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Direct and reverse inclusions for strongly multiple summing operators

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Abstract. We prove some direct and reverse inclusion results for strongly summing and strongly multiple summing operators under the assumption that the range has finite cotype.

Keywords. p-Summing; multilinear operator; multiple summing operator; strongly multiple summing operator.

1. Introduction

The role of absolutely summing linear operators in Banach spaces theory is well established as the reader can see in the beautiful and profound monographs [4, 6, 13, 14, 18]. Following the pioneering work of Pietsch [15], in the last decade several classes of multilinear maps have been investigated as extensions of the linear concept of absolutely summing operators (see [1–3, 5, 7, 9–12, 16, 17] and the references therein).

One of these classes is that of strongly multiple p-summing multilinear operators which was introduced by Botelho and Pellegrino in [3] and has not been explored more since then. Pellegrino and Santos [10] raise the question whether there are true inclusion results for the class of strongly multiple summing operators. The main purpose of this paper is to show that the results proven in [16] for multiple summing operators can be adapted to obtain similar results for strongly multiple summing operators (see Proposition 2.4, Theorem 2.6, Corollary 2.7 and Corollary 2.8).

Throughout this paper by X, X_1, etc. we denote Banach spaces, $L(X_1, \ldots, X_n; Y)$ denote the Banach space of all bounded n-linear operators $U : X_1 \times \cdots \times X_n \to Y$. We use standard notations and notions from Banach space theory, as presented in [4, 6, 13, 14, 18].

We recall some definitions and notations. Given $1 \leq p < \infty$, a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, for a finite system $(x_i)_{1 \leq i \leq n} \subset X$ we write $w_p(x_i \mid 1 \leq i \leq n)$ to denote $\sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{1/p}$. Let $X_1, \ldots, X_n$ be Banach spaces and $1 \leq p < \infty$. If $(x_i^1, \ldots, x_i^n) \in X_1 \times \cdots \times X_n$ $(1 \leq i \leq m)$, we write $w_p((x_i^1, \ldots, x_i^n) \mid 1 \leq i \leq m)$ to denote $\sup\{(\sum_{i=1}^m |\psi(x_i^1, \ldots, x_i^n)|^p)^{1/p} \mid \psi \in L(X_1, \ldots, X_n; \mathbb{K}), \|\psi\| \leq 1\}$; similarly if $(x_{i_j}^j)_{1 \leq i_j \leq m_j} \subset X_j$ $(1 \leq j \leq n)$ we write

$w_p((x^1_{i_1}, \ldots, x^n_{i_n}) \mid 1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n)$ to denote

$$\sup \left\{ \left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} |\psi(x^1_{i_1}, \ldots, x^n_{i_n})|^p \right)^{\frac{1}{p}} \middle| \psi \in L(X_1, \ldots, X_n; \mathbb{K}), \|\psi\| \le 1 \right\}.$$

In [7], Dimant introduces the concept of strongly p-summing operators.

Let $1 \le p < \infty$. A bounded n-linear operator $U : X_1 \times \cdots \times X_n \to Y$ is called strongly p-summing if and only if there exists $C \ge 0$ such that for each $(x^j_i)_{1 \le i \le m} \subset X_j$ $(1 \le j \le n)$ the following relation holds:

$$\left(\sum_{i=1}^{m} \|U(x^1_i, \ldots, x^n_i)\|^p \right)^{\frac{1}{p}} \le C w_p((x^1_i, \ldots, x^n_i) \mid 1 \le i \le m).$$

We denote by $\pi^{\text{strong}}_p(U) = \inf\{C \ge 0 \mid C \text{ as above}\}$ the strongly p-summing norm of U and by $\Pi^{\text{strong}}_p(X_1, \ldots, X_n; Y)$, we denote the class of all strongly p-summing operators from $X_1 \times \cdots \times X_n$ to Y.

As a natural extension of strongly summing operators in [3], Botelho and Pellegrino introduces the concept of strongly multiple (q, p)-summing operators. Let $1 \le p \le q < \infty$. A bounded n-linear operator $U : X_1 \times \cdots \times X_n \to Y$ is called strongly multiple (q, p)-summing if and only if there exists $C \ge 0$ such that for each $(x^j_{i_j})_{1 \le i_j \le m_j} \subset X_j (1 \le j \le n)$ the following relation holds:

$$\left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \|U(x^1_{i_1}, \ldots, x^n_{i_n})\|^q \right)^{\frac{1}{q}} \le C w_p((x^1_{i_1}, \ldots, x^n_{i_n}) \mid 1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n).$$

We denote by $\pi^{\text{strong-mult}}_{q,p}(U) = \inf\{C \ge 0 \mid C \text{ as above}\}$ the strongly multiple (q, p)-summing norm of U and by $\Pi^{\text{strong-mult}}_{q,p}(X_1, \ldots, X_n; Y)$, we denote the class of all strongly multiple (q, p)-summing operators from $X_1 \times \cdots \times X_n$ to Y. When $p = q$ we write $\Pi^{\text{strong-mult}}_p$ instead of $\Pi^{\text{strong-mult}}_{p,p}$.

Multiple summing operators introduced independently by Matos in [9], and independently by Bombal *et al* in [2], seem to be the most successful extension of the linear case (see [5, 11, 12, 16] and references therein).

Let $1 \le p \le q < \infty$. A bounded n-linear operator $U : X_1 \times \cdots \times X_n \to Y$ is called multiple (q, p)-summing, if there exists a constant $C \ge 0$ such that for every choice of systems $(x^j_{i_j})_{1 \le i_j \le m_j} \subset X_j$ $(1 \le j \le n)$ the following relation holds:

$$\left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \|U(x^1_{i_1}, \ldots, x^n_{i_n})\|^q \right)^{\frac{1}{q}} \le C w_p(x^1_{i_1} \mid 1 \le i_1 \le m_1) \cdots w_p(x^n_{i_n} \mid 1 \le i_n \le m_m).$$

We denote by $\pi^{\mathrm{mult}}_{q,p}(U) = \inf\{C \geq 0 \mid C \text{ as above}\}$ the multiple (q,p)-summing norm of U and by $\Pi^{\mathrm{mult}}_{q,p}(X_1,\ldots,X_n;Y)$, we denote the class of all multiple (q,p)-summing operators from $X_1 \times \cdots \times X_n$ to Y. When $p = q$ we write Π^{mult}_{p} instead of $\Pi^{\mathrm{mult}}_{p,p}$.

Recall that, if $0 < p < \infty$, Z a Banach space and $(z_i)_{1\leq i\leq n} \subset Z$, the Rademacher means are defined by $\rho_p(z_i \mid 1 \leq i \leq n) := \left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)\, z_i\right\|^p \mathrm{d}t\right)^{1/p}$ (see [4, 6, 13, 14, 18]). By $(r_n)_{n\in\mathbb{N}}$ we denote the sequence of Rademacher functions. We recall Khinchin's and Kahane's inequalities.

Khinchin's inequality. If $0 < p < \infty$, there are positive constants A_p, B_p, called Khinchin's constants, such that

$$A_p\left(\sum_{i=1}^n |a_i|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left|\sum_{i=1}^n a_i r_i(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq B_p\left(\sum_{i=1}^n |a_i|^2\right)^{\frac{1}{2}}$$

and each choice of scalars $a_1,\ldots,a_n$ (see [4, 6]).

Kahane–Khinchin's inequality. If $0 < p,q < \infty$, there exists a constant $K_{p,q} > 0$, called Kahane–Khinchin's constant such that

$$\left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)\, x_i\right\|^q \mathrm{d}t\right)^{\frac{1}{q}} \leq K_{p,q}\left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)\, x_i\right\|^p \mathrm{d}t\right)^{\frac{1}{p}}$$

for each Banach space X and each choice of elements $x_1,\ldots,x_n \in X$ (see [6]).

Let $2 \leq q < \infty$. A Banach space X is said to have cotype q if there exists a constant $C \geq 0$ such that for every elements $x_1,\ldots,x_n \in X$ the following relation holds: $\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leq C\rho_2(x_i \mid 1 \leq i \leq n)$ and the cotype q constant of X is denoted by $C_q(X) = \inf\{C \mid C \text{ as above}\}$. When we say that a Banach space has cotype q we always understand $2 \leq q < \infty$ (see [4, 6, 13, 14, 18]).

For $0 < p < \infty$, Z a Banach space, n a natural number, $\left(z_{i_1\cdots i_n}\right)_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \subset Z$ we define the multiple Rademacher means by

$$\rho_p\left(z_{i_1\cdots i_n} \mid 1 \leq i_1 \leq m_1, \ldots, 1 \leq i_n \leq m_n\right)$$
$$:= \left(\int_{[0,1]^n} \left\|\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} r_{i_1}(t_1)\cdots r_{i_n}(t_n)\, z_{i_1\cdots i_n}\right\|^p \mathrm{d}t_1\cdots \mathrm{d}t_n\right)^{\frac{1}{p}}.$$

We recall the following inequality.

Multiple Khinchin's inequality. If $0 < p < \infty$, then

$$[A_p]^n \left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} |a_{i_1\cdots i_n}|^2 \right)^{\frac{1}{2}}$$
$$\leq \left(\int_{[0,1]^n} \left| \sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} r_{i_1}(t_1)\cdots r_{i_n}(t_n)\, a_{i_1\cdots i_n} \right|^p \mathrm{d}t_1\cdots \mathrm{d}t_n \right)^{\frac{1}{p}}$$
$$\leq [B_p]^n \left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} |a_{i_1\cdots i_n}|^2 \right)^{\frac{1}{2}}$$

for each choice of scalars $\left(a_{i_1\cdots i_n}\right)_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n}$, where A_p, B_p are Khinchin's constants (see page 455 of [4] or chapter IX, Problem 3.6 of [8]).

Multiple Kahane–Khinchin's inequality. If $0 < p, q < \infty$, then

$$\rho_q(z_{i_1\cdots i_n} \mid 1 \leq i_1 \leq m_1, \ldots, 1 \leq i_n \leq m_n)$$
$$\leq [K_{p,q}]^n \rho_p(z_{i_1\cdots i_n} \mid 1 \leq i_1 \leq m_1, \ldots, 1 \leq i_n \leq m_n)$$

for each Banach space Z and each choice of elements $\left(z_{i_1\cdots i_n}\right)_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \subset Z$, where $K_{p,q}$ are Kahane–Khinchin's constants (see Proposition 3.1 of [1]).

2. The results

2.1 *The case of strongly summing operators*

In this section we test whether, the well-known inclusion in the linear case, $\Pi_p(X, Y) \subset \Pi_{q,2}(X, Y)$ if Y has cotype q (see Theorem 11.13, page 222 of [6]) is true in the case of strongly summing operators. To this end, as is natural, we follow ideas similar to those in the linear case.

Lemma 2.1. *Let $U : X_1 \times \cdots \times X_n \to Y$ be a bounded n-linear operator and $1 \leq r < \infty$. If $n \geq 2$ then, for each $(x_i^j)_{1\leq i\leq m} \subset X_j (1 \leq j \leq n)$ the following relation holds*:

$$\rho_r(U(x_i^1, \ldots, x_i^n) \mid 1 \leq i \leq m)$$
$$\leq \left(\int_{[0,1]^n} \|U(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_n(t_1, \ldots, t_{n-1}, s))\|^r \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}\mathrm{d}s \right)^{\frac{1}{r}},$$

where $x_1 : [0, 1] \to X_1, \ldots, x_{n-1} : [0, 1] \to X_{n-1}$, $x_n : [0, 1]^n \to X_n$ are defined by

$$x_1(t_1) = \sum_{i=1}^{m} x_i^1 r_i(t_1), \ldots, x_{n-1}(t_{n-1}) = \sum_{i=1}^{m} x_i^{n-1} r_i(t_{n-1}),$$
$$x_n(t_1, \ldots, t_{n-1}, s) = \sum_{i=1}^{m} x_i^n r_i(t_1) \cdots r_i(t_{n-1})\, r_i(s).$$

Proof. Define $h : [0, 1] \to Y$ by $h(s) = \sum_{i=1}^{m} r_i(s)\, U\left(x_i^1, \ldots, x_i^n\right)$. From a well-known result (see e.g. Lemma 2 of [17]), for each $s \in [0, 1]$ we have

$$\begin{aligned} h(s) &= \sum_{i=1}^{m} U(x_i^1, \ldots, x_i^{n-1}, r_i(s)\, x_i^n) \\ &= \int_{[0,1]^{n-1}} U\left(\sum_{i=1}^{m} x_i^1 r_i(t_1), \ldots, \sum_{i=1}^{m} x_i^{n-1} r_i(t_{n-1}),\right. \\ &\quad \left.\sum_{i=1}^{m} x_i^n r_i(t_1) \cdots r_i(t_{n-1})\, r_i(s)\right) \mathrm{d}\mathbf{t} \\ &= \int_{[0,1]^{n-1}} U\left(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_n(t_1, \ldots, t_{n-1}, s)\right) \mathrm{d}\mathbf{t}. \end{aligned}$$

In the above sequel, we write $\mathrm{d}\mathbf{t}$ instead of $\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}$. Using the known inequalities: if $f : S \to Z$ is a r-Bochner integrable function with respect to the probability measure μ, then

$$\left\| \int_S f \,\mathrm{d}\mu \right\| \le \int_S \|f\| \,\mathrm{d}\mu \le \left(\int_S \|f\|^r \,\mathrm{d}\mu\right)^{\frac{1}{r}}.$$

We deduce that for each $s \in [0, 1]$, we have

$$\|h(s)\| \le \left(\int_{[0,1]^{n-1}} \|U(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_n(t_1, \ldots, t_{n-1}, s))\|^r \,\mathrm{d}\mathbf{t}\right)^{\frac{1}{r}}.$$

Then by Fubini's theorem

$$\begin{aligned} \rho_r(U(x_i^1, \ldots, x_i^n) \mid 1 \le i \le m) &= \left(\int_0^1 \|h(s)\|^r \,\mathrm{d}s\right)^{\frac{1}{r}} \\ &\le \left(\int_{[0,1]^n} \|U(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_n(t_1, \ldots, t_{n-1}, s))\|^r \,\mathrm{d}\mathbf{t}\,\mathrm{d}s\right)^{\frac{1}{r}}. \end{aligned}$$

□

PROPOSITION 2.2

Let $X_1, \ldots, X_n$, Y be Banach spaces and $1 \le r < \infty$. If $U \in \Pi_r^{\mathrm{strong}}(X_1, \ldots, X_n; Y)$, then for each $(x_i^j)_{1 \le i \le m} \subset X_j$ $(1 \le j \le n)$ the following relation holds:

$$\begin{aligned} &\rho_r(U(x_i^1, \ldots, x_i^n) \mid 1 \le i \le m) \\ &\quad \le B_r\left[K_{2,r}\right]^{n-1} \pi_r^{\mathrm{strong}}(U)\, w_2((x_{i_1}^1, \ldots, x_{i_n}^n) \mid 1 \le i_1 \le m, \ldots, 1 \le i_n \le m). \end{aligned}$$

Proof. The case $n = 1$ is well known (see Proposition 12.5, page 234 of [6]). Let $n \geq 2$. From Lemma 2.1, we have

$$\rho_r(U(x_i^1, \dots, x_i^n) \mid 1 \leq i \leq m) \leq \left(\int_{[0,1]^n} \| U(x_1(t_1), \dots, x_{n-1}(t_{n-1}), x_n(t_1, \dots, t_{n-1}, s)) \|^r \, \mathrm{d}\mathbf{t} \, \mathrm{d}s \right)^{\frac{1}{r}}. \tag{1}$$

Since U is strongly r-summing from the domination theorem (see Proposition 1.2 of [7]) there exists a regular Borel probability μ on $\Psi = B_{L(X_1,\dots,X_n;\mathbb{K})}$ such that for each $(x^1, \dots, x^n) \in X_1 \times \cdots \times X_n$,

$$\| U(x^1, \dots, x^n) \| \leq \pi_r^{\text{strong}}(U) \left(\int_\Psi |\psi(x^1, \dots, x^n)|^r \, \mathrm{d}\mu(\psi) \right)^{\frac{1}{r}}. \tag{2}$$

From (1) and (2) we get

$$\begin{aligned}
\rho_r\big(U(x_i^1, \dots, x_i^n) \mid 1 \leq i \leq m\big) &\leq \pi_r^{\text{strong}}(U) \Bigg(\int_{[0,1]^n} \int_\Psi |\psi(x_1(t_1), \dots, x_{n-1}(t_{n-1}), \\
&\qquad x_n(t_1, \dots, t_{n-1}, s))|^r \, \mathrm{d}\mu(\psi) \, \mathrm{d}\mathbf{t} \mathrm{d}s \Bigg)^{\frac{1}{r}} \\
&= \pi_r^{\text{strong}}(U) \Bigg(\int_\Psi \int_{[0,1]^n} |\psi(x_1(t_1), \dots, x_{n-1}(t_{n-1}), \\
&\qquad x_n(t_1, \dots, t_{n-1}, s))|^r \, \mathrm{d}s \mathrm{d}\mathbf{t} \mathrm{d}\mu(\psi) \Bigg)^{\frac{1}{r}}.
\end{aligned} \tag{3}$$

In the last equality we have used Fubini's theorem.

Let $\psi \in \Psi$ and $(t_1, \dots, t_{n-1}) \in [0,1]^{n-1}$. From the linearity of ψ in the last variable, the definition of $x_n(t_1, \dots, t_{n-1}, s)$ and Khinchin's inequality we get

$$\begin{aligned}
&\left(\int_0^1 |\psi(x_1(t_1), \dots, x_{n-1}(t_{n-1}), x_n(t_1, \dots, t_{n-1}, s))|^r \, \mathrm{d}s \right)^{\frac{1}{r}} \\
&= \left(\int_0^1 \left| \sum_{i_n=1}^m r_{i_n}(s) \, r_{i_n}(t_1) \cdots r_{i_n}(t_{n-1}) \psi\big(x_1(t_1), \dots, x_{n-1}(t_{n-1}), x_{i_n}^n\big) \right|^r \mathrm{d}s \right)^{\frac{1}{r}} \\
&\leq B_r \left(\sum_{i_n=1}^m \left| \psi\big(x_1(t_1), \dots, x_{n-1}(t_{n-1}), x_{i_n}^n\big) \right|^2 \right)^{\frac{1}{2}} \\
&= B_r \big\| \big(\psi\big(x_1(t_1), \dots, x_{n-1}(t_{n-1}), x_1^n\big), \dots, \\
&\qquad \psi\big(x_1(t_1), \dots, x_{n-1}(t_{n-1}), x_m^n\big) \big) \big\|_{l_2^m}.
\end{aligned} \tag{4}$$

The linearity with respect to each variable $1, \ldots, n-1$ will give us

$$\left(\psi\left(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_1^n\right), \ldots, \psi\left(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_m^n\right)\right)$$
$$= \sum_{i_1,\ldots,i_{n-1}=1}^{m} a_{i_1\cdots i_{n-1}} r_{i_1}(t_1)\cdots r_{i_{n-1}}(t_{n-1}), \tag{5}$$

where $a_{i_1\cdots i_{n-1}} = \left(\psi\left(x_{i_1}^1, \ldots, x_{i_{n-1}}^{n-1}, x_1^n\right), \ldots, \psi\left(x_{i_1}^1, \ldots, x_{i_{n-1}}^{n-1}, x_m^n\right)\right) \in l_2^m$. If we use (5), the relation (4) becomes

$$\left(\int_0^1 \left|\psi\left(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_n(t_1, \ldots, t_{n-1}, s)\right)\right|^r \mathrm{d}s\right)^{\frac{1}{r}}$$
$$\leq B_r \left\|\sum_{i_1,\ldots,i_{n-1}=1}^{m} a_{i_1\cdots i_{n-1}} r_{i_1}(t_1)\cdots r_{i_{n-1}}(t_{n-1})\right\|_{l_2^m}$$

and, from here, we deduce

$$\left(\int_{[0,1]^{n-1}}\int_0^1 \left|\psi\left(x_1(t_1), \ldots, x_{n-1}(t_{n-1}), x_n(t_1, \ldots, t_{n-1}, s)\right)\right|^r \mathrm{d}s\mathrm{d}\mathbf{t}\right)^{\frac{1}{r}}$$
$$\leq B_r \left(\int_{[0,1]^{n-1}} \left\|\sum_{i_1,\ldots,i_{n-1}=1}^{m} a_{i_1\cdots i_{n-1}} r_{i_1}(t_1)\cdots r_{i_{n-1}}(t_{n-1})\right\|_{l_2^m}^r \mathrm{d}\mathbf{t}\right)^{\frac{1}{r}}. \tag{6}$$

From multiple Kahane–Khinchin's inequality we get

$$\left(\int_{[0,1]^{n-1}} \left\|\sum_{i_1,\ldots,i_{n-1}=1}^{m} a_{i_1\cdots i_{n-1}} r_{i_1}(t_1)\cdots r_{i_{n-1}}(t_{n-1})\right\|_{l_2^m}^r \mathrm{d}\mathbf{t}\right)^{\frac{1}{r}}$$
$$\leq \left[K_{2,r}\right]^{n-1}\left(\int_{[0,1]^{n-1}} \left\|\sum_{i_1,\ldots,i_{n-1}=1}^{m} a_{i_1\cdots i_{n-1}} r_{i_1}(t_1)\cdots r_{i_{n-1}}(t_{n-1})\right\|_{l_2^m}^2 \mathrm{d}\mathbf{t}\right)^{\frac{1}{2}}$$
$$= \left[K_{2,r}\right]^{n-1}\left(\sum_{i_1,\ldots,i_{n-1}=1}^{m} \left\|a_{i_1\cdots i_{n-1}}\right\|_{l_2^m}^2\right)^{\frac{1}{2}}$$
$$= \left[K_{2,r}\right]^{n-1}\left(\sum_{i_1,\ldots,i_n=1}^{m} \left|\psi\left(x_{i_1}^1, \ldots, x_{i_n}^n\right)\right|^2\right)^{\frac{1}{2}}. \tag{7}$$

Then from (3), (6), (7) and the fact that μ is a probability we get

$$\rho_r(U(x_i^1,\ldots,x_i^n)\mid 1\le i\le m)\le B_r[K_{2,r}]^{n-1}\pi_r^{\text{strong}}(U)$$

$$\cdot\sup\left\{\left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n}|\psi(x_{i_1}^1,\ldots,x_{i_n}^n)|^2\right)^{\frac{1}{2}}\middle|\,\psi\in L(X_1,\ldots,X_n:\mathbb{K}),\|\psi\|\le 1\right\}$$

and the proof will be completed. □

The next proposition is the multilinear analogue of the inclusion in the linear case, $\Pi_p(X,Y)\subset\Pi_{q,2}(X,Y)$ if Y has cotype q, in the case of strongly summing operators.

PROPOSITION 2.3

Let $X_1,\ldots,X_n$, Y be Banach spaces and $1\le r<\infty$. If Y has cotype q and $U\in\Pi_r^{\text{strong}}(X_1,\ldots,X_n;Y)$, then for each $(x_i^j)_{1\le i\le m}\subset X_j$ $(1\le j\le n)$ the following relation holds:

$$l_q(U(x_i^1,\ldots,x_i^n)\mid 1\le i\le m)$$
$$\le B_r[K_{2,r}]^nC_q(Y)\pi_r^{\text{strong}}(U)\,w_2((x_{i_1}^1,\ldots,x_{i_n}^n)\mid 1$$
$$\le i_1\le m,\ldots,1\le i_n\le m).$$

Proof. Using the fact that Y has cotype q and Kahane–Khinchin's inequality we have

$$l_q(U(x_i^1,\ldots,x_i^n)\,|\,1\le i\le m)\le C_q(Y)K_{2,r}\rho_r(U(x_i^1,\ldots,x_i^n)\,|\,1\le i\le m).$$

From Proposition 2.2 we get the statement. This proposition will be further improved, see Corollary 2.7. □

The case of strongly multiple summing operators.

In this section we test again whether, the well-known inclusion in the linear case, $\Pi_p(X,Y)\subset\Pi_{q,2}(X,Y)$ if Y has cotype q, is true in the case of strongly multiple summing operators. To this end, we follow ideas similar to those in [16]. The next result is similar to that shown in Proposition 2 of [16].

PROPOSITION 2.4

Let $X_1,\ldots,X_n,Y$ be Banach spaces and $1\le r<\infty$. If $U\in\Pi_r^{\text{strong-mult}}(X_1,\ldots,X_n;Y)$, then for each $(x_{i_j}^j)_{1\le i_j\le m_j}\subset X_j(1\le j\le n)$ the following relation holds:

$$\rho_r(U(x_{i_1}^1,\ldots,x_{i_n}^n)\mid 1\le i_1\le m_1,\ldots,1\le i_n\le m_n)$$
$$\le[B_r]^n\pi_r^{\text{strong-mult}}(U)\,w_2((x_{i_1}^1,\ldots,x_{i_n}^n)\,|\,1\le i_1\le m_1,\ldots,1\le i_n\le m_n).$$

Proof. We first prove the case $n = 2$. So, let $U \in \Pi_r^{\text{strong-mult}}(X, Y; Z)$. With the same notations as in Proposition 1(b) in [16], from the fact that U is strongly multiple r-summing we have

$$\rho_r(U(x_i, y_j) \mid 1 \le i \le n; 1 \le j \le m) = \left(\sum_{(\varepsilon,\eta)\in D_n\times D_m} \left\| U\left(\frac{x_\varepsilon}{2^{\frac{n}{r}}}, \frac{y_\eta}{2^{\frac{m}{r}}}\right) \right\|^r \right)^{\frac{1}{r}}$$
$$\le \pi_r^{\text{strong-mult}}(U)\, w_r\left(\left(\frac{x_\varepsilon}{2^{\frac{n}{r}}}, \frac{y_\eta}{2^{\frac{m}{r}}} \right) \middle| (\varepsilon, \eta) \in D_n \times D_m \right). \tag{8}$$

From the definition and double Khinchin's inequality we deduce

$$w_r\left(\left(\frac{x_\varepsilon}{2^{\frac{n}{r}}}, \frac{y_\eta}{2^{\frac{m}{r}}} \right) \middle| (\varepsilon, \eta) \in D_n \times D_m \right)$$
$$= \sup_{\|\psi\|\le 1} \left(\frac{1}{2^{n+m}} \sum_{(\varepsilon,\eta)\in D_n\times D_m} |\psi(x_\varepsilon, y_\eta)|^r \right)^{\frac{1}{r}}$$
$$= \sup_{\|\psi\|\le 1} \left(\iint_{[0,1]^2} \left| \sum_{i,j=1}^{n,m} r_i(t)\, r_j(s)\, \psi(x_i, y_j) \right|^r \mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{r}}$$
$$\le [B_r]^2 \sup_{\|\psi\|\le 1} \left(\sum_{i,j=1}^{n,m} |\psi(x_i, y_j)|^2 \right)^{\frac{1}{2}}$$
$$= [B_r]^2\, w_2((x_i, y_j) \mid 1 \le i \le n; 1 \le j \le m). \tag{9}$$

From (8) and (9) we obtain the statement.

The general case can be proved in the same way if we use the multilinear version of Proposition 1(b) in [16] and multiple Khinchin's inequality. We omit the details. □

Lemma 2.5. *If Y has cotype q, $1 \le r < \infty$ and $(y_{i_1\cdots i_n})_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \subset Y$, then*

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \|y_{i_1\cdots i_n}\|^q \right)^{\frac{1}{q}}$$
$$\le [C_q(Y) K_{r,2}]^n \rho_r(y_{i_1\cdots i_n} \mid 1 \le i_1 \le m_1, \dots, 1 \le i_n \le m_n),$$

where $K_{r,2}$ is the Kahane–Khinchin constant.

Proof. Indeed, from Lemma 3 of [16] or Lemma 1 of [5] and multiple Kahane–Khinchin's inequality we have

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \|y_{i_1\cdots i_n}\|^q \right)^{\frac{1}{q}}$$
$$\le [C_q(Y)]^n \rho_2(y_{i_1\cdots i_n} \mid 1 \le i_1 \le m_1, \dots, 1 \le i_n \le m_n)$$
$$\le [C_q(Y)]^n [K_{r,2}]^n \rho_r(y_{i_1\cdots i_n} \mid 1 \le i_1 \le m_1, \dots, 1 \le i_n \le m_n).$$

□

The next result is similar to that shown in Theorem 6, Theorem 8 of [16].

Theorem 2.6.

(i) *If Y has cotype q, then for any Banach spaces $X_1, \ldots, X_n$, all $1 \le r < \infty$,*

$$\Pi_r^{\text{strong-mult}}(X_1, \ldots, X_n; Y) \subset \Pi_{q,2}^{\text{strong-mult}}(X_1, \ldots, X_n; Y).$$

(ii) *If Y has cotype 2, then for any Banach spaces $X_1, \ldots, X_n$, all $1 \le r \le 2$,*

$$\Pi_r^{\text{strong-mult}}(X_1, \ldots, X_n; Y) \subset \Pi_2^{\text{strong-mult}}(X_1, \ldots, X_n; Y).$$

(iii) *If Y has cotype 2, then for any Banach spaces $X_1, \ldots, X_n$, all $2 \le r < \infty$,*

$$\Pi_r^{\text{strong-mult}}(X_1, \ldots, X_n; Y) \subset \Pi_2^{\text{strong-mult}}(X_1, \ldots, X_n; Y).$$

Proof.

(i) Let $U \in \Pi_r^{\text{strong-mult}}(X_1, \ldots, X_n; Y)$ and let also $(x_{i_j}^j)_{1 \le i_j \le m_j} \subset X_j$ $(1 \le j \le n)$. Since Y has cotype q from Lemma 2.5 we have

$$\left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \|U(x_{i_1}^1, \ldots, x_{i_n}^n)\|^q\right)^{\frac{1}{q}} \le [C_q(Y)\, K_{r,2}]^n \rho_r(U(x_{i_1}^1, \ldots, x_{i_n}^n) \,|\, 1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n). \tag{10}$$

Further from Proposition 2.4 we have

$$\begin{aligned}\rho_r(U(x_{i_1}^1, \ldots, x_{i_n}^n) \mid 1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n) \\ \le [B_r]^n \pi_r^{\text{strong-mult}}(U)\, w_2((x_{i_1}^1, \ldots, x_{i_n}^n) \mid 1 \\ \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n).\end{aligned} \tag{11}$$

Then from (10) and (11) we get

$$\left(\sum_{i_1,\ldots,i_n=1}^{m_1,\ldots,m_n} \|U(x_{i_1}^1, \ldots, x_{i_n}^n)\|^q\right)^{\frac{1}{q}} \le [C_q(Y)\, B_r K_{r,2}]^n \pi_r^{\text{strong-mult}}(U)$$

$$w_2((x_{i_1}^1, \ldots, x_{i_n}^n) \mid 1 \le i_1 \le m_1, \ldots, 1 \le i_n \le m_n),$$

i.e. $U \in \Pi_{q,2}^{\text{strong-mult}}(X_1, \ldots, X_n; Y)$ and

$$\pi_{q,2}^{\text{strong-mult}}(U) \le [C_q(Y)\, B_r K_{r,2}]^n \pi_r^{\text{strong-mult}}(U).$$

(ii) and (iii) are particular cases of (i).

From the obvious inclusion, $\Pi_r^{\text{strong}}(\cdots) \subset \Pi_r^{\text{strong-mult}}(\cdots)$ (see Proposition 5.2(ii) of [3]) and Theorem 2.6, we get the next result which is an improvement of Proposition 2.3. □

COROLLARY 2.7

(i) *If Y has cotype q, then for any Banach spaces $X_1, \ldots, X_n$, all $1 \leq r < \infty$,*

$$\Pi_r^{\text{strong}}(X_1, \ldots, X_n; Y) \subset \Pi_{q,2}^{\text{strong-mult}}(X_1, \ldots, X_n; Y).$$

(ii) *If Y has cotype 2, then for any Banach spaces $X_1, \ldots, X_n$, all $1 \leq r \leq 2$,*

$$\Pi_r^{\text{strong}}(X_1, \ldots, X_n; Y) \subset \Pi_2^{\text{strong-mult}}(X_1, \ldots, X_n; Y).$$

(iii) *If Y has cotype 2, then for any Banach spaces $X_1, \ldots, X_n$, all $2 \leq r < \infty$,*

$$\Pi_r^{\text{strong}}(X_1, \ldots, X_n; Y) \subset \Pi_2^{\text{strong-mult}}(X_1, ..., X_n; Y).$$

Since we deduce easily from the definitions that if $L(X_1, \ldots, X_n; \mathbb{K}) = \Pi_2^{\text{mult}}(X_1, \ldots, X_n; \mathbb{K})$, then $\Pi_{q,2}^{\text{strong-mult}}(X_1, \ldots, X_n; Y) = \Pi_{q,2}^{\text{mult}}(X_1, \ldots, X_n; Y)$ for each Banach space Y (see also Theorem 5.7 of [3]) and from Corollary 2.7 we get

COROLLARY 2.8

Let $X_1, \ldots, X_n$ be such that $L(X_1, \ldots, X_n; \mathbb{K}) = \Pi_2^{\text{mult}}(X_1, \ldots, X_n; \mathbb{K})$.

(i) *If Y has cotype q, then for all $1 \leq r < \infty$,*

$$\Pi_r^{\text{strong}}(X_1, \ldots, X_n; Y) \subset \Pi_{q,2}^{\text{mult}}(X_1, \ldots, X_n; Y).$$

(ii) *If Y has cotype 2, then for all $1 \leq r \leq 2$,*

$$\Pi_r^{\text{strong}}(X_1, \ldots, X_n; Y) \subset \Pi_2^{\text{mult}}(X_1, \ldots, X_n; Y).$$

(iii) *If Y has cotype 2, then for all $2 \leq r < \infty$,*

$$\Pi_r^{\text{strong}}(X_1, \ldots, X_n; Y) \subset \Pi_2^{\text{mult}}(X_1, \ldots, X_n; Y).$$

References

[1] Achour D and Mezrag L, On the Cohen strongly p-summing multilinear operators, *J. Math. Anal. Appl.* **327** (2007) 550–563

[2] Bombal F, Pérez-García D and Villanueva I, Multilinear extensions of Grothendieck's theorem, *Quart. J. Math.* **55** (2004) 441–450

[3] Botelho G and Pellegrino D, Coincidence situations for absolutely summing non-linear mappings, *Port. Math.* **64** (2007) 175–191

[4] Defant A and Floret K, Tensor norms and operator ideals, Math. Studies 176 (1993) (North-Holland)

[5] Defant A, Popa D and Schwarting U, Coordinatewise multiple summing operators in Banach spaces, *J. Funct. Anal.* **259** (2010) 220–242

[6] Diestel J, Jarchow H and Tonge A, Absolutely Summing Operators, Cambridge Stud. Adv. Math. 43 (1995) (Cambridge University Press)

[7] Dimant V, Strongly p-summing multilinear operators, *J. Math. Anal. Appl.* **278** (2003) 182–193

[8] Makarov B M, Goluzina M G, Lodkin A A and Podkoryotov A N, Selected problems in real analysis, Transl. Math. Monographs 107 A.M.S. (1992)

[9] Matos M C, Fully absolutely summing and Hilbert-Schmidt multilinear mappings, *Collect. Math.* **54** (2003) 111–136

[10] Pellegrino D and Santos J, Absolutely summing operators: a panorama, *Quaest. Math.* **34** (2011) 447–478

[11] Pérez-García D, The inclusion theorem for multiple summing operators, *Stud. Math.* **165** (2004) 275–290

[12] Pérez-García D, Comparing different classes of absolutely summing multilinear operators, *Arch. Math.* **85** (2005) 258–267

[13] Pisier G, Factorization of linear operators and geometry of Banach spaces, *Reg. Conf. Ser. Math.* **60** X (1986)

[14] Pietsch A, Operator ideals, Veb Deutscher Verlag der Wiss., Berlin (1978); North Holland (1980)

[15] Pietsch A, Ideals of multilinear functionals, in: Proc. Second Int. Conf. Operator Alg., Ideals and their Applications in Theoretical Physics (1983) (Leipzig: Teubner-Texte) pp. 185–199

[16] Popa D, Reverse inclusions for multiple summing operators, *J. Math. Anal. Appl.* **350** (2009) 360–368

[17] Popa D, Multilinear variants of Pietsch's composition theorem, *J. Math. Anal. Appl.* **370** (2010) 415–430

[18] Tomczak-Jagermann N, Banach-Mazur distances and finite dimensional operator ideals, Pitman Monographs (1989) (New York: John Wiley & Sons, Inc.) vol. 38