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Outgoing Cuntz scattering system for a coisometric lifting and transfer function

KALPESH J HARIA

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India
E-mail: kalpesh@math.iitb.ac.in

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Abstract. We study a coisometry that intertwines Popescu's presentations of minimal isometric dilations of a given operator tuple and of a coisometric lifting of the tuple. Using this we develop an outgoing Cuntz scattering system which gives rise to an input–output formalism. A transfer function is introduced for the system. We also compare the transfer function and the characteristic function for the associated lifting.

Keywords. Multivariate operator theory; row contraction; contractive lifting; outgoing Cuntz scattering system; transfer function; multi-analytic operator; input–output formalism; linear system; characteristic function.

1. Introduction

The model of repeated interaction between quantum systems has been recently studied in [6] and an outgoing Cuntz scattering system was associated to the model. In [4] the authors gave a vast generalization of Gohm's repeated interaction model using the theory of liftings of row contractions. We recall that in the generalized repeated interaction model of [4] we have the following operator theoretic data:

Let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces such that $\tilde{\mathcal{H}}$ be a subspace of $\mathcal{H}$ and $\Omega^{\mathcal{K}}$ be a distinguished unit vector of $\mathcal{K}$. Let $\mathcal{P}$ be a d-dimensional Hilbert space with an orthonormal basis $\{\epsilon_j\}_{j=1}^{d}$, and $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ and $\tilde{U} : \tilde{\mathcal{H}} \otimes \mathcal{K} \to \tilde{\mathcal{H}} \otimes \mathcal{P}$ be two unitaries such that

$$U(\tilde{h} \otimes \Omega^{\mathcal{K}}) = \tilde{U}(\tilde{h} \otimes \Omega^{\mathcal{K}}) \text{ for each } \tilde{h} \in \tilde{\mathcal{H}}. \tag{1.1}$$

The unitary $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ can be decomposed as

$$U(h \otimes \Omega^{\mathcal{K}}) = \sum_{j=1}^{d} E_j^* h \otimes \epsilon_j \text{ for each } h \in \mathcal{H},$$

where E_j's are some operators in $\mathcal{B}(\mathcal{H})$, for $j = 1, \ldots, d$. Likewise there exist some operators C_j's in $\mathcal{B}(\tilde{\mathcal{H}})$ such that

$$\tilde{U}(\tilde{h} \otimes \Omega^{\mathcal{K}}) = \sum_{j=1}^{d} C_j^* \tilde{h} \otimes \epsilon_j \text{ for each } \tilde{h} \in \tilde{\mathcal{H}}.$$

Observe that $\sum_{j=1}^{d} E_j E_j^* = I$ and $\sum_{j=1}^{d} C_j C_j^* = I$, i.e., $\underline{E}$ and $\underline{C}$ are coisometric tuples. Then equation (1.1) yields that $E_j^*\tilde{h} = C_j^*\tilde{h}$ for each $\tilde{h} \in \tilde{\mathcal{H}}$, i.e., the tuple $\underline{E}$ is a *lifting* (cf. [3]) of the tuple $\underline{C}$. In this article we proceed in the opposite way, viz. we start with coisometric d-tuples $\underline{E} = (E_1, \ldots, E_d)$ and $\underline{C} = (C_1, \ldots, C_d)$ such that the tuple $\underline{E}$ is a lifting of the tuple $\underline{C}$, then associate a pair of unitaries to it but satisfying similar relations as above and then study the model.

The article is organized as follows: After associating a pair of unitaries to a given coisometric lifting as stated above, we identify a coisometry which intertwines corresponding minimal isometric dilations in § 2. We investigate the properties of this coisometry and express it as a limit of certain compositions of these unitaries. The Cuntz scattering system was introduced in [2] using the generalization of Lax–Phillips scattering system to a multivariate operator setting. In § 3 we study the forward part of a Cuntz scattering system which is called as an outgoing Cuntz scattering system in [2]. Let $\tilde{\Lambda}$ denote the free semigroup with generators $1, \ldots, d$. Using an input–output formalism we define a colligation of operators [2] which gives rise to a $\tilde{\Lambda}$-linear system $\sum_{U,\tilde{U}}$. An application of the generalized Fourier transform to the $\tilde{\Lambda}$-linear system $\sum_{U,\tilde{U}}$ under zero initial condition leads to the input–output relation

$$\hat{y}(z) = \Theta_{U,\tilde{U}}\hat{u}(z)$$

between the Fourier transforms of input and output variables where $\Theta_{U,\tilde{U}}$ is the transfer function of the system. These transfer functions are multi-analytic operators. There are other approaches to transfer functions in [12] and [5].

Popescu introduced the characteristic function in [8] (cf. [11]) of a row contraction, and systematically developed an extensive theory of row contractions (cf. [9,10]). We use some of the concepts from Popescu's theory in this work. In [3], Dey and Gohm described a class of multi-analytic operators which classify certain class of liftings and called them characteristic functions for liftings. We find a relation between the transfer function and the characteristic function for the associated lifting.

The following *multi-index notation* will be used frequently in this article. Suppose $T_1, \ldots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$. If $\alpha \in \tilde{\Lambda}$ is a word $\alpha_1 \ldots \alpha_n$ with length $|\alpha| = n$ where each $\alpha_j \in \{1, \ldots, d\}$, then T_α denotes $T_{\alpha_1} \ldots T_{\alpha_n}$. For the empty word $\emptyset$ we define $|\emptyset| = 0$ and $T_\emptyset = I$. The full Fock space over $\mathbb{C}^d$ denoted by Γ is the Hilbert space

$$\Gamma := \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes^2} \oplus \cdots \oplus (\mathbb{C}^d)^{\otimes^m} \oplus \cdots.$$

The element $1 \oplus 0 \oplus \cdots$ of Γ is called the vacuum vector. Let $\{e_1, \ldots, e_d\}$ be the standard orthonormal basis of $\mathbb{C}^d$. For $\alpha = \alpha_1 \ldots \alpha_m \in \tilde{\Lambda}$, e_α will denote the vector $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$ in the full Fock space Γ and $e_\emptyset$ will denote the vacuum vector. $\{e_\alpha : \alpha \in \tilde{\Lambda}\}$ forms an orthonormal basis of the full Fock space.

2. Unitaries associated to coisometric lifting

In this section we construct a coisometry associated to coisometric lifting which plays an important role in this article. For simplicity we assume that d is finite throughout this article but all the results here can be derived also for $d = \infty$. A tuple $\underline{T} = (T_1, \ldots, T_d)$ of bounded linear operators on a Hilbert space $\mathcal{L}$ is said to be a *row contraction* if $\sum_{j=1}^{d} T_j T_j^* \leq I$. In particular if $\sum_{j=1}^{d} T_j T_j^* = I$, then the tuple $\underline{T} = (T_1, \ldots, T_d)$

is called a *coisometric*. If T_j's are isometries with orthogonal ranges, then the tuple $\underline{T} = (T_1, \ldots, T_d)$ is called a *row isometry*. Consider a row contraction $\underline{T}$ as a row operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$. Define $D_T := (I - \underline{T}^* \underline{T})^{\frac{1}{2}} : \bigoplus_1^d \mathcal{L} \to \bigoplus_1^d \mathcal{L}$ and let $\mathcal{D}_T := \overline{\text{Range } D_T}$. The (left) creation operators L_j's on Γ are defined by $L_j x = e_j \otimes x$ for $1 \leq j \leq d$ and $x \in \Gamma$. The tuple $\underline{L} = (L_1, \ldots, L_d)$ is a row isometry. Popescu in [7] gave the following explicit presentation of the minimal isometric dilation of $\underline{T}$ by $\underline{V}$ on $\hat{\mathcal{L}} = \mathcal{L} \oplus (\Gamma \otimes \mathcal{D}_T)$:

$$V_j\left(\ell \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha\right) = T_j \ell \oplus \left[e_\emptyset \otimes (D_T)_j \ell + e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha\right], \quad (2.1)$$

where $\ell \in \mathcal{L}$, $d_\alpha \in \mathcal{D}_T$, and $(D_T)_j : \mathcal{L} \to \bigoplus_1^d \mathcal{L}$ is defined for $j = 1, \ldots, d$ by the $(D_T)_j \ell = D_T(0, \ldots, \ell, \ldots, 0)$ with ℓ embedded at the j-th component. If in addition $\underline{T}$ is the coisometric tuple, then $\underline{V}$ is also the coisometric tuple.

Let $\underline{C} = (C_1, \ldots, C_d)$ be a coisometric tuple on a Hilbert space $\mathcal{H}_C$, $\underline{E} = (E_1, \ldots, E_d)$ be a coisometric lifting of $\underline{C} = (C_1, \ldots, C_d)$ on a Hilbert space $\mathcal{H}_E \supset \mathcal{H}_C$ and

$$E_j = \begin{pmatrix} C_j & 0 \\ B_j & A_j \end{pmatrix} \quad (2.2)$$

for $j = 1, \ldots, d$ with respect to the decomposition $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_C^\perp$. From now on we denote $\mathcal{H}_C^\perp$ by $\mathcal{H}_A$. Let $\underline{\hat{V}}^E = (\hat{V}_1^E, \ldots, \hat{V}_d^E)$ and $\underline{\hat{V}}^C = (\hat{V}_1^C, \ldots, \hat{V}_d^C)$ be minimal isometric dilations of the form given by equation (2.1) for tuples $\underline{E}$ and $\underline{C}$ on Hilbert spaces $\hat{\mathcal{H}}_E = \mathcal{H}_E \oplus (\Gamma \otimes \mathcal{D}_E)$ and $\hat{\mathcal{H}}_C = \mathcal{H}_C \oplus (\Gamma \otimes \mathcal{D}_C)$ respectively. Let $\mathcal{P}$ be a d-dimensional Hilbert space with an orthonormal basis $\{\epsilon_1, \ldots, \epsilon_d\}$. We define operators $\tilde{U} : \mathcal{H}_C \oplus \mathcal{D}_C \to \mathcal{H}_C \otimes \mathcal{P}$ and $U : \mathcal{H}_E \oplus \mathcal{D}_E \to \mathcal{H}_E \otimes \mathcal{P}$ as follows:

$$\tilde{U}(\tilde{h} \oplus y) := \sum_{j=1}^d (C_j^* \tilde{h} + (D_C)_j^* y) \otimes \epsilon_j = \sum_{j=1}^d (\hat{V}_j^C)^* (\tilde{h} \oplus e_\emptyset \otimes y) \otimes \epsilon_j,$$

$$U(h \oplus \eta) := \sum_{j=1}^d (E_j^* h + (D_E)_j^* \eta) \otimes \epsilon_j = \sum_{j=1}^d (\hat{V}_j^E)^* (h \oplus e_\emptyset \otimes \eta) \otimes \epsilon_j,$$

where $\tilde{h} \in \mathcal{H}_C,\ y \in \mathcal{D}_C,\ h \in \mathcal{H}_E$ and $\eta \in \mathcal{D}_E$.

We show that $\tilde{U}$ and U are unitaries. From the fact that $\underline{\hat{V}}^C = (\hat{V}_1^C, \ldots, \hat{V}_d^C)$ is a coisometric tuple, it follows that $\tilde{U}$ is an isometry. We claim that $\tilde{U}$ is a surjective map. For $\sum_{j=1}^d \tilde{h}_j \otimes \epsilon_j \in \mathcal{H}_C \otimes \mathcal{P}$,

$$\begin{aligned} \tilde{U}(\sum_{j=1}^d (C_j \tilde{h}_j \oplus (D_C)_j \tilde{h}_j) &= \sum_{j=1}^d \tilde{U}(C_j \tilde{h}_j \oplus (D_C)_j \tilde{h}_j) \\ &= \sum_{j=1}^d \left(\sum_{i=1}^d (C_i^* C_j \tilde{h}_j + (D_C)_i^* (D_C)_j \tilde{h}_j) \otimes \epsilon_i \right) \\ &= \sum_{j=1}^d \tilde{h}_j \otimes \epsilon_j, \end{aligned}$$

where the last equality uses

$$(D_C)_i^*(D_C)_j = \begin{cases} -C_i^* C_j, & \text{if } i \neq j; \\ I - C_j^* C_j, & \text{if } i = j. \end{cases}$$

Thus $\tilde{U}$ is unitary. Similarly it can be shown that U is also unitary.

For the main theorem of this section, we need to introduce some operators. Let us define $\tilde{U}_1 : \hat{\mathcal{H}}_C \to \mathcal{H}_C \otimes \mathcal{P} \oplus \big(\bigoplus_{m\geq 1}((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_C)\big)$ by the formula

$$\tilde{U}_1\Big(\tilde{h} \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes y_\alpha\Big) = \sum_{j=1}^{d}((C_j^*\tilde{h} + (D_C)_j^* y_\emptyset) \otimes \epsilon_j) \oplus \sum_{|\alpha|\geq 1} e_\alpha \otimes y_\alpha, \tag{2.3}$$

where $\tilde{h} \in \mathcal{H}_C$ and $y_\alpha \in \mathcal{D}_C$. For each $n \geq 2$, let

$$\tilde{U}_n : \mathcal{H}_C \otimes \mathcal{P}_{[1,n-1]} \oplus \Bigg(\bigoplus_{m\geq n-1}((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_C)\Bigg)$$
$$\to \mathcal{H}_C \otimes \mathcal{P}_{[1,n]} \oplus \Bigg(\bigoplus_{m\geq n}((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_C)\Bigg)$$

be defined for $\tilde{h}_{j_1,\ldots,j_{n-1}} \in \mathcal{H}_C$ and $y_{j_1\ldots j_{n-1}},\ y_\alpha \in \mathcal{D}_C$ by

$$\begin{aligned} &\tilde{U}_n\Bigg(\sum_{j_1,\ldots,j_{n-1}=1}^{d} (\tilde{h}_{j_1\ldots j_{n-1}} \otimes \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_{n-1}}) \\ &\quad \oplus \sum_{j_1,\ldots,j_{n-1}=1}^{d} (e_{j_1} \otimes \cdots \otimes e_{j_{n-1}} \otimes y_{j_1\ldots j_{n-1}}) \oplus \sum_{|\alpha|\geq n} e_\alpha \otimes y_\alpha\Bigg) \\ &\quad = \sum_{j_1,\ldots,j_n=1}^{d} \big((C_{j_n}^* \tilde{h}_{j_1\ldots j_{n-1}} + (D_C)_{j_n}^* y_{j_1\ldots j_{n-1}}) \otimes \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_n}\big) \\ &\quad \oplus \sum_{|\alpha|\geq n} e_\alpha \otimes y_\alpha. \end{aligned} \tag{2.4}$$

It can be seen that $\tilde{U}_n$ is unitary for each $n \geq 1$, with suitable modifications of the arguments which are used in proving that $\tilde{U}$ is unitary.

Similarly, we define the unitary operator $U_1 : \hat{\mathcal{H}}_E \to \mathcal{H}_E \otimes \mathcal{P} \oplus \big(\bigoplus_{m\geq 1}((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_E)\big)$ by the following formula:

$$U_1\Big(h \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big) = \sum_{j=1}^{d}((E_j^* h + (D_E)_j^* \eta_\emptyset) \otimes \epsilon_j) \oplus \sum_{|\alpha|\geq 1} e_\alpha \otimes \eta_\alpha, \tag{2.5}$$

where $h \in \mathcal{H}_E$ and $\eta_\alpha \in \mathcal{D}_E$. For each $n \geq 2$, define the unitary

$$U_n : \mathcal{H}_E \otimes \mathcal{P}_{[1,n-1]} \oplus \Big(\bigoplus_{m \geq n-1} ((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_E) \Big)$$
$$\to \mathcal{H}_E \otimes \mathcal{P}_{[1,n]} \oplus \Big(\bigoplus_{m \geq n} ((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_E) \Big)$$

for $h_{j_1 \dots j_{n-1}} \in \mathcal{H}_E$ and $\eta_{j_1 \dots j_{n-1}},\ \eta_\alpha \in \mathcal{D}_E$ by

$$\begin{aligned} U_n\Big(& \sum_{j_1,\dots,j_{n-1}=1}^{d} (h_{j_1\dots j_{n-1}} \otimes \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_{n-1}}) \\ & \oplus \sum_{j_1,\dots,j_{n-1}=1}^{d} (e_{j_1} \otimes \cdots \otimes e_{j_{n-1}} \otimes \eta_{j_1\dots j_{n-1}}) \oplus \sum_{|\alpha| \geq n} e_\alpha \otimes \eta_\alpha \Big) \\ = & \sum_{j_1,\dots,j_n=1}^{d} ((E_{j_n}^* h_{j_1\dots j_{n-1}} + (D_E)_{j_n}^* \eta_{j_1\dots j_{n-1}}) \otimes \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_n}) \\ & \oplus \sum_{|\alpha| \geq n} e_\alpha \otimes \eta_\alpha. \end{aligned} \tag{2.6}$$

Since $\underline{E}$ is a lifting of $\underline{C}$, $E_j^* \tilde{h} = C_j^* \tilde{h}$ for each $\tilde{h} \in \mathcal{H}_C$. It follows from equations (2.3), (2.4), (2.5) and (2.6) that

$$\tilde{U}_1 \tilde{h} = U_1 \tilde{h} \ \text{ and } \ \tilde{U}_n(\tilde{h} \otimes \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_{n-1}}) = U_n(\tilde{h} \otimes \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_{n-1}}) \tag{2.7}$$

for each $\tilde{h} \in \mathcal{H}_C$, $1 \leq j_1, \dots, j_{n-1} \leq d$, and $n \geq 2$. For each $n \geq 1$, let Q_n denote the orthogonal projection of $\hat{\mathcal{H}}_C$ onto $\mathcal{H}_C \oplus \big(\bigoplus_{m \leq n-1} ((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_C)\big)$, and let P_n denote the orthogonal projection of $\mathcal{H}_E \otimes \mathcal{P}_{[1,n]} \oplus \big(\bigoplus_{m \geq n} ((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_E)\big)$ onto $\mathcal{H}_C \otimes \mathcal{P}_{[1,n]}$.

We are now ready to prove the main result of this section.

Theorem 2.1. *If P_n and Q_n are the orthogonal projections as defined above for each $n \geq 1$, then*

$$\text{sot} - \lim_{n \to \infty} \tilde{U}_1^* \dots \tilde{U}_n^* P_n U_n \dots U_1$$

exists. This limit is a coisometry, say $\hat{W} : \hat{\mathcal{H}}_E \to \hat{\mathcal{H}}_C$. Its adjoint $\hat{W}^* : \hat{\mathcal{H}}_C \to \hat{\mathcal{H}}_E$ is given by

$$\text{sot} - \lim_{n \to \infty} U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n.$$

Here 'sot' *denotes the strong operator topology.*

Proof. Let us begin by the dense linear manifold $\bigcup_{l \geq 1} \Big(\mathcal{H}_C \oplus \big(\bigoplus_{m \leq l-1} ((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_C)\big)\Big)$ of $\hat{\mathcal{H}}_C$. Assume $\tilde{h} \in \tilde{\mathcal{H}}$ and $y_\alpha \in \mathcal{D}_C$ for all $\alpha \in \tilde{\Lambda}$ such that $|\alpha| \leq k-1$, for some positive integer k. We show that

$$\lim_{n \to \infty} U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n \Big(\tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big)$$

exists. For each $n \geq 1$, set $s_n = U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n \Big(\tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big)$. It follows from equation (2.7) that $s_k = s_{k+j}$ for each $j \geq 1$. Thus

$$\lim_{n \to \infty} U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n \Big(\tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big)$$

exists and it is s_k. We observe that

$$\begin{aligned} \Big\| \lim_{n \to \infty} U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n \Big(\tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big) \Big\| &= \| s_k \| \\ = \Big\| U_1^* \dots U_k^* \tilde{U}_k \dots \tilde{U}_1 Q_k \Big(\tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big) \Big\| & \\ = \Big\| U_1^* \dots U_k^* \tilde{U}_k \dots \tilde{U}_1 \Big(\tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big) \Big\| & \\ = \Big\| \tilde{h} \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes y_\alpha \Big\|. & \end{aligned}$$

Hence $\text{sot} - \lim_{n \to \infty} U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n$ defines an isometry on the dense linear manifold $\bigcup_{l \geq 1} \Big(\mathcal{H}_C \oplus \big(\bigoplus_{m \leq l-1} ((\mathbb{C}^d)^{\otimes^m} \otimes \mathcal{D}_C) \big) \Big)$ of $\hat{\mathcal{H}}_C$. By continuity, it extends to an isometry $\hat{R}$ from $\hat{\mathcal{H}}_C$ to $\hat{\mathcal{H}}_E$. Its adjoint $\hat{R}^* : \hat{\mathcal{H}}_E \to \hat{\mathcal{H}}_C$ is a coisometry and so we just need to rename $\hat{R}^*$ as $\hat{W}$.

Let $h \in \mathcal{H}_E$ and $\eta_\alpha \in \mathcal{D}_E$ for all $\alpha \in \tilde{\Lambda}$ such that $|\alpha| \leq k-1$, for some positive integer k. Let $n \geq k$, $\tilde{h} \in \mathcal{H}_C$ and $y_\beta \in \mathcal{D}_C$ for all $\beta \in \tilde{\Lambda}$, $|\beta| \leq n-1$. Then

$$\begin{aligned} &\Big\langle \hat{W} \Big(h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha \Big), \ \tilde{h} \oplus \sum_{|\beta| \leq n-1} e_\beta \otimes y_\beta \Big\rangle \\ &= \Big\langle h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha, \ U_1^* \dots U_n^* \tilde{U}_n \dots \tilde{U}_1 Q_n \Big(\tilde{h} \oplus \sum_{|\beta| \leq n-1} e_\beta \otimes y_\beta \Big) \Big\rangle \\ &= \Big\langle \tilde{U}_1^* \dots \tilde{U}_n^* P_n U_n \dots U_1 \Big(h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha \Big), \ \tilde{h} \oplus \sum_{|\beta| \leq n-1} e_\beta \otimes y_\beta \Big\rangle. \end{aligned}$$

It follows from the above calculations that

$$Q_n \hat{W} \Big(h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha \Big) = \tilde{U}_1^* \dots \tilde{U}_n^* P_n U_n \dots U_1 \Big(h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha \Big).$$

By the fact that $\text{sot} - \lim_{n \to \infty} Q_n = I$, we conclude that

$$\hat{W} \Big(h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha \Big) = \lim_{n \to \infty} \tilde{U}_1^* \dots \tilde{U}_n^* P_n U_n \dots U_1 \Big(h \oplus \sum_{|\alpha| \leq k-1} e_\alpha \otimes \eta_\alpha \Big).$$

Finally, we extend this formula to the whole of $\hat{\mathcal{H}}_E$ by continuity. □

The following result shows that the coisometry $\hat{W}$ and its adjoint $\hat{W}^*$ intertwine the tuples $\underline{\hat{V}}^E$ and $\underline{\hat{V}}^C$.

PROPOSITION 2.2

For $j = 1, \ldots, d$, $\hat{W}\hat{V}_j^E = \hat{V}_j^C\hat{W}$ *and* $\hat{V}_j^E\hat{W}^* = \hat{W}^*\hat{V}_j^C$.

Proof. Let $h \in \mathcal{H}_E$ and $\eta_\alpha \in \mathcal{D}_E$ for all $\alpha \in \tilde{\Lambda}$ such that $\sum_{\alpha\in\tilde{\Lambda}} \|\eta_\alpha\|^2 < \infty$. Suppose $\tilde{h} \in \mathcal{H}_C$ and $y_\beta \in \mathcal{D}_C$ for all $\beta \in \tilde{\Lambda}$ such that $|\beta| \leq n-1$, for some positive integer n. Then

$$
\begin{aligned}
&\Big\langle \hat{W}\hat{V}_j^E\Big(h \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big),\ \tilde{h} \oplus \sum_{|\beta|\leq n-1} e_\beta \otimes y_\beta\Big\rangle \\
&= \Big\langle \hat{V}_j^E\Big(h \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big),\ \hat{W}^*\Big(\tilde{h} \oplus \sum_{|\beta|\leq n-1} e_\beta \otimes y_\beta\Big)\Big\rangle \\
&= \Big\langle E_j h + e_\emptyset \otimes (D_E)_j h + e_j \otimes \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha, \\
&\quad U_1^* \ldots U_n^* \tilde{U}_n \ldots \tilde{U}_1\Big(\tilde{h} \oplus \sum_{|\beta|\leq n-1} e_\beta \otimes y_\beta\Big)\Big\rangle \\
&= \Big\langle U_1^*\Big(h \otimes \epsilon_j \oplus e_j \otimes \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big), \\
&\quad U_1^* \ldots U_n^* \tilde{U}_n \ldots \tilde{U}_2\Big(\sum_{i=1}^d ((C_i^*\tilde{h} + (D_C)_i^* y_\emptyset) \otimes \epsilon_i) \oplus \sum_{1\leq|\beta|\leq n-1} e_\beta \otimes y_\beta\Big)\Big\rangle \\
&= \Big\langle \Big(h \otimes \epsilon_j \oplus e_j \otimes \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big),\ U_2^* \ldots U_n^* \tilde{U}_n \ldots \tilde{U}_2 \\
&\quad \Big(\sum_{i=1}^d ((C_i^*\tilde{h} + (D_C)_i^* y_\emptyset) \otimes \epsilon_i) \oplus \sum_{1\leq|\beta|\leq n-1} e_\beta \otimes y_\beta\Big)\Big\rangle \\
&= \Big\langle h \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha,\ U_1^* \ldots U_{n-1}^* \tilde{U}_{n-1} \ldots \tilde{U}_1 \\
&\quad \Big(C_j^*\tilde{h} + (D_C)_j^* y_\emptyset \oplus L_j^* \otimes I\Big(\sum_{1\leq|\beta|\leq n-1} e_\beta \otimes y_\beta\Big)\Big)\Big\rangle \\
&= \Big\langle h \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha,\ \hat{W}^*(\hat{V}_j^C)^*\Big(\tilde{h} \oplus \sum_{|\beta|\leq n-1} e_\beta \otimes y_\beta\Big)\Big\rangle \\
&= \Big\langle \hat{V}_j^C\hat{W}\Big(h \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big),\ \tilde{h} \oplus \sum_{|\beta|\leq n-1} e_\beta \otimes y_\beta\Big\rangle.
\end{aligned}
$$

Thus $\hat{W}\hat{V}_j^E = \hat{V}_j^C\hat{W}$ for $j = 1, \ldots, d$.

$$\begin{aligned}\hat{W}^*\hat{V}_j^C\Big(\tilde{h} \oplus \sum_{|\alpha|\le n-1} e_\alpha \otimes y_\alpha\Big) &= \hat{W}^*\tilde{U}_1^*\Big(\tilde{h} \otimes \epsilon_j \oplus e_j \otimes \sum_{|\alpha|\le n-1} e_\alpha \otimes y_\alpha\Big)\\ &= U_1^* \ldots U_{n+1}^*\tilde{U}_{n+1} \ldots \tilde{U}_1\tilde{U}_1^*\Big(\tilde{h} \otimes \epsilon_j \oplus e_j \otimes \sum_{|\alpha|\le n-1} e_\alpha \otimes y_\alpha\Big)\\ &= U_1^*U_2^* \ldots U_{n+1}^*\tilde{U}_{n+1} \ldots \tilde{U}_2\Big(\tilde{h} \otimes \epsilon_j \oplus e_j \otimes \sum_{|\alpha|\le n-1} e_\alpha \otimes y_\alpha\Big)\\ &= \hat{V}_j^E\hat{W}^*\Big(\tilde{h} \oplus \sum_{|\alpha|\le n-1} e_\alpha \otimes y_\alpha\Big).\end{aligned}$$

By continuity, this extends to all of $\hat{\mathcal{H}}_C$. So $\hat{V}_j^E\hat{W}^* = \hat{W}^*\hat{V}_j^C$ for $j = 1, \ldots, d$. □

We have $(\hat{V}_j^E)^*\tilde{h} = E_j^*\tilde{h} = C_j^*\tilde{h}$ and $(\hat{V}_j^C)^*\tilde{h} = C_j^*\tilde{h}$ for each $\tilde{h} \in \mathcal{H}_C$, i.e., $\mathcal{H}_C$ is covariant under $\hat{V}_j^E$ and $\hat{V}_j^C$. Thus $\hat{V}_j^E(\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \subset \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$ and $\hat{V}_j^C(\Gamma \otimes \mathcal{D}_C) \subset \Gamma \otimes \mathcal{D}_C$. Define $V_j^E := \hat{V}_j^E|_{\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)} : \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E) \to \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$ and $V_j^C := \hat{V}_j^C|_{\Gamma \otimes \mathcal{D}_C} : \Gamma \otimes \mathcal{D}_C \to \Gamma \otimes \mathcal{D}_C$. In fact, $V_j^C = L_j \otimes I_{\mathcal{D}_C}$. Further note that $\hat{W}\tilde{h} = \tilde{h}$ and $\hat{W}^*\tilde{h} = \tilde{h}$ for each $\tilde{h} \in \mathcal{H}_C$. Define

$$W^* := \hat{W}^*|_{\Gamma \otimes \mathcal{D}_C} : \Gamma \otimes \mathcal{D}_C \to \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E).$$

It can be seen that W, the adjoint of W^*, is given by $\hat{W}|_{\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)} : \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E) \to \Gamma \otimes \mathcal{D}_C$. Then it follows from Proposition 2.2 that

$$WV_j^E = V_j^C W \text{ and } V_j^E W^* = W^* V_j^C \text{ for } j = 1, \ldots, d.$$

3. Outgoing Cuntz scattering system, transfer function and characteristic function of lifting

In order to define an outgoing Cuntz scattering system, we need the following:

DEFINITION 3.1

(1) Let $\underline{T} = (T_1, \ldots, T_d)$ be a row isometry on a Hilbert space $\mathcal{L}$. A subspace $\mathcal{M}$ of $\mathcal{L}$ is called *wandering subspace* with respect to $\underline{T}$ if

$$T_\alpha\mathcal{M} \perp T_\beta\mathcal{M} \text{ for distinct } \alpha, \beta \in \tilde{\Lambda}.$$

(2) A tuple $\underline{T} = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{L}$ is called a *row unitary* if $\underline{T}$ is a row isometry and $\overline{\text{span}}_{j=1,\ldots,d} T_j\mathcal{L} = \mathcal{L}$.
(3) A tuple $\underline{T} = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{L}$ is called *row shift* if $\underline{T}$ is a row isometry and there exits a wandering subspace $\mathcal{M}$ of $\mathcal{L}$ with respect to $\underline{T}$ such that $\mathcal{L} = \oplus_{\alpha\in\tilde{\Lambda}} T_\alpha\mathcal{M}$.

We omit the proofs of Theorems 3.3 and 3.5 in this section because they follow using similar arguments as those in §§ 4 and 5 of [4]. In Chapter 5 of [2] an outgoing Cuntz scattering system is defined as a collection

$$(\underline{V} = (V_1, \ldots, V_d),\ \mathcal{L},\ \mathcal{G}_*^+,\ \mathcal{G})$$

such that $\underline{V}$ is a row isometry on the Hilbert space $\mathcal{L}$, and $\mathcal{G}_*^+$ and $\mathcal{G}$ are subspaces of $\mathcal{L}$ such that

(a) $\mathcal{G}_*^+$ is the smallest $\underline{V}-$invariant subspace containing

$$\mathcal{E}_* := \mathcal{L} \ominus \overline{\operatorname{span}}_{j=1,\ldots,d} V_j \mathcal{L};$$

thus $\underline{V}|_{\mathcal{G}_*^+}$ is a row shift and $\mathcal{G}_*^+ = \bigoplus_{\alpha\in\tilde{\Lambda}} V_\alpha \mathcal{E}_*$.

(b) $\underline{V}|_{\mathcal{G}}$ is a row shift; thus $\mathcal{G} = \bigoplus_{\alpha\in\tilde{\Lambda}} V_\alpha \mathcal{E}$ where $\mathcal{E} := \mathcal{G} \ominus \overline{\operatorname{span}}_{j=1,\ldots,d} V_j \mathcal{G}$.

Our goal is to find an outgoing Cuntz scattering system inside our model. Let as before $\underline{E}$ be a coisometric lifting of a row contraction $\underline{C}$ by $\underline{A}$ and $\underline{\hat{V}}^E$ be the minimal isometric dilation of $\underline{E}$ of the form given by equation (2.1). First we show that tuples $\underline{\hat{V}}^E = (\hat{V}_1^E, \ldots, \hat{V}_d^E)$ and $\underline{V}^E = (V_1^E, \ldots, V_d^E)$ on Hilbert spaces $\hat{\mathcal{H}}_E$ and $\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$ are row unitary and row isometry respectively. Since $\underline{\hat{V}}^E = (\hat{V}_1^E, \ldots, \hat{V}_d^E)$ is the minimal isometric dilation of $\underline{E} = (E_1, \ldots, E_d)$, it follows that $\hat{V}_j^E$'s are isometries with orthogonal ranges. So V_j^E's are isometries with orthogonal ranges, because

$$V_j^E = \hat{V}_j^E|_{\mathcal{H}_A\oplus(\Gamma\otimes\mathcal{D}_E)} : \mathcal{H}_A\oplus(\Gamma\otimes\mathcal{D}_E) \to \mathcal{H}_A\oplus(\Gamma\otimes\mathcal{D}_E) \text{ for } j = 1, \ldots, d.$$

Thus $\underline{V}^E = (V_1^E, \ldots, V_d^E)$ is a row isometry. Also $\sum_{j=1}^d \hat{V}_j^E(\hat{V}_j^E)^* = I$, i.e., $\underline{\hat{V}}^E = (\hat{V}_1^E, \ldots, \hat{V}_d^E)$ is a row unitary. Define $\mathcal{E}_* := W^*(e_\emptyset \otimes \mathcal{D}_C)$. We claim that $\mathcal{E}_*$ is a wandering subspace with respect to $\underline{V}^E$. It is enough to prove that

$$W^*(e_\emptyset \otimes \mathcal{D}_C) \perp \overline{\operatorname{span}}_{j=1,\ldots,d} V_j^E(\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)),$$

since V_j^E's are isometries with orthogonal ranges. If $y \in \mathcal{D}_C$, $h_a \in \mathcal{H}_A$, and $\sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha \in \Gamma \otimes \mathcal{D}_E$, then

$$\begin{aligned}
&\Big\langle W^*(e_\emptyset \otimes y), V_j^E\Big(h_a \oplus \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes \eta_\alpha\Big)\Big\rangle \\
&\quad= \Big\langle U_1^*\tilde{U}_1(e_\emptyset \otimes y), E_j h_a \oplus e_\emptyset \otimes (D_E)_j h_a \oplus \sum_{\alpha\in\tilde{\Lambda}} e_j \otimes e_\alpha \otimes \eta_\alpha\Big\rangle \\
&\quad= \Big\langle \sum_{i=1}^d ((D_C)_i^* y_\emptyset \otimes \epsilon_i), U_1\Big(E_j h_a \oplus e_\emptyset \otimes (D_E)_j h_a \oplus \sum_{\alpha\in\tilde{\Lambda}} e_j \otimes e_\alpha \otimes \eta_\alpha\Big)\Big\rangle \\
&\quad= \Big\langle \sum_{i=1}^d ((D_C)_i^* y_\emptyset \otimes \epsilon_i), \sum_{i=1}^d ((E_i^* E_j h_a + (D_E)_i^*(D_E)_j h_a) \otimes \epsilon_i) \\
&\qquad\quad \oplus \sum_{\alpha\in\tilde{\Lambda}} e_j \otimes e_\alpha \otimes \eta_\alpha\Big\rangle \\
&\quad= \Big\langle \sum_{i=1}^d ((D_C)_i^* y_\emptyset \otimes \epsilon_i), h_a \otimes \epsilon_j \oplus \sum_{\alpha\in\tilde{\Lambda}} e_j \otimes e_\alpha \otimes \eta_\alpha\Big\rangle = 0,
\end{aligned}$$

for $j = 1, \dots, d$. The last equality holds because $(D_C)_i^* y_\emptyset \in \mathcal{H}_C$ for $i = 1, \dots, d$ and $h_a \in \mathcal{H}_A$. So our claim is established.

Our next aim is to prove that

$$\mathcal{E}_* = (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \ominus \overline{\operatorname{span}}_{j=1,\dots,d} V_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)).$$

The proof of the above claim shows that

$$\mathcal{E}_* = W^*(e_\emptyset \otimes \mathcal{D}_C) \subset (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \ominus \overline{\operatorname{span}}_{j=1,\dots,d} V_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)). \tag{3.1}$$

To show the reverse inclusion, let

$$x \in (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \ominus \overline{\operatorname{span}}_{j=1,\dots,d} V_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)).$$

We can write $x = u \oplus v$ where $u \in W^*(e_\emptyset \otimes \mathcal{D}_C)$ and $v \in (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \ominus W^*(e_\emptyset \otimes \mathcal{D}_C)$. It follows from equation (3.1) that

$$u \in (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \ominus \overline{\operatorname{span}}_{j=1,\dots,d} V_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)).$$

Then $v = x - u \in (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) \ominus \overline{\operatorname{span}}_{j=1,\dots,d} V_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E))$, i.e.,

$$v \perp \overline{\operatorname{span}}_{j=1,\dots,d} V_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)) (= \overline{\operatorname{span}}_{j=1,\dots,d} \hat{V}_j^E (\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E))). \tag{3.2}$$

Since $v \perp \mathcal{H}_C = \hat{W}^* \mathcal{H}_C$ and $v \perp W^*(e_\emptyset \otimes \mathcal{D}_C) = \hat{W}^*(e_\emptyset \otimes \mathcal{D}_C)$, we conclude that

$$v \perp \overline{\operatorname{span}}_{j=1,\dots,d} \hat{V}_j^E \mathcal{H}_C. \tag{3.3}$$

By equations (3.2) and (3.3), we see that $v \perp \overline{\operatorname{span}}_{j=1,\dots,d} \hat{V}_j^E \hat{\mathcal{H}}_E$. We get $v \perp \hat{\mathcal{H}}_E$, since $\underline{\hat{V}}^E = (\hat{V}_1^E, \dots, \hat{V}_d^E)$ is a row unitary. Thus $v = 0$, and therefore $x = u \in W^*(e_\emptyset \otimes \mathcal{D}_C)$. This proves the reverse inclusion. Define $\mathcal{G} := \Gamma \otimes \mathcal{D}_E$. Since $V_j^E|_{\mathcal{G}} = L_j \otimes I_{\mathcal{D}_E}$, it follows that $\underline{V}^E|_{\mathcal{G}} = (V_1^E|_{\mathcal{G}}, \dots, V_d^E|_{\mathcal{G}})$ is a row isometry, and $e_\emptyset \otimes \mathcal{D}_E$ is a wandering subspace of $\mathcal{G}$ with respect to $\underline{V}^E|_{\mathcal{G}}$ such that

$$\mathcal{G} = \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E (e_\emptyset \otimes \mathcal{D}_E).$$

Thus $\underline{V}^E|_{\mathcal{G}} = (V_1^E|_{\mathcal{G}}, \dots, V_d^E|_{\mathcal{G}})$ is a row shift.

We summarize the preceding discussion in the following:

Theorem 3.2. *A collection*

$$\left(\underline{V}^E = (V_1^E, \dots, V_d^E), \mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E), \mathcal{G}_*^+ = \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}_*, \mathcal{G} = \bigoplus_{\alpha \in \tilde{\Lambda}} V_\alpha^E \mathcal{E}\right)$$

is an outgoing Cuntz scattering system where $\mathcal{E}_* = W^*(e_\emptyset \otimes \mathcal{D}_C)$ *and* $\mathcal{E} = e_\emptyset \otimes \mathcal{D}_E$.

Take the input space as $\mathcal{D}_E$ and the output space as $\mathcal{D}_C$. If

$$\tilde{C} \;:=\; \sum_{j=1}^{d} (D_C)_j P_{\mathcal{H}_C} E_j^* : \mathcal{H}_E \to \mathcal{D}_C$$

and

$$\tilde{D} \;:=\; \sum_{j=1}^{d} (D_C)_j P_{\mathcal{H}_C} (D_E)_j^* : \mathcal{D}_E \to \mathcal{D}_C,$$

where $P_{\mathcal{H}_C}$ is the orthogonal projection of $\mathcal{H}_E$ onto $\mathcal{H}_C$, then we define a *colligation of operators* [2] as follows:

$$\mathcal{C}_{U,\tilde{U}} := \begin{pmatrix} E_1^* & (D_E)_1^* \\ \vdots & \vdots \\ E_d^* & (D_E)_d^* \\ \tilde{C} & \tilde{D} \end{pmatrix} : \mathcal{H}_E \oplus \mathcal{D}_E \to \bigoplus_{j=1}^{d} \mathcal{H}_E \oplus \mathcal{D}_C.$$

Consider the following $\tilde{\Lambda}$-*linear system* $\sum_{U,\tilde{U}}$ or non-commutative Fornasini–Marchesini system in [1] associated to the colligation $\mathcal{C}_{U,\tilde{U}}$:

$$x(j\alpha) = E_j^* x(\alpha) + (D_E)_j^* u(\alpha) \;\text{ and }\; y(\alpha) = \tilde{C}x(\alpha) + \tilde{D}u(\alpha),$$

where $j = 1, \ldots, d$ and α, $j\alpha$ are words in $\tilde{\Lambda}$, and

$$x : \tilde{\Lambda} \to \mathcal{H}_E, \quad u : \tilde{\Lambda} \to \mathcal{D}_E, \quad y : \tilde{\Lambda} \to \mathcal{D}_C.$$

Let $z = (z_1, \ldots, z_d)$ be a d-tuple of formal non-commuting indeterminates. Define Fourier transforms of x, u and y as

$$\hat{x}(z) = \sum_{\alpha \in \tilde{\Lambda}} x(\alpha) z^\alpha, \quad \hat{u}(z) = \sum_{\alpha \in \tilde{\Lambda}} u(\alpha) z^\alpha, \quad \hat{y}(z) = \sum_{\alpha \in \tilde{\Lambda}} y(\alpha) z^\alpha$$

respectively where $z^\alpha = z_{\alpha_n} \ldots z_{\alpha_1}$ for $\alpha = \alpha_n \ldots \alpha_1 \in \tilde{\Lambda}$. If we assume that $x(\emptyset) = 0$ and z-variables commute with the coefficients, then we get the input–output relation

$$\hat{y}(z) = \Theta_{U,\tilde{U}}(z)\hat{u}(z),$$

where $\Theta_{U,\tilde{U}}$ as a formal non-commutative power series is given by the following:

$$\Theta_{U,\tilde{U}}(z) := \sum_{\alpha \in \tilde{\Lambda}} \Theta_{U,\tilde{U}}^{(\alpha)} z^\alpha := \tilde{D} + \tilde{C} \sum_{\substack{\beta \in \tilde{\Lambda} \\ j=1,\ldots,d}} (E_{\bar{\beta}})^* (D_E)_j^* z^{\beta j}.$$

Here $\bar{\beta} = \beta_1 \ldots \beta_n$ is the reverse of $\beta = \beta_n \ldots \beta_1$ and $\Theta_{U,\tilde{U}}^{(\alpha)}$ are operators from $\mathcal{D}_E$ to $\mathcal{D}_C$. We refer $\Theta_{U,\tilde{U}}$ as *transfer function* associated to the unitaries U and $\tilde{U}$. For a Hilbert space $\mathcal{V}$, a non-commutative analogue of Hardy space is the space $\ell^2(\tilde{\Lambda}, \mathcal{V})$ of formal power series $g(z) = \sum_{\alpha \in \tilde{\Lambda}} g_\alpha z^\alpha$ with $\|g\|_{\ell^2}^2 = \sum_{\alpha \in \tilde{\Lambda}} \|g_\alpha\|^2 < \infty$ where $g_\alpha \in \mathcal{V}$. The following theorem shows that the formal non-commutative power series $\Theta_{U,\tilde{U}}$ turns out to be a contractive operator between Hilbert spaces.

Theorem 3.3. *The map* $M_{\Theta_{U,\tilde{U}}} : \ell^2(\tilde{\Lambda}, \mathcal{D}_E) \to \ell^2(\tilde{\Lambda}, \mathcal{D}_C)$ *defined by*

$$M_{\Theta_{U,\tilde{U}}} \hat{u}(z) := \Theta_{U,\tilde{U}}(z)\hat{u}(z)$$

is a contraction.

$M_{\Theta_{U,\tilde{U}}}$ intertwines with right translation i.e.,

$$M_{\Theta_{U,\tilde{U}}}\left(\sum_{\alpha\in\tilde{\Lambda}} u(\alpha)z^\alpha z^i\right) = M_{\Theta_{U,\tilde{U}}}\left(\sum_{\alpha\in\tilde{\Lambda}} u(\alpha)z^\alpha\right)z^i$$

for $i = 1, \ldots, d$. Thus $M_{\Theta_{U,\tilde{U}}}$ is a *multi-analytic operator* (cf. [9]). Since $M_{\Theta_{U,\tilde{U}}}$ is a contractive operator, the transfer function $\Theta_{U,\tilde{U}} \in \mathcal{S}_{nc,d}(\mathcal{D}_E, \mathcal{D}_C)$ (non-commutative d-variable Schur class, cf. section 2.4 of [2]) where

$$\mathcal{S}_{nc,d}(\mathcal{D}_E, \mathcal{D}_C) := \Bigg\{T(z) = \sum_{\alpha\in\tilde{\Lambda}} T_\alpha z^\alpha : M_T : \ell^2(\tilde{\Lambda}, \mathcal{D}_E) \to \ell^2(\tilde{\Lambda}, \mathcal{D}_C) \text{ satisfies } \|M_T\| \leq 1\Bigg\}.$$

Next we show that the transfer function coincides with the characteristic function of lifting [3]. Define unitaries $\Psi_C : \ell^2(\tilde{\Lambda}, \mathcal{D}_C) \to \Gamma \otimes \mathcal{D}_C$ and $\Psi_E : \mathcal{D}_E z^\emptyset \to e_\emptyset \otimes \mathcal{D}_E$ by

$$\Psi_C\left(\sum_{\alpha\in\tilde{\Lambda}} y_\alpha z^\alpha\right) = \sum_{\alpha\in\tilde{\Lambda}} e_{\tilde{\alpha}} \otimes y_\alpha \quad \text{and} \quad \Psi_E(\eta z^\emptyset) = e_\emptyset \otimes \eta$$

respectively where $y_\alpha \in \mathcal{D}_C$ and $\eta \in \mathcal{D}_E$. We observe that $\tilde{C}$ vanishes on $\mathcal{H}_C$ by the following argument. For $\tilde{h} \in \mathcal{H}_C$ we have

$$\tilde{C}\tilde{h} = \sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C} E_j^* \tilde{h} = \sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C} C_j^* \tilde{h} = \sum_{j=1}^d (D_C)_j C_j^* \tilde{h} = D_C \underline{C}^* \tilde{h}.$$

Since $\underline{C}$ is a coisometric tuple, the operator D_C is the orthogonal projection. So we have

$$\begin{aligned}\tilde{C}\tilde{h} &= D_C^2 \underline{C}^* \tilde{h} = (I - \underline{C}^*\underline{C})\underline{C}^*\tilde{h} = (\underline{C}^* - \underline{C}^*\underline{C}\,\underline{C}^*)\tilde{h} \\ &= (\underline{C}^* - \underline{C}^*)\tilde{h} = 0. \end{aligned} \tag{3.4}$$

The second last equality follows by $\underline{C}$ is a coisometric tuple. Further, for $h_a \in \mathcal{H}_A$,

$$\begin{aligned}\tilde{C}h_a &= \sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C} E_j^* h_a = \sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C}(B_j^* h_a \oplus A_j^* h_a) \\ &= \sum_{j=1}^d (D_C)_j B_j^* h_a = D_C \underline{B}^* h_a = \underline{B}^* h_a. \end{aligned} \tag{3.5}$$

The last equality holds because D_C is the orthogonal projection and Range $\underline{B}^* \subset \mathcal{D}_C$. Define $D_{*,A} := (I - \underline{A}\underline{A}^*)^2 : \mathcal{H}_A \to \mathcal{H}_A$ and $\mathcal{D}_{*,A} := \overline{\text{Range } D_{*,A}}$. Since $\underline{E}$ is a coisometric lifting of $\underline{C}$, it follows from Theorem 2.1 of [3] that there exist an isometry $\gamma : \mathcal{D}_{*,A} \to \mathcal{D}_C$ with $\gamma D_{*,A} h_a = \underline{B}^* h_a$ for each $h_a \in \mathcal{H}_A$. By equation (3.5), we have

$$\tilde{C} h_a = \gamma D_{*,A} h_a \quad \text{for each } h_a \in \mathcal{H}_A. \tag{3.6}$$

We recall the following expansion of the symbol of the characteristic function $M_{C,E} : \Gamma \otimes \mathcal{D}_E \to \Gamma \otimes \mathcal{D}_C$ of lifting $\underline{E}$ of $\underline{C}$ from [3]: For $h \in \mathcal{H}_C$,

$$\Theta_{C,E}(D_E)_i h = e_\emptyset \otimes [(D_C)_i h - \gamma D_{*,A} B_i h] - \sum_{|\alpha| \geq 1} e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* B_i h, \tag{3.7}$$

and for $h \in \mathcal{H}_A$,

$$\Theta_{C,E}(D_E)_i h = -e_\emptyset \otimes \gamma D_{*,A} A_i h + \sum_{j=1,\ldots,d} e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* (\delta_{ij} I - A_j^* A_i) h, \tag{3.8}$$

where $i = 1, \ldots, d$.

Theorem 3.4. *The transfer function $\Theta_{U,\tilde{U}}$ and the symbol of the characteristic function $\Theta_{C,E}$ are related by the formula*

$$\Psi_C \Theta_{U,\tilde{U}}(z) = \Theta_{C,E} \Psi_E.$$

In other words, the transfer function $\Theta_{U,\tilde{U}}$ coincides with the characteristic function $\Theta_{C,E}$.

Proof. Let $h \in \mathcal{H}_E$. For $i = 1, \ldots, d$,

$$\begin{aligned} &\Psi_C \Theta_{U,\tilde{U}}((D_E)_i h z^\emptyset) \\ &= \Psi_C \Big[\tilde{D}\, z^\emptyset + \sum_{\beta \in \tilde{\Lambda}, j=1,\ldots,d} \tilde{C} (E_{\bar{\beta}})^* (D_E)_j^* z^{\beta j} \Big] ((D_E)_i h z^\emptyset) \\ &= \Psi_C \Big[\tilde{D} (D_E)_i h\, z^\emptyset + \sum_{\beta \in \tilde{\Lambda}, j=1,\ldots,d} \tilde{C} (E_{\bar{\beta}})^* (D_E)_j^* (D_E)_i h\, z^{\beta j} \Big]. \end{aligned} \tag{3.9}$$

Case 1. For $h \in \mathcal{H}_C$ and $i = 1, \ldots, d$,

$$\begin{aligned} \tilde{D}(D_E)_i h &= \sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C} (D_E)_j^* (D_E)_i h = \sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C} (\delta_{ij} I - E_j^* E_i) h \\ &= (D_C)_i h - \Big(\sum_{j=1}^d (D_C)_j P_{\mathcal{H}_C} E_j^* \Big) E_i h = (D_C)_i h - \tilde{C} E_i h \\ &= (D_C)_i h - \tilde{C}(C_i h \oplus B_i h) = (D_C)_i h - \tilde{C} B_i h. \end{aligned} \tag{3.10}$$

The last equality follows by equation (3.4). The second part of equation (3.9) simplifies to

$$\begin{aligned}
&\sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^*(D_E)_j^*(D_E)_i h\, z^{\beta j}\\
&= \sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^*(\delta_{ij} I - E_j^* E_i) h\, z^{\beta j}\\
&= \sum_{\beta\in\tilde{\Lambda}} \tilde{C}(E_{\bar{\beta}})^* h\, z^{\beta i} - \sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* E_j^* E_i h\, z^{\beta j}.
\end{aligned}$$

By equation (3.4) it follows that

$$\begin{aligned}
&\sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^*(D_E)_j^*(D_E)_i h\, z^{\beta j}\\
&= -\sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* E_j^* E_i h\, z^{\beta j}\\
&= -\sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(E_{\bar{\beta}})^* \big((C_j^* C_i + B_j^* B_i) h \oplus A_j^* B_i h\big)\, z^{\beta j}\\
&= -\sum_{\beta\in\tilde{\Lambda},\, j=1,\ldots,d} \tilde{C}(A_{\bar{\beta}})^* A_j^* B_i h\, z^{\beta j}\\
&= -\sum_{|\alpha|\geq 1} \tilde{C}(A_{\bar{\alpha}})^* B_i h\, z^{\alpha}.
\end{aligned} \tag{3.11}$$

The equality which is second from below in the above equation array follows by equation (3.4). By equations (3.9), (3.10) and (3.11), we have for $i = 1, \ldots, d$ and $h \in \mathcal{H}_C$,

$$\begin{aligned}
\Psi_C \Theta_{U,\tilde{U}}((D_E)_i h z^{\emptyset}) &= \Psi_C\Big[(D_C)_i h - \tilde{C} B_i h)\, z^{\emptyset} - \sum_{|\alpha|\geq 1} \tilde{C}(A_{\bar{\alpha}})^* B_i h\, z^{\alpha}\Big]\\
&= e_{\emptyset} \otimes ((D_C)_i h - \tilde{C} B_i h) - \sum_{|\alpha|\geq 1} e_{\bar{\alpha}} \otimes \tilde{C}(A_{\bar{\alpha}})^* B_i h\\
&= e_{\emptyset} \otimes [(D_C)_i h - \gamma D_{*,A} B_i h]\\
&\quad - \sum_{|\alpha|\geq 1} e_{\bar{\alpha}} \otimes \gamma D_{*,A} (A_{\bar{\alpha}})^* B_i h.
\end{aligned}$$

By equation (3.7) we obtain

$$\begin{aligned}
\Psi_C \Theta_{U,\tilde{U}}((D_E)_i h z^{\emptyset}) &= \Theta_{C,E}(e_{\emptyset} \otimes (D_E)_i h)\\
&= \Theta_{C,E} \Psi_E((D_E)_i h z^{\emptyset}).
\end{aligned} \tag{3.12}$$

Case 2. For $h \in \mathcal{H}_A$ and $i = 1, \ldots, d$,

$$\begin{aligned}
\tilde{D}(D_E)_i h &= \sum_{j=1}^{d} (D_C)_j P_{\mathcal{H}_C} (D_E)_j^* (D_E)_i h = \sum_{j=1}^{d} (D_C)_j P_{\mathcal{H}_C} (\delta_{ij} I - E_j^* E_i) h\\
&= (D_C)_i P_{\tilde{\mathcal{H}}} h - \left(\sum_{j=1}^{d} D_j P_{\mathcal{H}_C} E_j^*\right) E_i h = -\tilde{C} A_i h.
\end{aligned} \tag{3.13}$$

Consider again the second part of equation (3.9).

$$\begin{aligned}
&\sum_{\beta\in\tilde\Lambda,\, j=1,\dots,d} \tilde C(E_{\bar\beta})^*(D_E)_j^*(D_E)_i h\, z^{\beta j}\\
&= \sum_{\beta\in\tilde\Lambda,\, j=1,\dots,d} \tilde C(E_{\bar\beta})^*(\delta_{ij} I - E_j^* E_i) h\, z^{\beta j}\\
&= \sum_{\beta\in\tilde\Lambda,\, j=1,\dots,d} \tilde C(A_{\bar\beta})^*(\delta_{ij} I - A_j^* A_i) h\, z^{\beta j}. \qquad (3.14)
\end{aligned}$$

The last equality follows from equation (3.4). By equations (3.9), (3.13) and (3.14), we have for $i = 1, \dots, d$ and $h \in \mathcal{H}_A$,

$$\begin{aligned}
&\Psi_C \Theta_{U,\tilde U}((D_E)_i h z^{\emptyset})\\
&= \Psi_C[-\tilde C A_i h\, z^{\emptyset} + \sum_{\beta\in\tilde\Lambda,\, j=1,\dots,d} \tilde C(A_{\bar\beta})^*(\delta_{ij} I - A_j^* A_i) h\, z^{\beta j}]\\
&= -e_{\emptyset} \otimes \tilde C A_i h + \sum_{\beta\in\tilde\Lambda,\, j=1,\dots,d} e_j \otimes e_{\bar\beta} \otimes \tilde C(A_{\bar\beta})^*(\delta_{ij} I - A_j^* A_i) h)\\
&= -e_{\emptyset} \otimes \gamma D_{*,A} A_i h + \sum_{\beta\in\tilde\Lambda,\, j=1,\dots,d} e_j \otimes e_{\bar\beta} \otimes \gamma D_{*,A}(A_{\bar\beta})^*(\delta_{ij} I - A_j^* A_i) h.
\end{aligned}$$

Equation (3.8) yields

$$\begin{aligned}
\Psi_C \Theta_{U,\tilde U}((D_E)_i h z^{\emptyset}) &= \Theta_{C,E}(e_{\emptyset} \otimes (D_E)_i h\\
&= \Theta_{C,E} \Psi_E((D_E)_i h z^{\emptyset}). \qquad (3.15)
\end{aligned}$$

We infer from equations (3.12) and (3.15) that

$$\Psi_C \Theta_{U,\tilde U} = \Theta_{C,E} \Psi_E.$$

□

We extend the unitary Ψ_E to a unitary (also denoted by Ψ_E) from $\mathcal{H}_A \oplus \ell^2(\tilde\Lambda, \mathcal{D}_E)$ onto $\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E)$ by

$$\Psi_E\Big(h_a \oplus \sum_{\alpha\in\tilde\Lambda} \eta_\alpha z^\alpha\Big) = \Big(h_a \oplus \sum_{\alpha\in\tilde\Lambda} e_{\bar\alpha} \otimes \eta_\alpha\Big),$$

where $h_a \in \mathcal{H}_A$ and $\eta_\alpha \in \mathcal{D}_E$. Using the unitaries Ψ_C, Ψ_E, and the coisometry W of § 2 we define Ψ_W by the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}_A \oplus (\Gamma \otimes \mathcal{D}_E) & \xrightarrow{\;W\;} & \Gamma \otimes \mathcal{D}_C\\
\Big\downarrow {\scriptstyle \Psi_E^{-1}} & & \Big\downarrow {\scriptstyle \Psi_C^{-1}}\\
\mathcal{H}_A \oplus \ell^2(\tilde\Lambda, \mathcal{D}_E) & \xrightarrow{\;\Psi_W\;} & \ell^2(\tilde\Lambda, \mathcal{D}_C),
\end{array} \qquad (3.16)$$

i.e., $\Psi_W = \Psi_C^{-1} W \Psi_E$. Similar to Theorem 5.1 of [4] we have

Theorem 3.5. *The operator Ψ_W satisfies the relation*

$$\Psi_W|_{\ell^2(\tilde{\Lambda},\mathcal{D}_E)} = M_{\Theta_{U,\tilde{U}}}.$$

Observe that we also obtain

$$W|_{e_\emptyset\otimes\mathcal{D}_E} = \Theta_{C,E}.$$

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