

In this article we consider (possibly degenerate) stochastic differential equations (SDEs) with non-Lipschitz coefficients. If the coefficients are Lipschitz, we can prove the existence of a unique strong solution (see [9]). But uniqueness fails in the case of non-Lipschitz coefficients. The literature on this topic is not exhaustive. However, in one-dimensional case, there is an extensive literature (see [3]).

Now we briefly recall the relevant literature. If the coefficients are continuous, then weak solution exists up to explosion time (see pp. 155–163 of [9]). Under additional linear growth conditions, weak solution exists for all time. In the case of non-degenerate diffusion coefficient, Stroock and Varadhan proved the existence of a unique weak solution for SDE with bounded, measurable drift and bounded, continuous diffusion coefficients (see Theorem 4.2, 5.6 of [11]). Krylov [8] relaxed the continuity assumption on diffusion coefficient and proved that there is a unique weak solution for $n \leq 2$, and for $n > 2$, SDE has a weak solution under the additional nondegeneracy condition. Engelbert and Schmidt studied SDE in one-dimensional case and formulated necessary and sufficient conditions to prove the existence of unique weak solution (see [3] for details).

The existence of a unique strong solution is known when the coefficients are locally Lipschitz continuous with linear growth. If the drift coefficient is bounded, measurable and diffusion coefficient is Lipschitz continuous and non-degenerate, then existence of a unique strong solution is known (see [16] for the one-dimensional case and [13] for the multidimensional case). In [16] Zvonkin proved the result if the diffusion coefficient is Hölder continuous with exponent $\frac{1}{2}$.

To the best of our knowledge, the above result of Zvonkin is not known in the multidimensional case. In fact, there exists examples of SDEs evolving in $\mathbb{R}^n$, $n \geq 3$ with zero drift and Hölder continuous diffusion coefficient possessing multiple solutions.

However there are some partial results. Swart [12] proved the pathwise uniqueness of an SDE evolving in unit ball of $\mathbb{R}^n$ with diffusion coefficient locally Lipschitz in the interior of the unit ball and Hölder continuous with exponent $\frac{1}{2}$ on the boundary and some special drift coefficient. Fang and Zhang [5] studied the existence of unique strong solution in the case when the Lipschitz assumption on the coefficients is relaxed by a logarithmic factor. This still does not give results for the Hölder continuous class, analogous to the one-dimensional case of Zvonkin. The technique used in [5] does not seem to work for Hölder continuous coefficients.

In this article we attempt to study SDE with Hölder continuous coefficients. Our approach is to use comparison type arguments. To the best of our knowledge, this approach is not used to prove the existence of strong solutions in multi-dimensional case. We now briefly discuss the content of our article. We consider the SDE

$$\begin{cases} \mathrm{d}X(t) &= \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \\ X(0) &= x \in \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i > 0,\ \forall i\}$, the positive orthant. Using comparison technique, we obtain sufficient condition for any solution $X(\cdot)$ of (1.1) to evolve in the positive orthant, $\mathbb{R}^n_+$. As an example we establish the existence of a unique strong solution to the SDE

$$\begin{cases} \mathrm{d}X_i(t) &= \mu_i(X(t))\,\mathrm{d}t + \sqrt{|X_i(t)|}\displaystyle\sum_{j=1}^{n} \sigma_{ij}(X(t))\,\mathrm{d}W_j(t), \\ X(0) &= x \in \mathbb{R}^n_+,\ i = 1, \ldots, n\,. \end{cases} \tag{1.2}$$

The SDE (1.2) is studied in [1,2] in connection with super-Markov chains. The existence of a unique weak solution is established under certain conditions and the existence of a unique strong solution is left as an open problem. In this article, we give a partial solution to this open problem.

Equation (1.2) can also be seen as a multidimensional version of CIR model in mathematical finance. The solution of SDE (1.1) or in particular (1.2) remaining in positive orthant have an important implication in finance. In [4], the authors studied the positivity of a class of one-dimensional SDEs and obtained the nonarbitrage nature of the SDE. Typically the price movements of assets (particularly equities) are modeled by certain SDEs. The solutions of such SDEs must evolve in the positive orthant. Indeed if the asset prices are allowed to hit zero, the model creates arbitrage possibility. It may be emphasized that in the mathematical finance literature, effectively there exists only one class of models which captures the above positivity phenomenon. It is the multidimensional analog of the Black–Scholes model and its nonlinear versions. Note that for modeling short rates, there exists a variant of Bessel processes [6], the so-called CIR model for short term interest rates. Multidimensional version of CIR model has not been studied in the mathematical finance literature. A possible reason may be the non-availability of a sufficient condition that ensures the evolution of the process given by the multidimensional CIR model in the positive orthant.

We further illustrate that our comparison theorem based technique can be used for other classes of SDEs. More precisely we consider the SDE

$$\begin{cases} \mathrm{d}X(t) &= c(\theta - X(t))\mathrm{d}t + \sqrt{2(1 - \|X(t)\|^2)}\,\mathrm{d}W(t), \\ X(0) &= x \in B(0,\ 1) \subseteq \mathbb{R}^n\,, \end{cases} \tag{1.3}$$

where $B(0, 1) = \{x \in \mathbb{R}^n : \|x\| < 1\}$. We show that under the assumption $c(1 - \sqrt{n}\,|\theta|) \geq 2$, the SDE (1.3) has a unique strong solution evolving in $B(0, 1)$. The SDE (1.3) is studied by Swart in [12] for the case $\theta = 0$. The method used in [12] does not seem to work for $\theta \neq 0$.

Our paper is organized as follows: In § 2, we prove the existence of a unique strong solution to the SDE (2.1) in one dimension under the assumption (A1). The results in § 2 is obtained by closely mimicking the classical technique of Yamada and Watanabe [14]. Section 3 contains a study of SDE in positive orthant. We apply our main result in § 3 to study a multidimensional variant of Bessel process in § 4. Finally we apply our technique developed in § 3 to study an SDE in the unit ball in § 5.

2. An auxiliary result

In this section, we prove a minor generalization of the result in [14] suited for our purpose. Consider the (possibly) degenerate one-dimensional SDE

$$\begin{cases} \mathrm{d}X(t) = \mu(X(t))\mathrm{d}t + \sigma(X(t)\,\mathrm{d}W(t)), \\ X(0) = x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $\mu : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$ and $W(\cdot)$ is an $\mathbb{R}$-valued Wiener process. We assume that

(A1)

(i) The function μ is locally Lipschitz continuous on $\mathbb{R}$.
(ii) There exists $\epsilon > 0$, a strictly increasing function $\rho : [0, \infty) \to \mathbb{R}$ satisfying $\rho(0) = 0$ and $\int_{0+} \frac{1}{\rho^2(s)}\,\mathrm{d}s = \infty$ such that

$$|\sigma(x) - \sigma(y)| \leq C_R\,\rho(|x - y|), \tag{2.2}$$

for some $C_R > 0$ and for all $x, y \in B(0, R)$ with $|x - y| \leq \epsilon$.
(iii) The functions μ and σ have linear growth.

The only essential difference is that in [14], the condition (2.2) is assumed for all $x, y \in \mathbb{R}$. We only assume (2.2) locally.

Following pp. 182–183 of [9], we introduce some notation. Define $\phi_k : \mathbb{R} \to \mathbb{R}$ as follows:

Let $1 = s_0 > s_1 > s_2 > \cdots > s_k > \cdots > 0$ be such that

$$\int_{s_k}^{s_{k-1}} \frac{1}{\rho^2(t)}\,\mathrm{d}t = k, \quad k \geq 1\,.$$

Let ψ_k be continuous functions with $\mathrm{supp}(\psi_k) \subseteq (s_k, s_{k-1})$, $k \geq 1$, and

$$0 \leq \psi_k(s) \leq \frac{2}{\rho^2(s)k}, \quad \int_{s_k}^{s_{k-1}} \psi_k(s)\,\mathrm{d}s = 1.$$

Set

$$\phi_k(t) = \int_0^{|t|} \int_0^s \psi_k(\theta)\, d\theta\, ds \quad \forall\, t \in \mathbb{R}\,.$$

Then ϕ_k satisfies

(i) $\phi_k \in C^2(\mathbb{R})$
(ii) $0 \le \phi_k^{'}(t) \le 1$
(iii) $0 \le \phi_k^{''}(t) \le \frac{2}{k\rho^2(|t|)}, \quad t \in \mathbb{R}, t \neq 0$
(iv) $\phi_k(t) \uparrow |t|$ as $k \to \infty$.

Our first result in this section is the following pathwise uniqueness result.

Lemma 2.1. *Assume* (A1). *Then pathwise uniqueness holds for* (2.1).

Proof. Let $X_i(\cdot)$, $i = 1, 2$ be two solutions of (2.1) with a prescribed Wiener process $W(\cdot)$. Set

$$X(t) = X_1(t) - X_2(t), \quad t \ge 0\,.$$

Then $X(\cdot)$ satisfies

$$dX(t) = (\mu(X_1(t)) - \mu(X_2(t)))\, dt + (\sigma(X_1(t)) - \sigma(X_2(t)))\, dW(t)\,.$$

Define

$$\tau_R = \inf\{t > 0 \mid |X(t)| \ge R \text{ or } |X(t)| \ge \epsilon\},$$

and

$$\tau = \inf\{t > 0 \mid \|X(t)\| \ge \epsilon\},$$

where $\epsilon > 0$ is from (A1)(ii). Observe that $\tau_R \to \tau$ as $R \to \infty$, since linear growth conditions of the coefficients of (2.1) guarantees the nonexplosion of any solution of (2.1) (see Theorem 2.4, pp. 163–164 of [9]). Using Ito–Dynkin's formula, we get

$$\begin{aligned} &E\phi_k(X(t \wedge \tau_R)) \\ &\quad = E\Big[\int_0^{t\wedge\tau_R} (\mu(X_1(s)) - \mu(X_2(s)))\, \phi_k^{'}(X(s))\, ds\Big] \\ &\qquad + \frac{1}{2} E\Big[\int_0^{t\wedge\tau_R} [\sigma(X_1(s)) - \sigma(X_2(s))]^2\, \phi_k^{''}(X(s))\, ds\Big] \\ &\quad \le K_R\, E\Big[\int_0^{t\wedge\tau_R} |X(s)|\, ds\Big] + C_R^2 E\Big[\int_0^{t\wedge\tau_R} \rho^2(|X(s)|)\phi_k^{''}(X(s))\, ds\Big], \end{aligned} \tag{2.3}$$

where $K_R > 0$ is the Lipschitz constant of μ in $B(0, R)$. Hence

$$E\phi_k(X(t \wedge \tau_R)) \le K_R\, E\Big[\int_0^{t\wedge\tau_R} |X(s)|\, ds\Big] + \frac{C_R^2\, t}{k}\,. \tag{2.4}$$

Let $k \to \infty$ in (2.4). We have

$$\begin{aligned} E|X(t \wedge \tau_R)| &\leq K_R\, E\Big[\int_0^{t\wedge\tau_R} |X(s)|\, \mathrm{d}s \Big] \\ &= K_R\, E\Big[\int_0^{t\wedge\tau_R} |X(s\wedge\tau_R)|\, \mathrm{d}s \Big] \\ &\leq K_R \int_0^t E|X(s\wedge\tau_R)|\, \mathrm{d}s\,. \end{aligned}$$

Using Gronwall's lemma, we now have

$$E|X(t\wedge\tau_R)| = 0, \quad \text{for all } t\,.$$

Letting $R \to \infty$ and using Fatou's lemma, we get

$$X(t\wedge\tau) = 0 \quad \text{for all } t. \tag{2.5}$$

To complete the proof, it is enough to show that $\tau = \infty$ a.s. If $P\{\tau < \infty\} > 0$, then there exists $T > 0$ such that $P\{\tau \leq T\} > 0$. Hence it follows from (2.5) that on $\{\tau \leq T\}$, $X(\tau) = 0$ which is false in view of the definition of τ. Hence $P\{\tau < \infty\} = 0$. This completes the proof. □

We now prove the main result of this section.

Theorem 2.1. *Under the Assumption* (A1), *the SDE* (2.1) *has a unique strong solution for all time.*

Proof. In view of Yamada–Watanabe theorem on pathwise uniqueness ([14], pp. 309–310 of [7]) to show the existence of a unique strong solution, it is sufficient to show the pathwise uniqueness and existence of a weak solution. Lemma 2.1 guarantees the pathwise uniqueness. Since the drift and the diffusion coefficients are continuous and have linear growth, the SDE (2.1) has a weak solution for all time (see Theorem 2.3, Theorem 2.4, pp. 159–164 of [9]). This completes the proof of the theorem. □

3. SDE in positive orthant

In this section, we study the SDE (2.1) in the positive orthant. It is evident from [15] that in general pathwise uniqueness of (1.1) is not true under the multidimensional version of (A1). Hence in particular, it is not sure whether one can prove pathwise uniqueness for the SDE (1.2) using the Lipschitz continuous assumption on σ because of the factor x_i^{β}. So we need to assume extra condition on the coefficients. So in this section we propose a sufficient condition for pathwise uniqueness for equations evolving in the orthant. Our method is to propose certain assumptions, which ensures that any solution of the SDE with initial condition in the positive orthant remains in the positive orthant for all time.

We introduce some notation before stating the assumptions.

Let $\rho_1 : [0, \infty) \to \mathbb{R}$ be a strictly increasing function satisfying $\rho_1(0) = 0$ and $\int_{0+} \frac{1}{\rho_1^2(s)}\, \mathrm{d}s = \infty$.

Define $p_i : \mathbb{R}^n_+ \to \mathbb{R}$, $a_i : \mathbb{R}^n_+ \to \mathbb{R}$ and $b_i : \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, \ldots, n$ by

$$p_i(x) = x_i,\ a_i(x) = m_{ii}(x),\ \ b_i(x) = \frac{\mu_i(x)}{a_i(x)},$$

where $(m_{ij}(x)) = \sigma(x)\sigma(x)^{\perp}$. Also define

$$a_i^+(r) = \sup_{x: p_i(x)=r} a_i(x),\ b_i^-(r) = \inf_{x: p_i(x)=r} b_i(x).$$

We assume the following:

(A2) The functions μ, σ are continuous with linear growth. Also $a_i^+(r) < \infty$ for all $r > 0$.

(A3)

(i) The function $\mu_i(x) > 0$ for all x in a neighborhood of $\partial\mathbb{R}^n_+$, say $\mathcal{D}$, $i = 1, 2, \ldots, n$.
(ii) The functions b_i^-, $i = 1, 2, \ldots, n$ satisfies

$$b_i^-(r) > \frac{1}{r} \quad \text{for all } r > 0.$$

(iii) There exists a function $\tilde{\sigma} : [0, \infty) \to \mathbb{R}$ such that

$$m_{ii}(x) \le n^3\tilde{\sigma}^2(p_i(x)) \quad \text{for all } x \in \mathcal{D},\ i = 1, 2, \ldots, n,$$

where $\tilde{\sigma}$ satisfies the following: there exists an $\epsilon > 0$ and a $C_R^1 > 0$ which depends on R such that for $r_1, r_2 \in [0, R]$ with $|r_1 - r_2| \le \epsilon$,

$$|\tilde{\sigma}^2(r_1) - \tilde{\sigma}^2(r_2)| \le C_R^1 |r_1 - r_2|,$$
$$|\sqrt{r_1}\tilde{\sigma}(r_1) - \sqrt{r_2}\tilde{\sigma}(r_2)| \le C_R^1 \rho_1(|r_1 - r_2|).$$

Example 3.1 When $\sigma_{ij}(x) = \sqrt{|x_i|}\,\hat{\sigma}_{ij}(x)$, $x \in \mathbb{R}^n$, then the assumption (A3)(ii) reads as

$$\inf_{x: x_i = r} \frac{\mu_i(x)}{\sum_{j=1}^n \hat{\sigma}_{ij}^2(x)} \ge 1,\ i = 1, \ldots, n.$$

A sufficient condition for the above is

$$\mu_i(x) \ge \sum_{j=1}^n \hat{\sigma}_{ij}^2(x),\ i = 1, \ldots, n,\ x \in \mathbb{R}^d.$$

We are now ready to state and prove the main theorem of our article.

Theorem 3.1. *Assume* (A2) *and* (A3). *Let* $X(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))$ *be a solution to* (2.1) *with the initial condition* $X_i(0) = x_i > 0$, $i = 1, 2, \ldots, n$. *Then with probability* 1, *paths of* $X_i(\cdot)$ *are positive*, $i = 1, 2, \ldots, n$.

Proof. Fix i. Consider the process $\hat{X}(\cdot)$ satisfying

$$\mathrm{d}\hat{X}(t) = \mu(\hat{X}(t))\,\mathrm{d}t + \hat{\sigma}(\hat{X}(t))\,\mathrm{d}W(t),$$

where $\hat{\sigma}$ satisfies

$$\hat{\sigma}(x) = \begin{cases} \sigma(x), & \text{if } x \in \mathcal{D} \\ \tilde{\sigma}(p_i(x)), & \text{if } x \in \mathcal{D}_1^c \end{cases}$$

for a neighborhood $\mathcal{D}_1$ of $\partial\mathbb{R}^n_+$ such that $\mathcal{D} \subseteq \mathcal{D}_1$ and $\hat{\sigma}$ satisfies

$$\sum_{k=1}^{n} \hat{\sigma}_{ik}(x)\,\hat{\sigma}_{ki}(x) \le n^3\,\tilde{\sigma}^2(p_i(x)) \text{ for } x \in \mathcal{D}_1 \setminus \mathcal{D}\,.$$

Such a $\hat{\sigma}$ can be constructed as follows: Using Urysohn's lemma, there exists a continuous function $f : \mathbb{R}^n \to [0,\ 1]$ such that $f(x) = 0$ on $\overline{\mathcal{D}}$ and $f(x) = 1$ on $\mathcal{D}_1^c$. Now choose

$$\hat{\sigma}(x) = (1 - f(x))\sigma(x) + f(x)\tilde{\sigma}(p_i(x)), \quad x \in \mathbb{R}^n\,.$$

Note that the process $\hat{X}(\cdot)$ given above and the process $X(\cdot)$ given by (2.1) behave identically in some neighborhood of $\partial\mathbb{R}^n_+$. Hence without the loss of generality we can assume that $\sigma = \hat{\sigma}$.

Set

$$\tau = \inf\{t \ge 0 \mid X(s) \in \partial\mathbb{R}^n_+\}$$

and define

$$\phi(t) = \int_0^{t\wedge\tau} \frac{a_i(X(s))}{a_i^+(p_i(X(s)))}\mathrm{d}s\,.$$

Then ϕ is pathwise smooth, strictly increasing and $\phi(0) = 0$. Let $\psi = \phi^{-1}$. Set $Y_i(t) = X_i(\psi(t))$. Using the time change of Brownian motion arguments as in pp. 183–185 of [9], we can show that $Y(\cdot) := (Y_1(\cdot), \ldots, Y_n(\cdot))$ satisfies

$$\begin{cases} \mathrm{d}Y_i(t) = \dfrac{a_i^+(p_i(Y(t)))}{a_i(Y(t))}\mu_i(Y(t))\mathrm{d}t \\ \qquad + \sqrt{\dfrac{a_i^+(p_i(Y(t)))}{a_i(Y(t))}} \displaystyle\sum_{j=1}^{n} \sigma_{ij}(Y(t))\mathrm{d}\bar{W}_j(t) \end{cases} \tag{3.1}$$

for some Brownian motion $\bar{W}(\cdot)$.

Using Ito's formula to $p_i(\cdot)$ and the process (3.1), we obtain

$$\begin{cases} \mathrm{d}p_i(Y(t)) = \dfrac{a_i^+(p_i(Y(t)))}{a_i(Y(t))}\mu_i(Y(t))\mathrm{d}t \\ \qquad + \sqrt{\dfrac{a_i^+(p_i(Y(t)))}{a_i(Y(t))}} \displaystyle\sum_{j=1}^{n} \sigma_{ij}(Y(t))\mathrm{d}\bar{W}_j(t)\,, \end{cases}$$

i.e.

$$\mathrm{d}p_i(Y(t)) = a_i^+(p_i(Y(t)))b_i(Y(t))\mathrm{d}t + \sqrt{a_i^+(p_i(Y(t)))}\mathrm{d}\tilde{W}(t), \tag{3.2}$$

for some one-dimensional Brownian motion $\tilde{W}(\cdot)$.

Now consider the following SDEs:

$$dZ(t) = \frac{a_i^+(Z(t))}{Z(t)}dt + \sqrt{a_i^+(Z(t))}d\tilde{W}(t) \tag{3.3}$$

and

$$dZ_1(t) = 3\,a_i^+(\sqrt{Z_1(t)})\,dt \;+\; 2\sqrt{Z_1(t)}\sqrt{a_i^+(\sqrt{Z_1(t)})}\,d\tilde{W}(t). \tag{3.4}$$

By applying Ito's formula we can see that $Z(\cdot)$ is a solution to (3.3) iff $Z_1(\cdot)$ is a solution to (3.4).

Using (A3)(iii), we have

$$a_i^+(r) \;=\; n^3\tilde{\sigma}^2(r), \qquad r > 0\,.$$

Observe that equation (3.4) satisfies all the assumptions in Theorem 2.1. Thus (3.4) has a unique strong solution. As a consequence, equation (3.3) will also have a unique strong solution for all time.

Now take

$$b^1(r) \;=\; a_i^+(r)\,b_i^-(r),\;\; b_i^2(r) \;=\; \frac{a_i^+(r)}{r}\,.$$

Then

$$b_i^1(r) \;>\; b_i^2(r), \qquad \text{for all } r > 0$$

and

$$a^+(p_i(Y(t))\,b_i(Y(t)) \;\geq\; b_i^1(p_i(Y(t)))\,.$$

Note that $b_i^1(0)$ may or may not be equal to $b_i^2(0)$. If $b_i^1(0) > b_i^2(0)$, then by a direct application of comparison theorem (see p. 437 of [9]), we have

$$Z(t) \;\leq\; p_i(Y(t)) \text{ a.s.} \tag{3.5}$$

If $b_i^1(0) = b_i^2(0)$, then apply comparison theorem up to the stopping time $\eta_n = \inf\{t > 0 \mid p_i(Y(t)) \leq \frac{1}{n}$ or $Z(t) \leq \frac{1}{n}\}$ and let $n \to \infty$. We have (3.5).

Set $M(t) = \log Z(t)$. Then Ito's formula implies that

$$dM(t) = \frac{1}{Z(t)}\sqrt{a^+(Z(t))}d\tilde{W}(t)\,,$$

i.e. for $\eta = \sup\{s \geq 0 || M(s)| < \infty\}$, $M(t) = \log Z(t)$ defines a local Martingale in $[0, \eta)$. Now using Corollary 34.13, p. 67 of [10], on $\{\eta < \infty\}$ we have

$$\overline{\lim}_{t\uparrow\eta} M(t) = \infty \text{ a.s.},$$

i.e.

$$\overline{\lim}_{t\uparrow\eta} Z(t) = \infty \text{ a.s.}$$

Now since (3.3) has a unique strong solution for all $t \geq 0$, $\{\eta < \infty\}$ has probability 0. That is $Z(t) > 0$ a.s., for all $t \geq 0$. Hence from (3.5) it follows that $Y_i(\cdot) > 0$ a.s. □

4. Multidimensional variant of Bessel process

In this section we prove the existence of unique strong solution to the SDE

$$\begin{cases} dX_i(t) &= \mu_i(X(t))\,dt + \sqrt{|X_i(t)|}\sum_{j=1}^{n}\sigma_{ij}(X(t))\,dW_j(t) \\ X(0) &= x \in \mathbb{R}^n_+ . \end{cases} \tag{4.1}$$

The SDE (4.1) can be seen as the multidimensional variant of the Bessel process given by

$$dX(t) = c\,dt + 2\sqrt{|X(t)|}dW(t), \qquad X(0) = x \geq 0 .$$

Positive diffusion processes play a central role in mathematical finance due to the arbitrage-free nature of it, see for example [4]. Equation (4.1) is studied in [1,2] in connection with super-Markov chains. Pathwise uniqueness is posed as an open problem in [1]. Pathwise uniqueness results for SDE with non-Lipschitz coefficients are very limited in the multidimensional case. Fang and Zhang [5] provided pathwise uniqueness results for a general class of SDEs. But these results can not be applied for SDEs whose coefficients are Hölder continuous. Thus (4.1) can not be studied using the results in [5]. Our objective in this section is to study the pathwise uniqueness of (4.1). We first prove a general theorem.

Theorem 4.1. *Assume that μ is locally Lipschitz in $\mathbb{R}^n_+$ with linear growth and satisfies* (A3)(i), (ii). *Also assume that σ is locally Lipschitz in $\mathbb{R}^n_+$ with linear growth and satisfies* (A3)(iii). *Then the SDE* (2.1) *has a unique strong solution with paths evolving in $\mathbb{R}^n_+$.*

Proof. Since the coefficients of the SDE (2.1) are locally Lipschitz continuous with linear growth, using classical result of Ito, existence of unique weak solution for all time and pathwise uniqueness up to hitting time $\tau = \inf\{t > 0 \mid X(t) \in \partial\mathbb{R}^n_+\}$ follows. Now using Theorem 3.1, we have $\tau = \infty$ a.s. Hence the SDE (2.1) has pathwise uniqueness for all time. Using Yamada–Watanabe theorem [14], we can conclude the existence of unique strong solution to (2.1) for all time. □

We now study (4.1) and prove the existence and uniqueness in the following theorem.

Theorem 4.2. *Assume that μ is locally Lipschitz with linear growth, μ is positive in a neighborhood of $\partial\mathbb{R}^n_+$, σ is bounded and locally Lipschitz and satisfies*

$$\mu_i(x) \geq \sum_{i=1}^{n}\sigma^2_{ij}(x),\ i = 1, \ldots, n, \quad x \in \mathbb{R}^d .$$

Then the SDE (4.1) *has a unique positive strong solution.*

Proof. In view of Theorem 4.1, it remains to show that the coefficients of the SDE (4.1) satisfies (A3)(i), (ii) and (iii). Since σ is bounded, locally Lipschitz continuous, we have

$$\begin{aligned} m_{ii}(x) &= \sum_{j=1}^{n} x_i\,\sigma^2_{ij}(x) \\ &\leq n\,K\,x_i = n^3\tfrac{K}{n^2}\,p_i(x), \end{aligned}$$

where $K > 0$ is a bound for σ. Hence σ satisfies (A3)(iii). Also from Example 3.1, it follows that the conditions (A3)(i), (ii) holds. This completes the proof. □

5. SDE in unit ball

Our aim in this section is to show that the method developed in § 3 can be adapted to study equations evolving in domains other than positive orthant. We restrict our attention to a variant of an SDE in unit ball studied by Swart [12].

Consider the SDE

$$\begin{cases} \mathrm{d}X(t) &= c(\theta - X(t))\mathrm{d}t + \sqrt{2(1-\|X(t)\|^2)}\,\mathrm{d}W(t), \\ X(0) &= x \in B(0,1) \subseteq \mathbb{R}^n . \end{cases} \tag{5.1}$$

The SDE (5.1) is studied in [12] with $\theta = 0$. In [12], the author showed the existence of a unique strong solution for $\theta = 0$ and $c \geq 1$.

We study equation (5.1) for any θ with the assumption that $c(1 - \sqrt{n}\,|\theta|) \geq 2$.

We now prove our result in the following theorem.

Theorem 5.1. *Assume $c(1 - \sqrt{n}\,|\theta|) \geq 2$. Then the SDE (5.1) has a unique strong solution. Moreover solution evolves in $B(0,1)$.*

Proof. In view of the arguments in the proof of Theorem 4.1, it suffices to show that solutions to (5.1) with initial condition in $B(0,1)$ remains in $B(0,1)$.

Let $X(\cdot)$ be a solution to (5.1) corresponding to the Wiener process $W(\cdot)$. We can assume without loss of generality that $X(0) = x \neq 0$. Define

$$p(x) = \|x\|^2, \; a(x) = 8\,p(x)\,(1 - p(x))$$

and

$$b(x) = \frac{2[n - (n+c)p(x)] + 2c\,\theta \sum_{i=1}^n x_i}{a(x)}, \quad x \in \mathbb{R}^n .$$

Set

$$a^+(r) = \sup_{p(x)=r} a(x), \; b^+(r) = \sup_{p(x)=r} b(x), \qquad r > 0 .$$

Then

$$a^+(r) = 8r(1-r), \; b^+(r) = \frac{2[n - (n + c(1 - \sqrt{n}\,|\theta|))r]}{a^+(r)} .$$

Let $\tau = \inf\{t \geq 0 \,|\, \|X(t)\| \geq 1\}$ and

$$\phi(t) = \int_0^{t\wedge\tau} \frac{a(X(s))}{a^+(p(X(s)))}\,\mathrm{d}s .$$

Now mimicking the arguments from Theorem 3.1, there exists one-dimensional Wiener process $\tilde{W}(\cdot)$ such that $Y(\cdot)$ given by $Y(t) = X(\psi(t))$, $\psi = \phi^{-1}$ satisfies

$$\begin{cases} \mathrm{d}p(Y(t)) = a^{+}(p(Y(t)))\, b(Y(t))\mathrm{d}t + \sqrt{8\, p(Y(t))(1-p(Y(t)))}\mathrm{d}\tilde{W}(t), \\ p(Y(0)) = p(x)\,. \end{cases} \tag{5.2}$$

Consider the SDE

$$\begin{cases} \mathrm{d}Z(t) = 2[n-(n+c(1-\sqrt{n}\,|\theta|))Z(t)]\mathrm{d}t + \sqrt{8Z(t)(1-Z(t))}\mathrm{d}\tilde{W}(t), \\ Z(0) = p(x) \in (0,\ 1)\,. \end{cases} \tag{5.3}$$

Since $\sqrt{8x(1-x)}$ is Hölder continuous with exponent $\frac{1}{2}$, the SDE (5.3) has a unique strong solution. Also note that for $n \geq 2$,

$$\int_{\frac{1}{2}}^{x} \mathrm{e}^{-2\int_{\frac{1}{2}}^{y} \frac{2[n-(n+c(1-\sqrt{n}|\theta|))u]}{8u(1-u)}\mathrm{d}u}\mathrm{d}y = 2^{\frac{c(\sqrt{n}|\theta|-1)-n}{2}} \int_{\frac{1}{2}}^{x} y^{-\frac{n}{2}}(1-y)^{-\frac{c(1-\sqrt{n}|\theta|)}{2}}\mathrm{d}y. \\ = \infty\,.$$

By Proposition 5.22, p. 345 of [7], we have $0 < Z(t) < 1$ a.s. for t. Now note that

$$a^{+}(p(y))b(y) \leq a^{+}(p(y))\, b^{+}(p(y)) = 2[n-(n+c(1-\sqrt{n}|\theta|))p(y)]\,.$$

Hence using comparison theorem (p. 437 of [9]), we have

$$p(Y(t)) \leq Z(t) < 1 \text{ a.s.} \quad \forall\, t\,, \tag{5.4}$$

i.e $\|Y(t)\| < 1$ a.s. for all t. Hence the process $X(\cdot)$ never hits the boundary of $B(0,\ 1)$. □

Remark 5.1. From (5.4), the process $X(\cdot)$ given by SDE (5.1) can be in boundary only when the process $Z(\cdot)$ given by (5.3) is at 1. Hence by looking at the behavior of $Z(\cdot)$ at 1, we can get a sufficient condition for the process $X(\cdot)$ spending zero time on the boundary. Now we can apply the argument of [12] to show the pathwise uniqueness of (5.1) under the case when (5.1) spends zero time on the boundary.

6. Conclusion

In this article, we prove the existence of a unique strong solution for a class of multidimensional SDEs with non-Lipschitz diffusion coefficients. The analogous result for the one-dimensional case was known for a long time but the multidimensional version was not available. We prove the result under the assumptions (A2) and (A3). The proofs of our results are based on comparison theorem arguments. We believe that it is for the first time that comparison theorems are exploited to derive such results.

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