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On a class of smooth Frechet subalgebras of C^*-algebras

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Abstract. The paper contributes to understanding the differential structure in a C^*-algebra. Refining the Banach (D_p^*)-algebras investigated by Kissin and Shulman as noncommutative analogues of the algebra $C^p[a, b]$ of p-times continuously differentiable functions, we investigate a Frechet (D_∞^*)-subalgebra $\mathcal{B}$ of a C^*-algebra as a noncommutative analogue of the algebra $C^\infty[a, b]$ of smooth functions. Regularity properties like spectral invariance, closure under functional calculi and domain invariance of homomorphisms are derived expressing $\mathcal{B}$ as an inverse limit over n of Banach (D_n^*)-algebras. Several examples of such smooth algebras are exhibited.

Keywords. Smooth subalgebra of a C^*-algebra; spectral invariance; closure under functional calculus; Arens–Michael decomposition of a Frechet algebra; Banach (D_p^*)-algebra; Frechet (D_∞^*)-algebra.

1. Introduction and examples

Recent developments in noncommutative geometry and topology require understanding differential structures in a C^*-algebra $\mathcal{A}$ generally specified by a dense *-subalgebra $\mathcal{B}$ satisfying smoothness properties like spectral invariance, closure under holomorphic and C^∞-functional calculi, K-theory isomorphism and completeness under a suitable locally convex or normed topology finer than the relative C^*-topology. Very many examples of such smooth subalgebras of C^*-algebras have been discovered and investigated. These include noncommutative torus, noncommutative sphere and noncommutative cylinder. On the other hand, there are at least two fairly general approaches to general constructs of such smooth algebras; viz. one originated by Blackadar and Cuntz [8] based on differential seminorms defining differential Banach algebras [8], differential Frechet algebras [4], and ultimately leading to smooth subalgebras [5, 8] and C^∞-subalgebras [5] of C^*-algebras. Another general approach developed simultaneously with [8] is due to Kissin and Shulman [14] wherein the following class of Banach algebras has been investigated.

DEFINITION 1.1

Let $(\mathcal{A}, \|\cdot\|_0)$ be a C^*-algebra. Let $\mathcal{B}$ be a dense *-subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is called a *Banach (D_p^*)-subalgebra of $\mathcal{A}$* if there exists a family of seminorms $\{\|\cdot\|_i : 0 \le i \le p\}$ such that the following hold:

(1) For all $x, y \in \mathcal{B}$ and for all i, $1 \le i \le p$, $\|xy\|_i \le \|x\|_i \|y\|_i$, $\|x^*\|_i = \|x\|_i$.

(2) For each i, $1 \leq i \leq p$, there exists $D_i > 0$ such that $\|xy\|_i \leq D_i(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i)$ for all x, y in $\mathcal{B}$.
(3) $\|\cdot\|_p$ is a norm and $(\mathcal{B}, \|\cdot\|_p)$ is a Banach algebra.

A Banach (D_p^*)-subalgebra of a C^*-algebra is envisaged as a noncommutative analogue of the commutative Banach algebra $C^p[a,b]$ of p-times continuously differentiable functions on a closed and bounded interval $[a,b]$. The smoothness properties of $(\mathcal{B}, \{\|\cdot\|_i : 0 \leq i \leq p\})$ as well as ideal structure have been investigated in [14]; and in the particular case of $p = 1$, Banach (D_1^*)-subalgebras of the C^*-algebra of bounded operators (and also of compact operators) have been studied in detail in [15–17] by analysing the first-order differential structure defined by a closed symmetric operator. As a natural sequel to Banach (D_p^*)-algebras, the following infinite order version has been proposed in [2] as a noncommutative analogue of $C^\infty[a,b]$.

DEFINITION 1.2

Let $(\mathcal{A}, \|\cdot\|_0)$ be a C^*-algebra. Let $\mathcal{B}$ be a dense *-subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is called a *Frechet* (D_∞^*)*-subalgebra of* $\mathcal{A}$ if there exists a sequence of seminorms $\{\|\cdot\|_i : 0 \leq i < \infty\}$ such that the following hold:

(1) For all i, $1 \leq i < \infty$, for all x, y in $\mathcal{B}$, $\|xy\|_i \leq \|x\|_i\|y\|_i$, $\|x^*\|_i = \|x\|_i$.
(2) For each i, $1 \leq i < \infty$, there exists $D_i > 0$ such that $\|xy\|_i \leq D_i(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i)$ holds for all x, y in $\mathcal{B}$.
(3) $\mathcal{B}$ is a Hausdorff Frechet *-algebra with the topology τ defined by the seminorms $\{\|\cdot\|_i : 0 \leq i < \infty\}$.

The purpose of the present paper is to understand the differential structure in $\mathcal{A}$ defined by $\mathcal{B}$ by investigating smoothness properties of $\mathcal{B}$. The paper is organized as follows. In § 2, it is shown that if a *-subalgebra $\mathcal{C}$ of $\mathcal{A}$ contains a Frechet (D_∞^*)-subalgebra $(\mathcal{B}, \{\|\cdot\|_i : 0 \leq i < \infty\})$, and is a Banach algebra with some norm, then it contains the completion of $\mathcal{B}$ in some $\|\cdot\|_p$. This provides an analogue of an old result of Silov [21] on algebras of smooth functions. It is shown that a Frechet (D_∞^*)-subalgebra of a C^*-algebra cannot be a Banach algebra under any norm, unless it is a Banach (D_p^*)-algebra for some p, $1 \leq p < \infty$. The property of being a (D_∞^*)-subalgebra of a C^*-algebra is preserved by amalgamation of identity, tensoring with matrices, completion and by direct sums modelled on sequence spaces c_0 and c. In § 3, a Frechet (D_∞^*)-subalgebra $\mathcal{B}$ of a C^*-algebra $\mathcal{A}$ is realized as $\mathcal{B} = \varprojlim \tilde{\mathcal{B}}_k$ inverse limit of a sequence $(\tilde{\mathcal{B}}_k)$ of Banach *-algebras such that each $\tilde{\mathcal{B}}_k$ is a Banach (D_k^*)-subalgebra of $\mathcal{A}$. This is used to show that $\mathcal{B}$ is spectrally invariant in $\mathcal{A}$; $\mathcal{B}$ is closed under the holomorphic functional calculus of $\mathcal{A}$ as well the C^∞-functional calculus for self-adjoint elements of $\mathcal{A}$; and that the quotient of a Frechet (D_∞^*)-subalgebra of a C^*-algebra by an appropriate ideal is a Frechet (D_∞^*)-subalgebra of a C^*-algebra. It is also shown that a (D_∞^*)-algebra behaves like a C^*-algebra with respect to homomorphisms. As a whole, the paper is intended to be a supplement to [14, 15]. We end this section with some examples of Frechet (D_∞^*)-subalgebras of C^*-algebras.

Example 1.3. Let $C[a,b]$ be the supnorm C^*-algebra of continuous functions on $[a,b]$. Let $C^\infty[a,b]$ be the Frechet subalgebra of all C^∞-functions on $[a,b]$, the Frechet topology being defined by the sequence of norms $\{p_k\}$, where $p_k(f) = \sum_{i=0}^k \frac{\|f^{(i)}\|_\infty}{i!}$.

The algebra $C^\infty[a, b]$ is a Frechet (D^*_∞)-subalgebra of $C[a, b]$. Let $\mathcal{B}$ be a Frechet (D^*_∞)-subalgebra of a C^*-algebra $\mathcal{A}$. Then with appropriate sequences of norms, each of the vector valued function algebras $C([a, b], \mathcal{B})$, $C^\infty([a, b], \mathcal{A})$ and $C^\infty([a, b], \mathcal{B})$ is a Frechet (D^*_∞)-subalgebra of the C^*-algebra $C([a, b], \mathcal{A})$. One can also consider the multivariate analogue of this.

Example 1.4. *Differential Frechet algebras.* This class of noncommutative smooth algebras has been considered in [4, 5] as a natural sequel to differential Banach algebras considered in [8, 14]. Let $\mathcal{U}$ be a dense *-subalgebra of a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$. Let ω be the collection of all sequences of nonnegative real numbers. A *differential norm* on $\mathcal{U}$ is a map $T : \mathcal{U} \to \omega$, where $x \in \mathcal{U} \mapsto (T_0(x), T_1(x), T_2(x), \dots) \in \omega$, such that the following hold for all x, y in $\mathcal{U}$, for all scalars $\lambda \in \mathbb{C}$.

(1) $T_0(x) = \|x\|_0$.
(2) $T(x + y) \le T(x) + T(y)$ in the sense that for each k, $T_k(x + y) \le T_k(x) + T_k(y)$.
(3) $T(\lambda x) = |\lambda| T(x)$.
(4) $T(xy) \le kT(x)T(y)$; i.e. there exist nonnegative numbers $k = (k_{i,j})$ such that for each k, $T_k(xy) \le \sum_{i+j=k} k_{i,j} T_i(x) T_j(y)$.
(5) $T_k(x^*) = T_k(x)$ for all k.

Contrary to [8], we do not assume the $l^1(\mathbb{N})$-summability of T. Differential norms arise from powers of derivations and *-homomorphisms into graded normed *-algebras. Taking differential norms as basis, a general theory of smooth subalgebras was initiated in [8]; which was further developed in [5] at the level of generality of the present definition. Let $r_k = \max_{i+j\le k}(k_{i,j})$; let $p_k(x) = r_k \sum_{j=0}^{k} T_j(x), 0 \le k < \infty$. Then for each k, there exist $D_k > 0, m_k > 0$ such that $p_k(xy) \le m_k p_k(x) p_k(y)$ as well as $p_k(xy) \le D_k\{p_k(x)p_{k-1}(y) + p_{k-1}(x)p_k(y)\}$ holds for all x, y. The differential Frechet *-algebra $C^\infty(\mathcal{U}, T)$ defined by T is the completion of $\mathcal{U}$ in the sequence of seminorms (p_k); and is realized as $C^\infty(\mathcal{U}, T) = \varprojlim C^k(\mathcal{U}, T)$, the inverse limit of Banach *-algebras $C^k(\mathcal{U}, T)$, which are completions of $\mathcal{U}$ in the norms p_k. If T is closable with respect to $\|\cdot\|_0$ (i.e. for each k, the identity map $(\mathcal{U}, T_k) \to (\mathcal{U}, \|\cdot\|_0)$ is closable), then $C^\infty(\mathcal{U}, T)$ is a Frechet (D^*_∞)-subalgebra of $\mathcal{A}$; and each $C^k(\mathcal{U}, T)$ is a Banach (D^*_k)-subalgebra of $\mathcal{A}$.

Example 1.5. *Frechet algebras defined by derivations*

(i) Let $\delta : D(\delta) \to \mathcal{A}$ be a closed *-derivation defined on a dense *-subalgebra $D(\delta)$ in a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$. Let the domain $C^n(\delta) := D(\delta^n)$ be the domain of δ^n which is a dense Banach *-subalgebra of $\mathcal{A}$ with norm $\|x\|_n := \sum_{j=0}^{n}(\|\delta^j(x)\|_0/j!)$ [9]. The C^∞-domain $C^\infty(\delta) := \cap_{n=0}^{\infty} C^n(\delta)$ is a dense Frechet *-subalgebra of $\mathcal{A}$ with the topology τ defined by the seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\cup\{0\}\}$ satisfying for each n and for x, y, $\|xy\|_n \le \|x\|_n\|y\|_n$ and for each $n \ge 1$, $\|xy\|_n \le \|x\|_n\|y\|_{n-1} + \|x\|_{n-1}\|y\|_n$. Thus $(\mathcal{B}, \tau)$ is a Frechet (D^*_∞)-subalgebra of $\mathcal{A}$. Notice that if δ is a generator, then by Theorem 9, p. 412 of [14], the norms $\|\cdot\|_n$ are the first order norms in the sense that there exist $D_n > 0$ such that $\|xy\|_n \le D_n\{\|x\|_n\|y\|_0 + \|x\|_0\|y\|_n\}$ for all x, y in $C^\infty(\delta)$.
(ii) This is a (D^*_∞)-analogue of Example 2, p. 406 of [14]. Let $(\mathcal{A}, \|\cdot\|)$ be a C^*-algebra. Let (δ_k) be a sequence of closed *-derivations on $\mathcal{A}$, δ_k having domain $D(\delta_k)$,

a dense *-subalgebra of $\mathcal{A}$. Let $1 \leq p < \infty$ and let $K_p = (\delta_1, \delta_2, \ldots, \delta_p)$ be a set of p number of derivations from this. The following Banach (D_p^*)-algebra has been constructed in [14]. For any subset $S = \{k_1, k_2, k_3, \ldots, k_m\}, k_1 > k_2 > k_3 > \cdots > k_m$ of $\{p, p-1, p-2, \ldots, 2, 1\}$, let $\delta_S(x) = \delta_{k_1}\delta_{k_2}\cdots\delta_{k_m}(x)$ having domain $D(\delta_S)$. Let $D(\delta_p\delta_{p-1}\cdots\delta_2\delta_1) = \cap_S\{D(\delta_S)\}$, S varying over all decreasing subsets of $\{p, p-1, \ldots, 2, 1\}$. For $x \in D(\delta_p\delta_{p-1}\cdots\delta_1)$, define $\|x\|_0 = \|x\|, \|x\|_i = \sum_S\{\|\delta_S(x)\| :$ S as above$\}$. For all appropriate x, y, the following are noted in [14].

(1) For any $S \subset \{p, p-1, \ldots, 2, 1\}$, $\delta_S(xy) = \sum_Q \delta_Q(x)\delta_{S-Q}(y)$.
(2) For each i, $\|xy\|_i \leq \sum_S \sum_Q \|\delta_Q(x)\|\|\delta_{S-Q}(y)\|$, where S ranges over all subsets of $\{i, i-1, \ldots, 2, 1\}$ and Q ranges over all subsets of S.
(3) $\|xy\|_i \leq \|x\|_i\|y\|_i$ for all i.
(4) $\|xy\|_i \leq \|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i$ for all $i \geq 1$.
(5) $(D(\delta_p\delta_{p-1}\cdots\delta_1), \|\cdot\|_p)$ is a Banach *-algebra.

Thus $D(\delta_p\delta_{p-1}\cdots\delta_1)$ is a Banach (D_p^*)-subalgebra of the C^*-algebra $\mathcal{A}$. Now let $(D(\{\delta_k\}_0^\infty) := \cap_{p=0}^\infty D(\delta_p\delta_{p-1}\cdots\delta_1)$ with the topology defined by the sequence of norms $\{\|.\|_i : 0 \leq i < \infty\}$. Then $D(\{\delta_k\}_0^\infty)$ is a Frechet (D_∞^*)-subalgebra of the C^*-algebra $\mathcal{A}$.

Example 1.6. *Smooth elements of a C^*-algebra defined by a Lie group action.* Let G be a Lie group acting on a unital C^*-algebra $\mathcal{A}$ by continuous *-automorphisms defined by the homomorphism $\alpha : G \to \mathrm{aut}(G)$ satisfying $\alpha_{st} = \alpha_s\alpha_t$ for all s, t in G. Let $\triangle$ be the collection of all infinitesimal generators of actions of one parameter subgroups of G; viz. $\triangle = \{(\mathrm{d}/\mathrm{d}t)\alpha_{u(t)} : t \to u(t)$ is a continuous *-homomorphism from $\mathbb{R}$ to $G\}$ consisting of derivations on $\mathcal{A}$. By a theorem of Malliavin (Chapter 1 of [9]), $\triangle$ is a finite dimensional vector space. Let $\{\delta_1, \delta_1, \ldots, \delta_d\}$ be a basis for $\triangle$. The class $C^n(\mathcal{A})$ of C^n-elements of $\mathcal{A}$ consists of all $x \in \mathcal{A}$ such that $x \in D(\delta_{i_1}\delta_{i_2}\cdots\delta_{i_n})$ where $\{\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_n}\}$ is an ordered n-tuple from $\{\delta_1, \delta_2, \ldots, \delta_d\}$. It is a Banach *-algebra with norm

$$\|x\|_n := \|x\|_0 + \sum_{k=1}^{n} \sum_{i_1,i_2,\ldots,i_k=1}^{d} (1/k!)\|\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}x\|.$$

Let $C^\infty(\mathcal{A}) = \varprojlim C^n(\mathcal{A}) = \cap_{n=0}^\infty C^n(\mathcal{A})$ with the topology defined by the sequence of norms $\{\|\cdot\|_n : n \in \mathbb{N}\cup\{0\}\}$. It is a Frechet (D_∞^*)-subalgebra of $\mathcal{A}$.

Example 1.7. *Schatten–von Neumann classes and Fredholm module*

(a) Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Let $\mathcal{K}(\mathcal{H})$ be the C^*-algebra of all compact operators; and let $\mathcal{C}^p(\mathcal{H})$, $1 \leq p < \infty$, be the dense Banach *-subalgebra consisting of operators of Schatten–von Neumann class with Schatten norm $\|\cdot\|_p$. Then

$$\mathcal{C}^1 \subset \mathcal{C}^p \subset \mathcal{C}^q \subset \mathcal{K}(\mathcal{H})$$

whenever $1 \leq p \leq q < \infty$ and $p, q \in \mathbb{R}^+$, and $\|\cdot\|_0 \leq \|\cdot\|_q \leq \|\cdot\|_p \leq \|\cdot\|_1$. For all x, y in $\mathcal{C}^p$, $\|xy\|_p = \frac{1}{2}\{\|xy\|_p + \|xy\|_p\} \leq \frac{1}{2}\{\|x\|_0\|y\|_p + \|y\|_0\|x\|_p\}$ showing that the pair $\{\|\cdot\|_0, \|\cdot\|_p\}$ is a differential norm of order 1 on $\mathcal{C}^p(\mathcal{H})$. Let $\mathcal{C}^{1+} = \bigcap_{p=1}^{\infty} \mathcal{C}^{1+\frac{1}{p}}(\mathcal{H})$.

For $x \in \mathcal{C}^{1+}$, let $|x|_p = \|x\|_{1+\frac{1}{p}}$ for $p = 1, 2, 3, \ldots$. Then on $\mathcal{C}^{1+}$, $|\cdot|_1 \leq |\cdot|_2 \leq \cdots \leq |\cdot|_n \leq |\cdot|_{n+1} \leq \cdots$. Also,

$$|xy|_n \leq \|x\|_0 |y|_n + |x|_n \|y\|_0 \leq |x|_{n-1}|y|_n + |x|_n|y|_{n-1}, \quad \text{for all } x, y \in \mathcal{C}^{1+};$$

and $\mathcal{C}^1(\mathcal{H}) \subset \mathcal{C}^{1+}(\mathcal{H}) \subset \mathcal{C}^2(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ are all continuous embeddings with $\|\cdot\|_0 \leq |\cdot|_n \leq \|\cdot\|_1$ on $\mathcal{C}^1(\mathcal{H})$. The algebra $\mathcal{C}^{1+}$ is a Frechet (D^*_∞)-subalgebra of $\mathcal{K}(\mathcal{H})$ with norms $\{|\cdot|_n : n = 1, 2, 3, \ldots\}$. We can analogously consider $\mathcal{C}^{p+}$ for $p > 1$.

(b) Let $(\mathcal{A}, \|\cdot\|_0)$ be a C^*-algebra of operators on a Hilbert space $\mathcal{H}$. Let $(\mathcal{A}, \mathcal{H}, D)$ be a *Fredholm module* [10] so that

(1) $D : \mathcal{H} \to \mathcal{H}$ is a bounded self-adjoint operator; and
(2) for all $x \in \mathcal{A}$, the commutator $[D, x] \in \mathcal{K}(\mathcal{H})$.

Let $\delta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the derivation defined by D as $\delta(x) = i[D, x]$. For $1 \leq p < \infty$, let $\mathcal{A}_p := \{x \in \mathcal{A} : \delta(x) \in \mathcal{C}^p(\mathcal{H})\}$ be the *p-summable part* of $\mathcal{A}$. It is a Banach *-algebra with norm $|x|_p := \|x\|_0 + \|\delta(x)\|_p$. Assume that each $\mathcal{A}_p$ is dense in $\mathcal{A}$. Let $\mathcal{A}_{p+} := \cap_{p<r\leq\infty} \mathcal{A}_r$ with the topology τ_p defined by norms $\{|.|_r : p < r \leq \infty\}$. Notice that $\mathcal{A}_{p+} = \cap_{k=1}^{\infty} \mathcal{A}_{p+\frac{1}{k}}$; the topology τ_p is also defined by the increasing sequence of norms $N_k(x) = \|x\|_0 + \|\delta(x)\|_{p+\frac{1}{k}}$, $k = 1, 2, 3, \ldots$ satisfying, for all k, $N_k(xy) \leq N_k(x)N_k(y)$ and $N_k(xy) \leq \|x\|_0 N_k(y) + \|y\|_0 N_k(x) \leq N_{k-1}(x)N_k(y) + N_k(x)N_{k-1}(y)$ for all x, y. Also, for all p, $\mathcal{A}_p \subset \mathcal{A}_{p+} \subset \mathcal{A}_{p+1} \subset \mathcal{A}$; and $\|x_0\| + \|\delta(x)\|_{p+1} \leq N_k(x) \leq \|x\|_0 + \|\delta(x)\|_p$ for all $x \in \mathcal{A}_p$. Thus $(\mathcal{A}_{p+}, \{N_k(\cdot) : k = 1, 2, 3, \ldots\})$ is a Frechet (D^*_∞)-subalgebra of $\mathcal{A}$.

Example 1.8. *C^k-algebras defined by an unbounded Fredholm module.* Let $(\mathcal{A}, \|\cdot\|)$ be a unital C^*-normed algebra having C^*-algebra completion $\tilde{\mathcal{A}}$. Let $(\mathcal{A}, \mathcal{H}, D)$ be an *unbounded Fredholm module* (*spectral triple*) [10] so that $\mathcal{H}$ is a separable Hilbert space with $\mathcal{A}$ being realized as a *-algebra of bounded operators on $\mathcal{H}$ and D is an unbounded self-adjoint operator on $\mathcal{H}$ having dense domain dom D so that the following hold:

(1) For all $x \in \mathcal{A}$, the closed commutator $[D, x]^-$ is a bounded operator on $\mathcal{H}$.
(2) For all $\lambda \notin \mathrm{sp}(D)$, $(\lambda 1 - D)^{-1}$ is a compact operator.

Let ad $D : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the derivation defined by D as $(\mathrm{ad}\ D)(x) := i[D, x]^-$ having the maximal domain in $\tilde{\mathcal{A}}$ given by $\mathrm{dom}(\mathrm{ad}\ D) := \{x \in \tilde{\mathcal{A}} : (\mathrm{ad}\ D)(x) \in \mathcal{B}(\mathcal{H})\}$, a Banach *-algebra with norm $\|x\|_1 := \|x\| + \|[D, x]\|$. We consider the following operator algebras with suggestive notations. Let $W^*(\tilde{\mathcal{A}})$ be the von Neumann algebra generated by $\tilde{\mathcal{A}}$:

$C^1(\mathrm{ad}\, D) := \{x \in \mathrm{dom}(\mathrm{ad}\, D) : \mathrm{ad}\, D(x) \in \tilde{\mathcal{A}}\}$ with the norm $\|\cdot\|_1$.
$\mathrm{Lip}(\mathrm{ad}\, D) := \{x \in W^*(\tilde{\mathcal{A}}) : \mathrm{ad}\, D(x) \in \mathcal{B}(\mathcal{H})\}$ with the norm $\|\cdot\|_1$.
$C^2(\mathrm{ad}\, D) := \{x \in \mathrm{dom}(\mathrm{ad}\, D) : \mathrm{ad}\, D(x) \in \mathrm{dom}(\mathrm{ad}\, D)\}$ with norm $\|x\|_2 := \|x\| + \|\mathrm{ad}\, D(x)\| + \|(\mathrm{ad}\, D)^2(x)\|$.
$\mathrm{Lip}^2(\mathrm{ad}\, D) := \{x \in W^*(\tilde{\mathcal{A}}) : \mathrm{ad}\, D(x) \in \mathrm{dom}(\mathrm{ad}\, D), (\mathrm{ad}\, D)^2(x) \in \mathcal{B}(\mathcal{H})\}$ with the norm $\|\cdot\|_2$.

We can inductively define the Banach *-algebras $C^k(\mathrm{ad}\ D)$ and $\mathrm{Lip}^k(\mathrm{ad}\ D)$ with appropriate norms. The Frechet algebras $C^\infty(\mathrm{ad}\ D) := \cap_{1\leq k<\infty} C^k(\mathrm{ad}\ D)$ and Lip^∞

(ad D) $:= \cap_{1\leq k<\infty} \mathrm{Lip}^k(\text{ad } D)$ with topology defined by the norms $\{\|\cdot\|_k : k \in \mathbb{N}\}$ representing respectively the C^∞-regularity and Lip^∞-regularity are (D^*_∞)-subalgebras of C^*-algebras.

2. Elementary properties

Let $(\mathcal{B}, \{\|\cdot\|_i\}_0^p)$ be a Banach (D^*_p)-subalgebra of a C^*-algebra $\mathcal{A}$. Let $q \geq p$. Then $\mathcal{B}$ is a Banach (D^*_q)-algebra with the same topology. This follows by defining $|\cdot|_i = \|\cdot\|_i$ for $0 \leq i \leq p$ and $|\cdot|_i = \|\cdot\|_p$ for $i > p$. Also $(\mathcal{B}, \{\|\cdot\|_i : 0 \leq i < \infty\})$ is a Frechet (D^*_∞)-algebra with norms $\|x\|_i = \|x\|_p$ for all $i > p$. This corresponds to the obvious statement that a Banach algebra is a Frechet algebra. However, in the present case, the topologies defined by the complete norm $\|\cdot\|_p$ and the seminorms $\{\|\cdot\|_i : 0 \leq i \leq p\}$ could be distinct. In fact, let $\|\cdot\| := \sum_{i=0}^p \|\cdot\|_i$. Let $\tilde{\mathcal{B}}$ be the Banach *-algebra obtained by completing $(\mathcal{B}, \|\cdot\|)$. Now the identity map on $\mathcal{B}$ extends as a continuous *-homomorphism $\phi : \tilde{\mathcal{B}} \to (\mathcal{B}, \|\cdot\|_p)$. Then $\|\cdot\|_0 \leq \|\cdot\|_p$ on $\mathcal{B}$. Indeed $\mathcal{B}$ is spectrally invariant in $\mathcal{A}$ by Theorem 5 of [14]; with the result, for any $x \in \mathcal{B}$, $\|x\|_0^2 = \|x^*x\|_0 = r_{\mathcal{A}}(x^*x) = r_{\mathcal{B}}(x^*x) \leq \|x^*x\|_p \leq \|x\|_p^2$. Thus $\ker\phi$ is $\|\cdot\|$-closed; the quotient $\tilde{\mathcal{B}}/\ker\phi$ is a Banach *-algebra with the quotient norm $|x+\ker\phi| = \inf\{\|x+y\| : \phi(y) = 0\}$; and ϕ induces the continuous isomorphism $\tilde{\phi}$, $\tilde{\phi}(x+\ker\phi) = \phi(x)$. Since $\mathcal{B}$ is semisimple being an A^*-algebra, Johnson's theorem on uniqueness of complete norm topology implies that $\tilde{\phi}$ is a homeomorphism. Thus the $\|\cdot\|_p$-topology on $\mathcal{B}$ is the quotient of $\|\cdot\|$-topology; and they coincide if each $\|\cdot\|_i$, $1 \leq i < p$ is closable with respect to $\|\cdot\|_0$.

On the other hand, the following shows that if a Frechet (D^*_∞)-algebra is a Banach algebra with some norm, then it is a Banach (D^*_p)-algebra for some p, $1 \leq p < \infty$. This corresponds to the fact that there is no norm on $C^\infty[a, b]$ making it a Banach algebra.

A topological algebra is a *Q-algebra* [11] if the set of its quasi regular elements is an open set.

PROPOSITION 2.1

Let $\mathcal{B}$ be a Frechet (D^*_∞)-subalgebra of a C^*-algebra $\mathcal{A}$. Let $\|\cdot\|$ be a norm on $\mathcal{B}$ making $\mathcal{B}$ a Banach *-algebra. Then the Frechet topology of $\mathcal{B}$ is determined by $\|\cdot\|$, and $\mathcal{B}$ is a Banach (D^*_p)-algebra for some $p < \infty$.

Proof. In view of Theorem (4.1.19), p. 188 of [19], $\mathcal{B}$ is semisimple. Thus the Frechet algebra $(\mathcal{B}, \tau)$ is a semisimple Q-algebra. Hence by Corollary 2.8(iii) of [11], the norm topology $\tau(\|\cdot\|)$ induced by $\|\cdot\|$ and the Frechet topology τ are the same. Hence $\|\cdot\|$ is continuous in τ, i.e. there exist $\alpha > 0$ and an integer $j > 0$ such that $\|x\| \leq \alpha\|x\|_j$ for all $x \in \mathcal{B}$. Also for every $i \geq 0$, there exists β_i such that $\|x\|_i \leq \beta_i\|x\|$. Thus $\|\cdot\|$ is equivalent to $\|\cdot\|_j$. It also follows that for any sequence (x_n) in $\mathcal{B}$, $\|x_n - x\| \to 0$ if and only if $\|x_n - x\|_i \to 0$ for all i. Consequently,

$$\begin{aligned}\|xy\| \leq \alpha\|xy\|_j &\leq \alpha D_j(\|x\|_j\|y\|_{j-1} + \|x\|_{j-1}\|y\|_j)\\ &\leq \alpha D_j\beta_j(\|x\|\|y\|_{j-1} + \|x\|_{j-1}\|y\|).\end{aligned}$$

Thus $\mathcal{B}$ is a (D^*_j)-subalgebra of $\mathcal{A}$ with $\{\|\cdot\|_i : 0 \leq i \leq j-1\} \cup \{\|\cdot\|\}$. Clearly the $\|\cdot\|$-topology of $\mathcal{B}$ coincides with the topology of $\{\|\cdot\|_i : 0 \leq i \leq j-1\} \cup \{\|\cdot\|\}$. This completes the proof. □

The following shows that the assumptions on the norms and $\{D_i\}_{i\geq 1}$ can be simplified in the (D_p^*)-subalgebra or a (D_∞^*)-subalgebra of a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$.

PROPOSITION 2.2

Let $(\mathcal{B}, \{\|\cdot\|_i\})$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$. Then there exists a sequence $\{|\cdot|_i\}$ of submultiplicative *-norms on $\mathcal{B}$ and $D > 0$ such that the following hold for all $x, y \in \mathcal{B}$:

(1) $|\cdot|_0 = \|\cdot\|_0$ *and* $|\cdot|_0 \leq |\cdot|_1 \leq |\cdot|_2 \leq \cdots$.
(2) $\{\|\cdot\|_i\}$ *and* $\{|\cdot|_i\}$ *determine the same topology.*
(3) $(\mathcal{B}, \{|\cdot|_i\})$ *is a Frechet* (D_∞^*)*-subalgebra of* $\mathcal{A}$.
(4) $|xy|_1 \leq D(|x|_0|y|_1 + |x|_1|y|_0)$.
(5) $|xy|_i \leq |x|_{i-1}|y|_i + |x|_i|y|_{i-1}$ *for* $i \geq 2$.

Proof. First we assume that $D_i \geq 1$. Keep $\|x\|_0$ as it is, define $K_i = \max\{D_j : 1 \leq j \leq i+1\}$. Also replace $\|\cdot\|_i$ by an equivalent norm $K_i\|\cdot\|_i$. Since $K_i \geq D_i \geq 1$, the new norms are submultiplicative seminorms. Taking $D = D_1$, $\|xy\|_1 \leq D(\|x\|_0\|y\|_1 + \|x\|_1\|y\|_0)$. Also, $\|xy\|_i \leq \|x\|_{i-1}\|y\|_i + \|x\|_i\|y\|_{i-1}$ for $i \geq 2$. Now define $|x|_0 = \|x\|_0$, $|x|_i = |x|_{i-1} + \|x\|_i$, $x \in \mathcal{B}$, $i \geq 1$. Then (1) is obvious. (2) Follows by noting that $\|x\|_i \leq |x|_i = \|x\|_0 + \|x\|_1 + \cdots \|x\|_i$. (3) For all x, y in $\mathcal{B}$,

$$\begin{aligned}
|xy|_1 &= |xy|_0 + \|xy\|_1 \leq |x|_0|y|_0 + \|x\|_1\|y\|_1 \\
&= \|x\|_0\|y\|_0 + \|x\|_1\|y\|_1 \\
&\leq \|x\|_0(\|y\|_0 + \|y\|_1) + \|x\|_1(\|y\|_0 + \|y\|_1) \\
&= (\|x\|_0 + \|x\|_1)(\|y\|_0 + \|y\|_1) \\
&= (|x|_0 + \|x\|_1)(|y|_0 + \|y\|_1) \\
&= |x|_1|y|_1.
\end{aligned}$$

Now suppose that $|\cdot|_{i-1}$ is submultiplicative. Then

$$\begin{aligned}
|xy|_i &= |xy|_{i-1} + \|xy\|_i \leq |x|_{i-1}|y|_{i-1} + \|x\|_i\|y\|_i \\
&\leq |x|_{i-1}(|y|_{i-1} + \|y\|_i) + \|x\|_i(|y|_{i-1} + \|y\|_i) \\
&= |x|_i|y|_i.
\end{aligned}$$

The statements (4) and (5) follow by simple computation. □

It is an old interesting result due to Silov [21] (see also the Preface of [20]) that if $\mathcal{C}$ is an algebra of continuous functions on $[0, 1]$ such that $C^\infty[0, 1] \subset \mathcal{C}$ and $\mathcal{C}$ is a Banach algebra with some norm, then there exists p such that $C^p[0, 1] \subset \mathcal{C}$. The following gives a noncommutative analogue of this in the present set up.

PROPOSITION 2.3

Let $(\mathcal{B}, \{\|\cdot\|_i : 0 \leq i < \infty\})$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $\mathcal{A}$ such that $\|\cdot\|_0 \leq \|\cdot\|_j$ for all j. Let $\mathcal{C}$ be a *-subalgebra of $\mathcal{A}$ such that $\mathcal{C}$ is a Banach *-algebra with some norm $|\cdot|$ and $\mathcal{B} \subset \mathcal{C}$. Then there exists p such that the completion $\mathcal{B}_p$ of $\mathcal{B}$ in the norm $\|\cdot\|_p$ is contained in $\mathcal{C}$.

Proof. Since $\|\cdot\|_0$ is a C^*-norm on $\mathcal{C}$, by Theorem (4.1.19), p. 188 of [19], $\mathcal{C}$ is semisimple. Also for $x \in \mathcal{C}$, $\|x\|_0 \leq |x|$. Indeed $\|x\|_0^2 = \|x^*x\|_0 = r_{\mathcal{A}}(x^*x) \leq r_{\mathcal{C}}(x^*x) \leq |x^*x| \leq |x|^2$. Now the identity map $(\mathcal{B}, \{\|\cdot\|_i\}_0^\infty) \to (\mathcal{C}, |\cdot|)$ is continuous as $\mathcal{B}$ is semisimple Frechet; and $\mathcal{C}$ is semisimple and a Q-algebra. Indeed let π be an irreducible *-representation of $\mathcal{C}$ on a Hilbert space. Then π also defines an irreducible *-representation $\pi \circ id$ of $\mathcal{B}$, where $id : \mathcal{B} \to \mathcal{C}$ is the inclusion. By semisimplicity, both π and $\pi \circ id$ are continuous. We use the closed graph theorem to show that id is continuous. Let $x_n \to x$ in $\mathcal{B}$ and $x_n \to y$ in $\mathcal{C}$. Then $\pi(x_n) \to \pi(y)$, $\pi(x_n) \to \pi(x)$, and so $\pi(x) = \pi(y)$ for all irreducible *-representations π of $\mathcal{C}$. As $\mathcal{C}$ is *-semisimple, $x = y$. Hence $id : \mathcal{B} \to \mathcal{C}$ is continuous. Thus there exist $k > 0$ and $j \in \mathbb{N}$ such that $|x| \leq k\|x\|_j$ for all $x \in \mathcal{B}$. Hence $id : (\mathcal{B}, \|\cdot\|_j) \to (\mathcal{C}, |\cdot|)$ extends as a continuous *-homomorphism $\varphi : \mathcal{B}_j \to \mathcal{C}$, where $\mathcal{B}_j$ is the completion of $(\mathcal{B}, \|\cdot\|_j)$. Now $\|\cdot\|_0 \leq \|\cdot\|_j$ on $\mathcal{B}_j$ and $\|\cdot\|_0 \leq |\cdot|$ on $\mathcal{C}$. This implies that φ is injective. In fact, as in the preceding discussion for $C^\infty[0, 1]$, φ is the identity map. This proves that $\mathcal{B}_j \subset \mathcal{C}$ completing the proof. □

We discuss the unitification. Let $(\mathcal{A}, \|\cdot\|_0)$ be a C^*-algebra; and let $(\mathcal{B}, \{\|\cdot\|_i\})$ be a (D_∞^*)-subalgebra of $\mathcal{A}$. Let $\mathcal{A}_e = \mathcal{A} \oplus \mathbb{C}\mathbf{1}$ be the C^*-algebra obtained by adjoining identity to $\mathcal{A}$ having the C^*-norm $\|x + \lambda\mathbf{1}\|_{\text{op}} = \sup\{\|xy + \lambda y\|_0 : y \in \mathcal{A}, \|y\|_0 \leq 1\}$. Then $\mathcal{B}_e = \{y + \lambda\mathbf{1} : y \in \mathcal{B}, \lambda \in \mathbb{C}\}$ is a dense *-subalgebra of $\mathcal{A}_e$. The Frechet topology of the unitification $\mathcal{B}_e$ of the Frechet algebra $(\mathcal{B}, \{\|\cdot\|_i : 0 \leq i < \infty\})$ is defined by the submultiplicative *-seminorms $\|y + \lambda\mathbf{1}\|_i = \|y\|_i + |\lambda|$, $i \geq 1$ and $\|\cdot\|_0 = \|\cdot\|_{\text{op}}$. We shall need the following.

Lemma 2.4. *Let $(\mathcal{X}, \|\cdot\|)$ be a C^*-normed algebra. On $\mathcal{X}_e$, let*

$$\|x+\lambda\mathbf{1}\|_{\text{op}} = \sup\{\|xy+\lambda y\| : y \in \mathcal{X}, \|y\| \leq 1\} \quad \textit{and} \quad \|x+\lambda\mathbf{1}\|_1 = \|x\| + |\lambda|.$$

Then $\|x + \lambda\mathbf{1}\|_1 \leq 6e\|x + \lambda\mathbf{1}\|_{\text{op}}$ for all $x \in \mathcal{X}, \lambda \in \mathbb{C}$.

Proof. Let $\mathcal{A}$ be the C^*-algebra completion of $\mathcal{X}$. Now a C^*-norm $\|\cdot\|$ is regular in the sense that for all $x \in \mathcal{X}$, $\|x\|_{\text{op}} = \|x\|$. Hence by [12] (see also [1]), for all $x + \lambda\mathbf{1} \in \mathcal{A}_e$,

$$\begin{aligned}\|x + \lambda\mathbf{1}\|_1 \leq 6e\|x + \lambda\mathbf{1}\|_{\text{op}} &= 6e \sup\{\|xy + \lambda y\| : y \in \mathcal{A}, \|y\| \leq 1\} \\ &= 6e \sup\{\|xy + \lambda y\| : y \in \mathcal{X}, \|y\| \leq 1\}\end{aligned}$$

since $\mathcal{X}$ is dense in $\mathcal{A}$. □

PROPOSITION 2.5

The Frechet algebra $\mathcal{B}_e$ is a (D_∞^*)-subalgebra of $\mathcal{A}_e$.

Proof. We can assume $D_i = 1$ for $i > 1$ and for $i = 1$, $D_i = D$. Let $i > 1$ and $x + \lambda\mathbf{1} \in \mathcal{B}_e$. Then for all $y \in \mathcal{B}$,

$$\begin{aligned}\|(x + \lambda\mathbf{1})y\|_i &= \|xy + \lambda y\|_i \leq \|xy\|_i + |\lambda|\|y\|_i \\ &\leq \|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i + |\lambda|\|y\|_i \\ &\leq (\|x\|_i + |\lambda|)\|y\|_{i-1} + (\|x\|_{i-1} + |\lambda|)\|y\|_i \\ &= \|x + \lambda\mathbf{1}\|_i\|y\|_{i-1} + \|y\|_i\|x + \lambda\mathbf{1}\|_{i-1}.\end{aligned}$$

Then for any $y + \mu\mathbf{1} \in \mathcal{B}_e$,

$$\begin{aligned}
\|(x + \lambda\mathbf{1})(y + \mu\mathbf{1})\|_i &= \|(x + \lambda\mathbf{1})y + \mu(x + \lambda\mathbf{1})\|_i \\
&\leq \|x + \lambda\mathbf{1}\|_i\|y\|_{i-1} + \|x + \lambda\mathbf{1}\|_{i-1}\|y\|_i \\
&\quad + |\mu|\|x + \lambda\mathbf{1}\|_i \\
&= \|x + \lambda\mathbf{1}\|_i(\|y\|_{i-1} + |\mu|) + \|x + \lambda\mathbf{1}\|_{i-1}\|y\|_i \\
&\leq \|x + \lambda\mathbf{1}\|_i\|y + \mu\mathbf{1}\|_{i-1} + \|x + \lambda\mathbf{1}\|_{i-1}(\|y\|_i + |\mu|) \\
&= \|x + \lambda\mathbf{1}\|_i\|y + \mu\mathbf{1}\|_{i-1} + \|x + \lambda\mathbf{1}\|_{i-1}\|y + \mu\mathbf{1}\|_i.
\end{aligned}$$

Thus the D_i-property holds for all $i > 1$. Take $i = 1$. For $x + \lambda\mathbf{1}$, $y + \mu\mathbf{1}$ in $\mathcal{B}_e$, we show that

$$\begin{aligned}
\|(x + \lambda\mathbf{1})(y + \mu\mathbf{1})\|_1 \;\leq\; &\{\max(6e, 6eD_1)\}(\|x + \lambda\mathbf{1}\|_1\|y + \mu\mathbf{1}\|_0 \\
&+ \|x + \lambda\mathbf{1}\|_0\|y + \mu\mathbf{1}\|_1).
\end{aligned}$$

Notice that $\|\cdot\|_0$ on $\mathcal{B}_e$ is $\|x + \lambda\mathbf{1}\|_0 = \sup\{\|xy + \lambda y\| : y \in \mathcal{B}, \|y\|_0 \leq 1\}$. Now

$$\begin{aligned}
\|(x + \lambda\mathbf{1})y\|_1 &\leq \|xy\|_1 + \|\lambda y\|_1 \\
&\leq D_1\{\|x\|_0\|y\|_1 + \|x\|_1\|y\|_0\} + |\lambda|\|y\|_1 \\
&\leq (D_1\|x\|_0 + |\lambda|)\|y\|_1 + D_1\|x + \lambda\mathbf{1}\|_1\|y\|_0 \\
&\leq 6e\max\{1, D_1\}\|x + \lambda\mathbf{1}\|_0\|y\|_1 + \|x + \lambda\mathbf{1}\|_1\|y\|_0 \\
&\leq 6e\max\{1, D_1\}\{\|x + \lambda\mathbf{1}\|_0\|y\|_1 + \|x + \lambda\mathbf{1}\|_1\|y\|_0\}
\end{aligned}$$

using the above lemma. Then

$$\begin{aligned}
\|(x + \lambda\mathbf{1})(y + \mu\mathbf{1})\|_1 &= \|xy + \lambda y + \mu x + \lambda\mu\mathbf{1}\|_1 \\
&= \|(x + \lambda\mathbf{1})y + \mu(x + \lambda\mathbf{1})\|_1 \\
&\leq \|(x + \lambda\mathbf{1})y\|_1 + |\mu|\|(x + \lambda\mathbf{1})\|_1 \\
&\leq 6e\max\{1, D_1\}\{\|x + \lambda\mathbf{1}\|_0\|y\|_1 + \|x + \lambda\mathbf{1}\|_1\|y\|_0\} \\
&\quad + |\mu|\|x + \lambda\mathbf{1}\|_1 \\
&\leq 6e\max\{1, D_1\}\{\|x + \lambda\mathbf{1}\|_0\|y\|_1 \\
&\quad + \|x + \lambda\mathbf{1}\|_1(\|y\|_0 + |\mu|)\} \\
&\leq 6e\max\{1, D_1\}\{\|x + \lambda\mathbf{1}\|_0(\|y\|_1 + |\mu|) \\
&\quad + 6e\|x + \lambda\mathbf{1}\|_1\|y + \mu\mathbf{1}\|_0\} \\
&\leq 36e^2\max\{1, D_1\}\{\|x + \lambda\mathbf{1}\|_0\|y + \mu\mathbf{1}\|_1 \\
&\quad + \|x + \lambda\mathbf{1}\|_1\|y + \mu\mathbf{1}\|_0\}
\end{aligned}$$

This completes the proof. □

We consider tensoring with matrices which is important in K-theory. Given a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$, let $M_n(\mathcal{A}) = M_n(\mathbb{C}) \bigotimes \mathcal{A}$ be the C^*-algebra of all $n\times n$ matrices having entries from $\mathcal{A}$. It admits a unique C^*-norm denoted by $\|\cdot\|_0$. Then $\|\cdot\|_0 = \|\cdot\| :=$ the maximum C^*-norm on $M_n(\mathbb{C}) \bigotimes \mathcal{A}$ induced by the C^*-norm on $M_n(\mathbb{C})$ and the C^*-norm $\|\cdot\|_0$ on $\mathcal{A}$. Let $(\mathcal{B}, \{\|\cdot\|_i : 0 \le i < \infty\})$ be a (D^*_∞)-subalgebra of $\mathcal{A}$. Then $M_n(\mathcal{B})$ is a dense *-subalgebra of $M_n(\mathcal{A})$; and it is also a Frechet *-algebra with the projective tensorial topology defined by the family of seminorms $\|\cdot\|_{\gamma,i}$, which is the projective cross norm given by $\|\cdot\|_i$ on $\mathcal{B}$ and the C^*-norm $\|\cdot\|$ on $M_n(\mathbb{C})$.

PROPOSITION 2.6

The algebra $M_n(\mathcal{B})$ is a (D^*_∞)-subalgebra of $M_n(\mathcal{A})$.

Proof. For $t, s \in M_n(\mathcal{B})$, consider an expression

$$t = \sum_u x_u \otimes a_u, \quad s = \sum_v y_v \otimes b_v,$$

$$\begin{aligned}
\|ts\|_{\gamma,i} &= \|\textstyle\sum_{u,v} x_u y_v \otimes a_u b_v\|_{\gamma,i} \\
&\le \sum_{u,v} \|x_u y_v\|_i \|a_u b_v\| \\
&\le \sum_{u,v} (D_i(\|x_u\|_i \|y_v\|_{i-1} + \|x_u\|_{i-1}\|y_v\|_i))\|a_u b_v\| \\
&\le D_i \left(\sum_{u,v} \|x_u\|_i \|y_v\|_{i-1} \|a_u\|\|b_v\| + \sum_{u,v} \|x_u\|_{i-1}\|y_v\|_i \|a_u\|\|b_v\| \right) \\
&= D_i \left[\left(\sum_u \|x_u\|_i \|a_u\|\right)\left(\sum_v \|y_v\|_{i-1}\|b_v\|\right) \right. \\
&\qquad \left. + \left(\sum_u \|x_u\|_{i-1}\|a_u\|\right)\left(\sum_v \|y_v\|_i \|b_v\|\right)\right].
\end{aligned}$$

Taking infimum over all expressions for tensorial representations for t, s, we get

$$\|ts\|_{\gamma,i} \le D_i(\|t\|_{\gamma,i}\|s\|_{\gamma,i-1} + \|t\|_{\gamma,i-1}\|s\|_{\gamma,i}).$$

This completes the proof. □

PROPOSITION 2.7

Let $\mathcal{B}$ be a dense *-subalgebra of a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$. Let $\{\|\cdot\|_i : 0 \le i < \infty\}$ be a separating sequence of submultiplicative *-seminorms on $\mathcal{B}$ such that each $\|\cdot\|_p$, $1 \le p < \infty$, satisfies the (D^*_p)-condition on $\mathcal{B}$. Let $\tilde{\mathcal{B}}$ be the completion of $\mathcal{B}$ in $\{\|\cdot\|_i : 0 \le i < \infty\}$. Then $\tilde{\mathcal{B}}$ is a Frechet (D^*_∞)-subalgebra of $\mathcal{A}$.

We omit the easy proof. Also the c_0-direct sum and the c-direct sum of a sequence of Frechet (D^*_∞)-subalgebras of C^*-algebras are Frechet (D^*_∞)-subalgebras of respectively the c_0-direct sum and the c-direct sum of the respective C^*-algebras.

3. Arens–Michael decomposition

The Arens–Michael decomposition [11] expresses a Frechet *-algebra as an inverse limit of a sequence of Banach *-algebras. The problems in Frechet algebras can be transferred to corresponding problems in Banach algebras using the Arens–Michael decomposition. In what follows, we obtain Arens–Michael decomposition of a Frechet (D^*_∞)-subalgebra of a C^*-subalgebra as an inverse limit of (D^*_p)-subalgebras of the C^*-algebra (which was announced in [2]). We use the same to investigate the differential properties of Frechet (D^*_∞)-subalgebras.

Theorem 3.1. *Let* $(\mathcal{B}, \{\|\cdot\|_i\}_0^\infty)$ *be a Frechet* (D^*_∞)*-subalgebra of a* C^**-algebra* $(\mathcal{A}, \|\cdot\|_0)$*. Then there exists a sequence* $(\tilde{\mathcal{B}}_k, \|\cdot\|_k)$ *of dense Banach* **-subalgebras of* $\mathcal{A}$ *such that the following hold*:

(1) *Each* $\tilde{\mathcal{B}}_k$ *is a Banach* (D^*_k)*-subalgebra of* $\mathcal{A}$ *continuously embedded in* $\mathcal{A}$*.*

(2) *The sequence* $\{\tilde{\mathcal{B}}_k\}$ *forms an inverse limit sequence of Banach *-algebras and* $\mathcal{B} = \varprojlim \tilde{\mathcal{B}}_k$*, the inverse limit of* $\tilde{\mathcal{B}}_k$*.*

Proof. We can assume that $D_i = 1$ for all $i \geq 2$ and $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \cdots$ so that all $\|\cdot\|_i$ are actually norms. For each i, let $\mathcal{B}_i = (\mathcal{B}, \|\cdot\|_i)^\sim$, the completion of the normed *-algebra $(\mathcal{B}, \|\cdot\|_i)$. Since $\|\cdot\|_{i-1} \leq \|\cdot\|_i$, the identity map $x \in (\mathcal{B}, \|\cdot\|_i) \mapsto x \in (\mathcal{B}, \|\cdot\|_{i-1})$ is a continuous *-homomorphism for each $i \geq 1$. Denote the extension of this map by $\psi_i : \mathcal{B}_i \to \mathcal{B}_{i-1}$. Then $\|\psi_i(z)\|_{i-1} \leq \|z\|_i$ for all $z \in \mathcal{B}_i$, $\psi_i|_{\mathcal{B}} = id$, and ψ_i is a norm decreasing *-homomorphism with dense range in $\mathcal{B}_{i-1}$. Hence by the general theory of Frechet algebras [11], the sequence

$$\mathcal{A} = \mathcal{B}_0 \xleftarrow{\psi_1} \mathcal{B}_1 \xleftarrow{\psi_2} \cdots \mathcal{B}_{i-1} \xleftarrow{\psi_i} \mathcal{B}_i \cdots$$

is an inverse limit sequence (in fact, a dense inverse limit sequence) of Banach *-algebras, and

$$(\mathcal{B}, \tau) = \varprojlim(\mathcal{B}_i, \|\cdot\|_i),$$

where τ is the topology of $\mathcal{B}$. Thus $(\mathcal{B}, \tau)$ is realized as a *-subalgebra of the product $\prod \mathcal{B}_i$ with the product topology; and if $(x_i) \in \prod \mathcal{B}_i$ is such that $\psi(x_{i+1}) = x_i$ for all $i \geq 1$, then there exists a unique $x \in \mathcal{B}$ such that $x_i = \pi_i(x)$ for all i, π_i denoting the projection map $\mathcal{B} \to \mathcal{B}_i$. For each i and $0 \leq k \leq i$, set

$$\|z\|_{i,k} = \|z\|_k, \qquad \text{for all } z \in \mathcal{B}.$$

As $\|z\|_{i,k} = \|z\|_k \leq \|z\|_i = \|z\|_{i,i}$, for all $0 \leq k \leq i$ and all $z \in \mathcal{B}$, the norms $\|z\|_{i,k}$ can be extended to $\mathcal{B}_i$. Denoting the extension of $\|\cdot\|_i$ by $\|\cdot\|_i^{'}$, the extension of $\|\cdot\|_{i,k}$ by $\|\cdot\|_{i,k}^{'}$, we prove that for each i, the algebra $(\mathcal{B}_i, \{\|\cdot\|_{i,k}^{'}\}_{k=0}^i)$ is a Banach D_i^*-algebra. In general, $\|\cdot\|_{i,j}^{'}$ is a submultiplicative *-seminorm and $(\mathcal{B}_i, \|\cdot\|_{i,j}^{'})$ need not be complete for $j < i$, but $\|\cdot\|_{i,i}^{'}$ coincides with $\|\cdot\|_i$. Then the following hold:

(i) $(\mathcal{B}_i, \|\cdot\|_i^{'})$ is a Banach *-algebra.

(ii) For each i, $(\mathcal{B}_i, \{\|\cdot\|_{i,k}^{'} : 0 \leq k \leq i\})$ has (D_i^*)-property. Indeed, fix $0 < j \leq i$.

Let $x, y \in \mathcal{B}_i$. Choose $x_n, y_n \in \mathcal{B}$ such that $x_n \to x$, $y_n \to y$ in the norm $\|\cdot\|_j$. Since $\|\cdot\|_j, \|\cdot\|_{j-1}$ are usual norms on $\mathcal{B}$, satisfying the D_i-inequality, we have

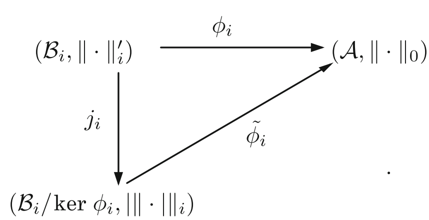
$$\begin{aligned}\|xy\|_{i,j}^{'} &= \lim_{n\to\infty}\|x_n y_n\|_j \leq \lim_{n\to\infty}\{\|x_n\|_j\|y_n\|_{j-1} + \|x_n\|_{i,j-1}\|y_n\|_{i,j}\}\\ &= \|x\|_{i,j}^{'}\|y\|_{i,j-1}^{'} + \|x\|_{i,j-1}^{'}\|y\|_{i,j}^{'}.\end{aligned}$$

Thus it follows that $(\mathcal{B}_i, \{\|\cdot\|_{i,j}^{'}\}_{j=0}^{i})$ is a Banach (D_i^*)-algebra.

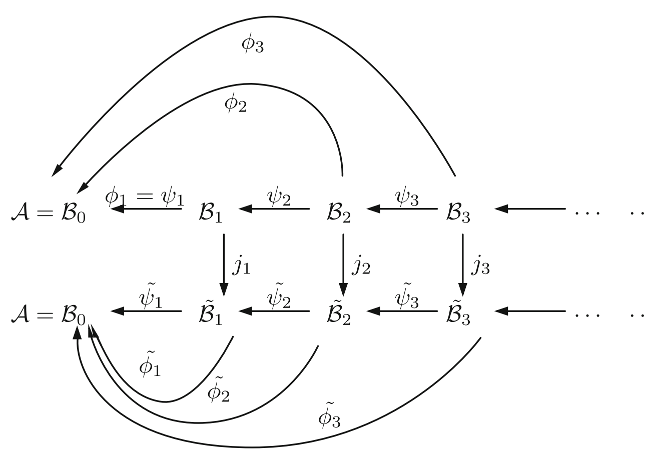
(iii) $\|\cdot\|_0^{'}$ is a C^*-seminorm on $\mathcal{B}_i$; it need not be a norm.

(iv)Since $\|\cdot\|_0 \leq \|\cdot\|_i$ on $\mathcal{B}$, there exists a continuous *-homomorphism $\phi_i : \mathcal{B}_i \to \mathcal{A}$ having dense range such that $\phi_i|\mathcal{B} = id$, $\|\phi_i(z)\|_0 \leq \|z\|_{i,i}^{'}$ for all $z \in \mathcal{B}_i$ and $\phi_i = \psi_1 \circ \psi_2 \circ \cdots \circ \psi_i$. Thus the Banach (D_i^*)-algebra $(\mathcal{B}_i, \{\|\cdot\|_{i,k}^{'}\}_{0\leq k\leq i})$ need not be a Banach (D_i^*)-subalgebra of the C^*-algebra $\mathcal{A}$. This is overcome as follows:

For each i, let $\ker\phi_i$ be the kernel of ϕ_i which is a $\|\cdot\|_i^{'}$-closed (i.e. $\|\cdot\|_{i,i}^{'}$-closed) *-ideal of the Banach *-algebra $\mathcal{B}_i$. Let $\tilde{\mathcal{B}_i} = \mathcal{B}_i/\ker\phi_i$ a Banach *-algebra with the quotient norm $|||\cdot|||_i$ from $(\mathcal{B}_i, \|\cdot\|_i^{'})$.



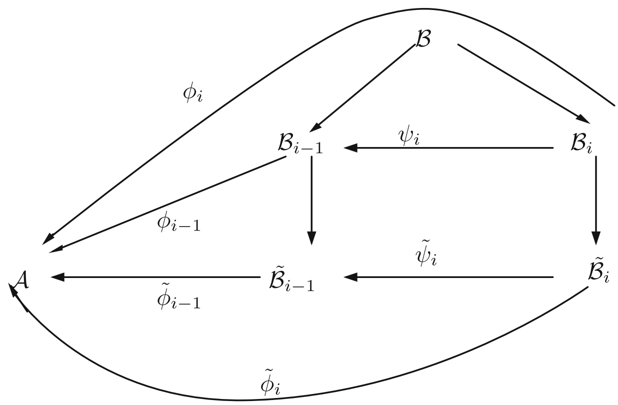
Let $j_i : (\mathcal{B}_i, \|\cdot\|_i^{'}) \to (\tilde{\mathcal{B}_i}, |||\cdot|||_i)$ be the natural quotient map. Then there exists an injective *-homomorphism $\tilde{\phi_i} : (\tilde{\mathcal{B}_i}, |||\cdot|||_i) \to (\mathcal{A}, \|\cdot\|_0)$ having dense range such that $\|\tilde{\phi_i}(z)\|_0 \leq |||\tilde{z}|||_i$, $\tilde{\phi_i} \circ j_i = \phi_i$, and $\tilde{\phi_i}(\tilde{z}) = \phi_i(z)$, $\tilde{z} = z + \ker\phi_i$.



Notice that for each i, $\phi_{i-1}(\psi_i(z)) = \phi_i(z)$ for all $z \in \mathcal{B}_i$. Indeed, let $z \in \mathcal{B}_i$. There exists a sequence (x_n) in $\mathcal{B}$ such that $\|x_n - z\|_i \to 0$. Hence $\|x_n - \psi_i(z)\|_{i-1} =$

$\|\psi_i(x_n) - \psi_i(z)\|_{i-1} \to 0$ and $\|x_n - \phi_i(z)\|_0 = \|\phi_i(x_n) - \phi_i(z)\|_0 \to 0$. Therefore $\|x_n - \phi_{i-1}(\psi_i(z))\|_0 = \|\phi_{i-1}(x_n) - \phi_{i-1}(\psi_i(z))\|_0 \to 0$. It follows that $\phi_{i-1}(\psi_i(z)) = \phi_i(z)$. We also have for each k, $\phi_k = \psi_1 \circ \psi_2 \circ \psi_3 \circ \cdots \circ \psi_k$. We need to define appropriate seminorms with (D_i^*)-properties on $\tilde{\mathcal{B}}_i$.

Let $\tilde{j}_i : \mathcal{B} \to \tilde{\mathcal{B}}_i$, $\tilde{j}_i(x) = x + \ker\phi_i$; viz., $\tilde{j}_i = j_i|_{\mathcal{B}}$. The homomorphism $\tilde{j}_i$ is one-to-one, since for $x \in \mathcal{B}$, $x + \ker\phi_i = 0$ implies that $\phi_i(x) = 0$, and so $x = \phi(x) = 0$.



Define $\tilde{\psi}_i : \tilde{\mathcal{B}}_i \to \tilde{\mathcal{B}}_{i-1}$, $\tilde{\psi}_i(z + \ker\phi_i) = \psi_i(z) + \ker\phi_{i-1}$. Then the following hold:

(v) The map $\tilde{\psi}_i$ is well defined. Let $z \in \mathcal{B}_i$ be such that $z + \ker\phi_i = 0$. Hence $\phi_i(z) = 0$. Since $\phi_{i-1} \circ \psi_i = \phi_i$, $\phi_{i-1}(\psi_i(z)) = 0$. Hence $\psi_i(z) \in \ker\phi_{i-1}$ and $\psi_i(z) + \ker\phi_{i-1} = 0$.
(vi) The map $\tilde{\psi}_i$ is a norm decreasing *-homomorphism, and so is ψ_i.
(vii) The map $\tilde{\psi}_i$ is one-to-one. Indeed, if $\tilde{\psi}_i(z + \ker\phi_i) = 0$, then $\psi_i(z) + \ker\phi_{i-1} = 0$, $\psi_i(z) \in \ker\phi_{i-1}$, and $\phi_i(z) = \phi_{i-1}(\psi_i(z)) = 0$. Hence $z + \ker\phi_i = 0$.
(viii) $\tilde{\psi}_i$ has dense range. Let $[z] = z + \ker\phi_{i-1} \in \tilde{\mathcal{B}}_{i-1}$ so that $z \in \mathcal{B}_{i-1}$. Since ψ has dense range, given $\epsilon > 0$, there exists $x \in \mathcal{B}_i$ such that $\|\psi_i(x) - z\|_{\mathcal{B}_{i-1}} < \epsilon$. Then

$$\begin{aligned} \||\tilde{\psi}_i(x + \ker\phi_i) - (z + \ker\phi_{i-1})\||_{i-1} &= \||\psi_i(x) - z + \ker\phi_{i-1}\||_{i-1} \\ &\le \|\psi_i(x) - z\|_{\mathcal{B}_{i-1}} < \epsilon. \end{aligned}$$

Thus $\tilde{\psi}_i$ has dense range. Now we have for $1 \le k \le i$, $\tilde{\phi}_k = \tilde{\phi}_{k-1} \circ \tilde{\psi}_k = \tilde{\psi}_1 \circ \tilde{\psi}_2 \circ \tilde{\psi}_3 \circ \cdots \circ \tilde{\psi}_k$. Let i be arbitrarily fixed. For $0 \le k \le i$, define $\||.\||_k' = \||.\||_{i,k}'$ (depending on i in general) on $\tilde{\mathcal{B}}_i$ as

$$\begin{aligned} \||[z]\||_k' &:= \||\tilde{\psi}_{k+1}(\tilde{\psi}_{k+2}(\cdots(\tilde{\psi}_i([z])\cdots)))\||_k, \qquad \text{for } 0 \le k \le i-1; \\ \||[z]\||_i' &:= \||[z] = z + \ker\phi_i\||_i \end{aligned}$$

Note that $\||[z]\||_i' = \|z\|_i$ if $z \in \mathcal{B}$. We show that the submultiplicative *-seminorms $\{\||.\||_k' : 0 \le k \le i\}$ have the (D_i^*)-property. We shall do this by arguments analogous to those done earlier for $\|.\|_k'$.

Notice that $\mathcal{B} \cap \ker\phi_i = (0)$. This is because the map ϕ_i restricted to $\mathcal{B}$ is the identity map. Thus $x \in \mathcal{B} \to x + \ker\phi_i \in \mathcal{B}_i/\ker\phi_i$ is a one-to-one

*-homomorphism on $\mathcal{B}$; and $\mathcal{B}$ is regarded as a subalgebra of $\mathcal{B}_i$. For $x \in \mathcal{B}$, we shall identify x with $x + \ker \phi_i$. Also, since

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|_k \leq \cdots,$$

we have

$$|||\cdot|||_0 \leq |||\cdot|||_1 \leq |||\cdot|||_2 \leq \cdots \leq |||\cdot|||_k \leq \cdots.$$

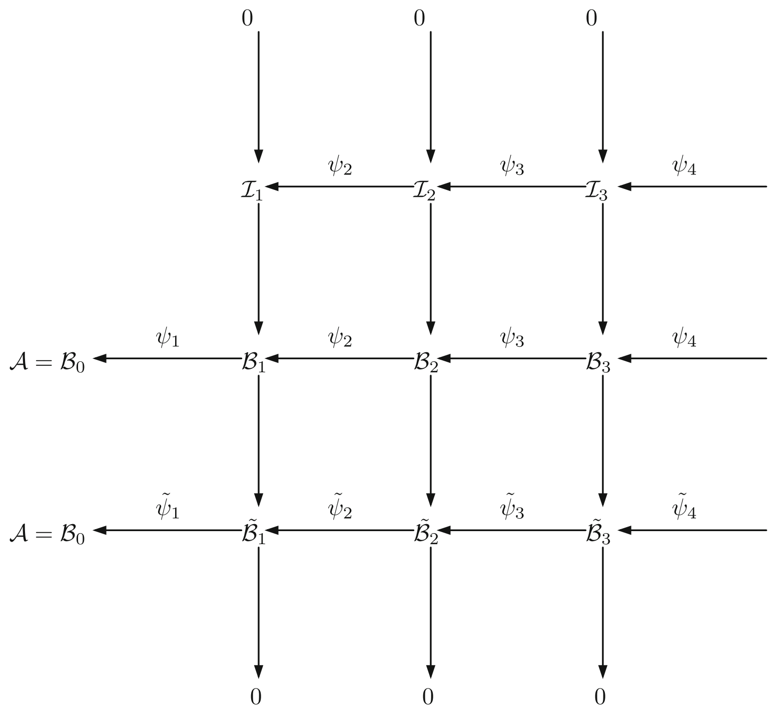
Thus, since $\mathcal{B}$ is dense in $\mathcal{B}_i$ and also in $\|\cdot\|_k, 0 \leq k \leq i$; $\mathcal{B}$ is dense in $\tilde{\mathcal{B}_i} = \mathcal{B}/\ker \phi_i$ in $|||\cdot|||_i$, and also in $|||\cdot|||_k, 0 \leq k \leq i$.

Let $\tilde{u} = u + \ker \phi_i$, $\tilde{v} = v + \ker \phi_i$ be in $\tilde{\mathcal{B}_i}$. There exist sequences (x_n), (y_n) in $\mathcal{B}$ such that $x_n \to \tilde{u}$, $y_n \to \tilde{v}$ in $(\tilde{\mathcal{B}_i}, |||\cdot|||_i)$, and so also in $|||\cdot|||_k, 0 \leq k \leq i$. Thus $|||x_n - \tilde{u}|||_k \to 0$, $|||y_n - \tilde{v}|||_k \to 0$ for all k, $0 \leq k \leq i$. Then for all such k,

$$\begin{aligned}
|||\tilde{u}\tilde{v}|||_k' &= |||\tilde{\psi}_{k+1}(\tilde{\psi}_{k+2}(\cdots(\tilde{\psi}_i(\tilde{u}\tilde{v}))\cdots))|||_k \\
&= |||\tilde{\psi}_{k+1}(\tilde{\psi}_{k+2}(\cdots(\tilde{\psi}_i(\lim_n(x_n y_n + \ker \phi_i)))\cdots))|||_k \\
&= |||\lim_n \tilde{\psi}_{k+1}(\tilde{\psi}_{k+2}(\cdots(\tilde{\psi}_i(\lim(x_n y_n + \ker \phi_i)))\cdots)|||_k, \\
&\quad \text{by the continuity of } \tilde{\psi}_k. \\
&= |||\lim_n x_n y_n + \ker \phi_k|||_k \text{ as } \tilde{\psi}_k|_{\mathcal{B}} = id, \\
&\quad \text{in an appropriate sense.} \\
&= \lim_n |||x_n y_n|||_k \\
&\leq \lim_n \{|||x_n|||_k |||y_n|||_{k-1} + |||x_n|||_{k-1}|||y_n|||_k\} \\
&= \{(\lim_n |||x_n|||_k)(\lim_n |||y_n|||_{k-1}) \\
&\quad + (\lim_n |||x_n|||_{k-1})(\lim_n |||y_n|||_k)\} \\
&= |||\tilde{\psi}_{k+1}(\tilde{\psi}_{k+2}(\cdots(\tilde{\psi}_i(\tilde{u}))))|||_k |||\tilde{\psi}_k \\
&\quad \times (\tilde{\psi}_{k+1}(\cdots(\tilde{\psi}_i(\tilde{v}))\cdots|||_{k-1} + |||\tilde{\psi}_k \\
&\quad \times (\tilde{\psi}_{k+1}(\cdots(\tilde{\psi}_i(\tilde{u}))\cdots|||_{k-1}|||\tilde{\psi}_{k+1}(\tilde{\psi}_{k+2}(\cdots(\tilde{\psi}_i(\tilde{v}))|||_k \\
&= |||\tilde{u}|||_k'.|||\tilde{v}|||_{k-1}' + |||\tilde{u}|||_{k-1}'.|||\tilde{v}|||_k'.
\end{aligned}$$

For $k = 0$, $|||\cdot|||'$ is a C^*-norm. For $k = 1$, it follows, as above, that $|||\tilde{u}\tilde{v}|||' \leq D\{|||\tilde{u}|||_1'|||\tilde{v}|||_0' + |||\tilde{v}|||_1'|||\tilde{u}|||_0'\}$. It follows that for a fixed i, $1 \leq i < \infty$, $\{|||\cdot|||_k' = |||\cdot|||_{i,k}' : 0 \leq k \leq i\}$ on $\tilde{\mathcal{B}_i}$ has the (D_i^*)-property.

Finally, we show that $\mathcal{B} = \varprojlim \tilde{\mathcal{B}}_i$. Let $\mathcal{I}_i = \ker \phi_i$. Since $\{(\mathcal{B}_i, \psi_i), 1 \le i < \infty\}$ forms an inverse limit sequence with $\mathcal{B} = \varprojlim \mathcal{B}_i$, the chains $\{(\mathcal{I}_i, \psi_i)\}$ and $\{(\tilde{\mathcal{B}}_i, \tilde{\psi}_i)\}$ also form inverse limit sequences of Frechet algebras; and we have the following:



Let $\tilde{\mathcal{B}} := \varprojlim \tilde{\mathcal{B}}_i$. We establish the desired assertion in the following three steps:

Claim 1. There exists a one-to-one homomorphism $\Theta : \mathcal{B} \to \tilde{\mathcal{B}}$ having dense range.

Let $x \in \mathcal{B} = \varprojlim \mathcal{B}_i$, say $x = (x_i)$, where $x_i \in \mathcal{B}_i$, $\psi_i(x_i) = x_{i-1}$ for all i. Take $\tilde{x}_i (= (\tilde{x})_i) = j_i(x_i) = x_i + \ker \phi_i \in \tilde{\mathcal{B}}_i$ for all i. Then $\tilde{\psi}_i(\tilde{x}_i) = \tilde{\psi}_i(x_i + \ker \phi_i) = \psi_i(x_i) + \ker \phi_{i-1} = x_{i-1} + \ker \phi_{i-1} = \tilde{x}_{i-1}$. Hence by the definition of inverse limit, there exists an element $y (= \tilde{x}$ say$) \in \varprojlim \tilde{\mathcal{B}}_i = \tilde{\mathcal{B}}$ such that $y_i (= \tilde{\phi}_i(y) = \tilde{\phi}_i(\tilde{x}) = (\tilde{x})_i) = \tilde{x}_i$, say $\tilde{x} = (\tilde{x}_i)$. This gives a map $\Theta : \mathcal{B} \to \tilde{\mathcal{B}}$, $\Theta(x) = \tilde{x}$ as above. Clearly Θ is a *-homomorphism. Further Θ is one-to-one. For let $x \in \mathcal{B}$ be such that $\Theta(x) = \tilde{x} = 0$. Then $\tilde{x}_i = 0$ for all i. Thus $x + \ker \phi_i = 0, \phi_i(x) = 0$ for all i. Since $x \in \mathcal{B}, x = (\phi_i(x)(= \phi_i(x_i)) = 0$. Thus Θ is one-to-one; and $\mathcal{B} = \varprojlim \mathcal{B}_i$ is embedded, via the *-isomorphism Θ, in the Frechet algebra $\varprojlim \tilde{\mathcal{B}}_i = \tilde{\mathcal{B}}$ with Θ having dense range. Clearly Θ is continuous.

Claim 2. There exists a one-to-one continuous *-homomorphism $\Phi : \varprojlim \tilde{\mathcal{B}}_i \to \mathcal{A}$ having dense range.

Indeed, let $\tilde{x} = (\tilde{x}_i) \in \varprojlim \tilde{\mathcal{B}}_i$. Then $\tilde{x}_i = x_i + \ker \phi_i$ for some $x_i \in \mathcal{B}_i$ and $\tilde{\psi}_i(\tilde{x}_i) = \tilde{x}_{i-1}$. Hence $\psi_i(x_i) + \ker \phi_{i-1} = x_{i-1} + \ker \phi_{i-1}$, and so, for all i, $\phi_i(x_i) = \phi_{i-1}(\psi_i(x_i)) = \phi_{i-1}(x_{i-1}) = y$ (say) in $\mathcal{A}$. This defines a *-homomorphism Φ :

$\varprojlim \tilde{\mathcal{B}}_i \to \mathcal{A}$, $\Phi(\tilde{x}) = y$. Further, if $y = 0$, then for all i, $\phi_i(x_i) = 0$, $x_i \in \ker\phi_i$, $\tilde{x}_i = 0$; and so $\tilde{x} = (\tilde{x}_i) = 0$ showing that Φ is one-to-one.

Claim 3. $\Theta = \Phi = id$ the identity map.

For any i, let $z \in \mathcal{I}_i$. Then $\phi_i(z) = 0$. Since $\phi_i = \phi_{i-1} \circ \psi_i$, $\phi_{i-1}(\psi_i(z)) = 0$, and so $\psi_i(z) \in \ker\phi_{i-1}$. Thus $\psi_i(\mathcal{I}_i) \subset \mathcal{I}_{i-1}$ for all i. Hence the sequence

$$\mathcal{I}_1 \xleftarrow{\psi_2} \mathcal{I}_2 \xleftarrow{\psi_3} \mathcal{I}_3 \xleftarrow{\psi_4} \mathcal{I}_4 \longleftarrow \cdots$$

is an inverse limit sequence. Let $\mathcal{I} = \varprojlim \mathcal{I}_i$. Then $\mathcal{I}$ is a closed two-sided *-ideal of $\mathcal{B} = \varprojlim \mathcal{B}_i$. Also

$$\mathcal{B}/\mathcal{I} = \varprojlim \mathcal{B}_i/\mathcal{I}_i = \varprojlim \tilde{\mathcal{B}}_i = \tilde{\mathcal{B}}.$$

The natural quotient map $\Pi : \mathcal{B} \to \mathcal{B}/\mathcal{I}$ is the map $\Pi(x) = x + \mathcal{I} = (x_i + \mathcal{I}_i) = (\tilde{x}_i)$. Thus $\Pi = \Theta$.

Thus Π is one-to-one; and Θ is onto. It follows that $\mathcal{I} = 0$; and so $\Pi = \Theta = id$. As a result, the map $\Phi = id$; and it follows that $\mathcal{B} = \varprojlim \mathcal{B}_i = \varprojlim \tilde{\mathcal{B}}_i$. This completes the proof of the theorem.

We apply the above theorem to discuss the smoothness properties of Frechet (D_∞^*)-algebras. When $\mathcal{B}$ is realized as an inverse limit of Banach *-algebras $\mathcal{B}_k$ as $\mathcal{B} = \varprojlim \mathcal{B}_k$, the enveloping σ-C^*-algebra $C^*(\mathcal{B})$ of $\mathcal{B}$ (which is the universal object, in the catagory of inverse limits of C^*-algebras, for bounded Hilbert space operator representations of $\mathcal{B}$) is realized as the inverse limit of C^*-algebras as $C^*(\mathcal{B}) = \varprojlim C^*(\mathcal{B}_k)$ [11]. The Frechet algebra $\mathcal{B}$ is an *algebra with a C^*-enveloping algebra* if $C^*(\mathcal{B})$ is a C^*-algebra [7, 11]. This class of Frechet algebras arises in the study of several aspects in C^*-algebras [3]. Part (1) of the following gives (D_∞^*)-algebra analogue of Theorem 5 of [14]. A subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$ is *spectrally invariant* in $\mathcal{A}$ if for every $x \in \mathcal{B}$, $\mathrm{sp}_{\mathcal{B}}(x) = \mathrm{sp}_{\mathcal{A}}(x)$

COROLLARY 3.2

Let $\mathcal{B}$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $\mathcal{A}$.

(1) *If $\mathcal{A}$ has identity* 1, *then* $1 \in \mathcal{B}$ *and* $(\mathcal{B}, \|\cdot\|_0)$ *is a Q-normed algebra.*
(2) *The algebra $\mathcal{B}$ is hermitian, spectrally invariant in $\mathcal{A}$, is closed under the holomorphic functional calculus of $\mathcal{A}$ and is an algebra with a C^*-enveloping algebra having $C^*(\mathcal{B}) = \mathcal{A}$.*

Proof. In view of the above theorem, $\mathcal{B} = \varprojlim \tilde{\mathcal{B}}_k$ is an inverse limit of Banach (D_k^*)-subalgebras $\tilde{\mathcal{B}}_k$ of $\mathcal{A}$. By Theorem 5, p. 404 of [14], $\mathbf{1} \in \tilde{\mathcal{B}}_k$ for all k, hence $\mathbf{1} \in \mathcal{B}$. Also by the same result, the Banach algebra $\tilde{\mathcal{B}}_k$ is spectrally invariant in $\mathcal{A}$; and for any $y \in \tilde{\mathcal{B}}_k$, the spectra are same as $\mathrm{sp}_{\tilde{\mathcal{B}}_k}(y) = \mathrm{sp}_{\mathcal{A}}(y)$ and the spectral radius in $\tilde{\mathcal{B}}_k$ satisfies $r_{\tilde{\mathcal{B}}_k}(y) \leq \|y\|_0$. Now let $x \in \mathcal{B}$ be realized as the coherent sequence $x = (x_k)$ with each $x_k \in \tilde{\mathcal{B}}_k$, $\mathrm{sp}_{\mathcal{B}}(x) = \cup_k \mathrm{sp}_{\tilde{\mathcal{B}}_k}(x_k)$ by [18]. Thus $\mathrm{sp}_{\mathcal{B}}(x) = \mathrm{sp}_{\mathcal{A}}(x)$; and $r_{\mathcal{B}}(x) = r_{\mathcal{A}}(x) \leq \|x\|_0$. Hence the normed algebra $(\mathcal{B}, \|\cdot\|_0)$ is a Q-algebra. As the Frechet topology on $\mathcal{B}$ is finer than the $\|\cdot\|_0$-topology, the Frechet algebra $(\mathcal{B}, \tau)$ is also a Q-algebra. Since a

Frechet Q-algebra is an algebra with a C^*-enveloping algebra [7], $C^*(\mathcal{B})$ is a C^*-algebra; since $C^*(\tilde{\mathcal{B}}_k) = \mathcal{A}$, we have $C^*(\mathcal{B}) = \mathcal{A}$. It also follows from [4] that $\mathcal{B}$ is closed under the holomorphic functional calculus of $\mathcal{A}$. □

By Theorem 12 of [14] and [5], each Banach (D_p^*)-subalgebra of a C^*-algebra as well as a differential Frechet *-algebra defined by a differential norm is closed under the C^∞-functional calculus for self-adjoint elements. The following includes both these results.

COROLLARY 3.3

Let $\mathcal{B}$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $\mathcal{A}$. Then $\mathcal{B}$ is closed under the C^∞-functional calculus for self-adjoint elements of $\mathcal{A}$.

Proof. In the notations of the proof of Theorem 3.1, $\mathcal{B} = \varprojlim \tilde{\mathcal{B}}_k$ where each $\tilde{\mathcal{B}}_k$ is a Banach (D_k^*)-subalgebra of $\mathcal{A}$, and $\{\tilde{\psi}_k\}_{k=1}^\infty$ is a sequence of linking homomorphisms $\tilde{\psi}_k : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_{k-1}$. Let $x = x^* \in \mathcal{B}$ be described by a coherent sequence (x_k) with each $x_k = x_k^* \in \mathcal{B}_k$ and $\tilde{\psi}_k(x_k) = x_{k-1}$. Since each of $\mathcal{B}$ and each $\tilde{\mathcal{B}}_k$ are spectrally invariant and Q-algebras, there is a closed bounded interval $[a, b]$ in $\mathbb{R}$ such that $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_{\tilde{\mathcal{B}}_k}(x_k) = \mathrm{sp}_{\mathcal{B}}(x) \subset [a, b] \subset \mathbb{R}$. We shall apply Theorem 12 of [14] to each $\tilde{\mathcal{B}}_k$. Let f be a C^∞-function on $[a, b]$. By appropriate extension, we can assume that $f \in \mathcal{C}_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space. Hence its Fourier transform $\widehat{f} \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$. For any linear differential operator $P(D)$ with constant co-efficients, $(\widehat{P(D)f})(s) = P(s)\widehat{f}(s)$; and $g = P(D)f \in C^\infty(\mathbb{R})$. Thus the conditions of Theorem 12 of [14] are satisfied. Set $y_k = f(x_k)$. Then by the standard C^*-algebra theory [20] and as in Theorem 12 of [14],

$$f(x) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{isx}\mathrm{d}s \in \mathcal{A};$$

$$y_k = f(x_k) = (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{isx_k}\mathrm{d}s \in \tilde{\mathcal{B}}_k.$$

Now notice that (y_k) is a coherent sequence satisfying $\tilde{\psi}_k(y_k) = y_{k-1}$ for all k. Indeed by continuity

$$\begin{aligned}
\tilde{\psi}_k(y_k) &= \tilde{\psi}_k(f(x_k)) \\
&= (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{is\tilde{\psi}_k(x_k)}\mathrm{d}s \\
&= (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} \hat{f}(s)\mathrm{e}^{isx_{k-1}}\mathrm{d}s \\
&= f(x_{k-1}) = y_{k-1} = f(\tilde{\psi}_k(x_k)).
\end{aligned}$$

Hence by the definition of inverse limit, there exists $y \in \mathcal{B}$ such that $(y)_k = y_k = f(x_k)$. Thus $y = f(x) \in \mathcal{B}$.

PROPOSITION 3.4

Let $\mathcal{B}$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $\mathcal{A}$. Let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$.

(1) *Let $\mathcal{A}$ have identity* **1**. *Then $\mathcal{J} = \mathcal{I} \cap \mathcal{B}$ is a Frechet (D_∞^*)-subalgebra of $\mathcal{I}$.*

(2) *Let* $\{u_i\}$ *be a bai for* $\mathcal{I}$ *contained in* $\mathcal{J}$ *which is also a bai for* $\mathcal{J}$ *in the Frechet topology of* $\mathcal{J}$ *such that* $\|\mathbf{1} - u_i\|_j \leq 1$ *for all* i, j. *Then* $\mathcal{B}/\mathcal{J}$ *is a Frechet* (D_∞^*)*-subalgebra of the* C^**-algebra* $\mathcal{A}/\mathcal{I}$.

Proof.

(1) By Corollary 3.2, $\mathbf{1} \in \mathcal{B}$. By Corollary 3.3, $\mathcal{B}$ is closed under C^∞-functional calculus of $\mathcal{A}$. Hence by Remark, p. 274 of [13], $\mathcal{B}$ is a locally normal Q^*-algebra, and Theorem 13 of [13] applies by showing that $\mathcal{J}$ is dense in $\mathcal{I}$, and (1) follows.
(2) Define $\varphi : \mathcal{B}/\mathcal{J} \to \mathcal{A}/\mathcal{I}$ by $\varphi(x + \mathcal{J}) = x + \mathcal{I}$. Clearly, φ is an injective *-homomorphism. Thus we identify $\mathcal{B}/\mathcal{J}$ with the *-subalgebra $\varphi(\mathcal{B}/\mathcal{J})$ of $\mathcal{A}/\mathcal{I}$. The C^*-norm on $\mathcal{A}/\mathcal{I}$ is $\|x + \mathcal{I}\|_0 = \inf\{\|x + y\|_0 : y \in \mathcal{I}\}$, which induces a C^*-norm on $\mathcal{B}/\mathcal{J}$. As the ideal $\mathcal{J}$ is closed in $\mathcal{B}$ in each norm $\|\cdot\|_j$, the quotient norms on $\mathcal{B}/\mathcal{J}$ defined as $\|x + \mathcal{J}\|_i = \inf\{\|x + y\|_i : y \in \mathcal{J}\}$ make $\mathcal{B}/\mathcal{J}$ a Frechet algebra. Since $\mathcal{B}$ is dense in $\mathcal{A}$, $\mathcal{B}/\mathcal{J}$ is dense in $\mathcal{A}/\mathcal{I}$. Let $x \in \mathcal{B}$ and $j \geq 0$ be fixed. Then for all $z \in \mathcal{I} \cap \mathcal{B}$, the fact that u_i is a bounded approximate identity for the Frechet topology on $\mathcal{J}$ implies that

$$\begin{aligned}\limsup_i \|x - xu_i\|_j &= \limsup_i \|x - xu_i - zu_i + z\|_j \\ &= \limsup_i \|(x + z)(\mathbf{1} - u_i)\|_j \leq \|x + z\|_j.\end{aligned}$$

Consequently, for each $x \in \mathcal{B}$,

$$\begin{aligned}\|x + \mathcal{J}\|_j &= \inf\{\|x + z\|_j : z \in \mathcal{J}\} \\ &\leq \liminf_i \|x - xu_i\|_j \\ &\leq \limsup_i \|x - xu_i\|_j \leq \inf\{\|x + z\|_j : z \in \mathcal{J}\} = \|x + \mathcal{J}\|_j\end{aligned}$$

and hence

$$\|x + \mathcal{J}\|_j = \inf\{\|x + z\|_j : z \in \mathcal{J}\} = \lim_i \|x - xu_i\|_j \tag{3.1}$$

the limit exists due to the above. Thus for $x, y \in \mathcal{B}$,

$$\begin{aligned}\|(x + J)(y + J)\|_j = \|xy + J\|_j &= \inf\{\|xy + z\|_j : z \in \mathcal{J}\} \\ &\leq \|xy - xu_i y - xyu_i + xu_i yu_i\|_j \\ &= \|(x - xu_i)(y - yu_i)\|_j \\ &\leq D_j(\|x - xu_i\|_j \|y - yu_i\|_{j-1} \\ &\quad + \|x - xu_i\|_{j-1} \|y - yu_i\|_j).\end{aligned}$$

Taking limits over i and using (3.1),

$$\|(x + J)(y + J)\|_j \leq D_j(\|x + J\|_j \|y + J\|_{j-1} + \|x + J\|_{j-1} \|y + J\|_j). \tag{3.2}$$

This completes the proof. □

In fact, by Theorem 13(ii) of [13], if $\mathcal{A}$ and $\mathcal{B}$ are as above and if $\mathcal{A}$ is unital, then $\mathcal{I} \to \mathcal{I} \cap \mathcal{B}$ is a one-to-one correspondence between the set of all closed ideals of $\mathcal{A}$ and the set of all $\|\cdot\|_0$-closed ideals of $\mathcal{B}$. The following explains what happens in case of $C^\infty[0,1] = \cap_{k=0}^{\infty} C^k[0,1]$. Recall that a norm $|\cdot|$ on a normed algebra $(\mathcal{X}, \|\cdot\|)$ is closable with respect to $\|\cdot\|$ if the identity map $id : (\mathcal{X}, |\cdot|) \to (\mathcal{X}, \|\cdot\|)$ is closable.

PROPOSITION 3.5

Let $(\mathcal{B}, \{\|\cdot\|_n\})$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $(\mathcal{A}, \|\cdot\|_0)$ such that the following hold:

(1) $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|_n \leq \cdots$
(2) *Each $\|\cdot\|_n$ is closable with respect to $\|\cdot\|_0$.*

Then there exists a decreasing sequence $\mathcal{B}_n$ of Banach *-algebras such that for each n, $\mathcal{B}_n$ is a Banach (D_n^*)-subalgebra of $\mathcal{A}$ and $\mathcal{B} = \cap_{n=0}^{\infty} \mathcal{B}_n$ topologically as well.

Proof. Let the Banach *-algebra $\mathcal{B}_n$ be the completion of $(\mathcal{B}, \|\cdot\|_n)$. Let $\psi_n : \mathcal{B}_{n+1} \to \mathcal{B}_n$ be the contractive *-homomorphism obtained by the extension by continuity of the identity map on $\mathcal{B}$. By the general theory of Frechet algebras [18], $\mathcal{B} = \varprojlim \mathcal{B}_n$. The closability condition implies that each $\mathcal{B}_n$ is a subalgebra of $\mathcal{A}$. Indeed, since $\|\cdot\|_0 \leq \|\cdot\|_n$ for all n, the identity map on $\mathcal{B}$ extends as continuous *-homomorphism $\phi_n : \mathcal{B}_n \to \mathcal{A}$. Let $x \in \ker \phi_n$. Then for some sequence (x_k) in $\mathcal{B}$, $\|x_k - x\|_n \to 0$ and $x_k = \phi_n(x_k) \to \phi_n(x) = 0$ in $\|\cdot\|_0$. By closability, $\|x_k\|_n \to 0$ as $k \to \infty$. It follows that $x = 0$ so that $\ker \phi_n = 0$, and $\mathcal{B}_n \subset \mathcal{A}$. Further the closability of each $\|\cdot\|_n$ with respect to $\|\cdot\|_0$ implies that for any $m < n$, $\|\cdot\|_n$ is closable with respect to $\|\cdot\|_m$; hence $\mathcal{B}_n \subset \mathcal{B}_m$. Thus the above inverse limit sequence become

$$\cdots \subset \mathcal{B}_{k+1} \subset \mathcal{B}_k \subset \cdots \subset \mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{B}_0 = \mathcal{A}$$

and $\mathcal{B} = \varprojlim \mathcal{B}_n = \bigcap_{n=0}^{\infty} \mathcal{B}_n$. □

PROPOSITION 3.6

For $i = 1, 2$, let $\mathcal{B}_i$ be a Frechet (D_∞^*)-subalgebra of a C^*-algebra $\mathcal{A}_i$. Let $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ be a homomorphism. Then the following hold:

(1) *ϕ is C^*-norm decreasing.*
(2) *ϕ extends uniquely to a C^*-algebra homomorphism from $\mathcal{A}_1 \to \mathcal{A}_2$.*
(3) *ϕ is continuous in the (D_∞^*) structures of $\mathcal{B}_1$ and $\mathcal{B}_2$.*
(4) *If ϕ is injective, then ϕ is an isometry for the respective C^*-norms.*
(5) *$\mathcal{B}_2 \cap \phi(\mathcal{A}_1)$ is a Frechet (D_∞^*)-subalgebra of the C^*-algebra $\phi(\mathcal{A}_1)$. Also $\phi(\mathcal{B}_1)$ is a Frechet (D_∞^*)-subalgebra of $\phi(\mathcal{A}_1)$ in the quotient topology from $\mathcal{B}_1/\ker\phi$.*

Proof. For any $x \in \mathcal{B}_1$ and with $\|\cdot\|_0$ denoting the respective C^*-norms, we have $\|\phi(x)\|_0^2 = \|\phi(x^*x)\|_0 = r_{\mathcal{A}_2}(\phi(x^*x)) = r_{\mathcal{B}_2}(\phi(x^*x)) \leq r_{\mathcal{B}_1}(x^*x) = r_{\mathcal{A}_1}(x^*x) = \|x^*x\|_0 = \|x\|_0^2$ (by the spectral invariance of $\mathcal{B}_2$ in $\mathcal{A}_2$). This proves (1) from which (2) is immediate.

The assertion (3) follows from a closed graph argument. Indeed, let (x_n) be a sequence in $\mathcal{B}_1$ such that $x_n \to x \in \mathcal{B}_1$ and $\phi(x_n) \to y \in \mathcal{B}_2$ in their respective Frechet topologies. Then $\|x_n - x\|_0 \to 0$. By (1), $\|\phi(x_n) - y\|_0 \to 0$. Hence $y = \phi(x)$ and ϕ is continuous in the respective (D_∞^*)-structures.

For (4), let ϕ be injective. Now ϕ extends as a C^*-norm decreasing *-homomorphism $\tilde{\phi} : \mathcal{A}_1 \to \mathcal{A}_2$ and $\phi(\mathcal{A}_1)$ is a C^*-subalgebra of $\mathcal{A}_2$. Let $\mathcal{J} = \ker\tilde{\phi}$. By Proposition 3.4, $\mathcal{B}_1 \cap \mathcal{J}$ is dense in $\mathcal{J}$. Since $\mathcal{B}_1 \cap \mathcal{J} = 0$, $\mathcal{J} = 0$. Thus $\tilde{\phi}$ is injective; by standard C^*-theory, $\tilde{\phi}$ and so ϕ is an isometry for C^*-norms.

For (5), notice that $\mathcal{B}_2 \cap \phi(\mathcal{A}_1)$ is complete in the relative topology τ_2 from the Frechet algebra topology of $\mathcal{B}_2$. For, let $(y_n) \subset \mathcal{B}_2 \cap \phi(A_1)$ be a τ_2-Cauchy sequence. Since $\phi(\mathcal{A}_1)$ is a C^*-subalgebra of $\mathcal{A}_2$, there exists $y \in \phi(\mathcal{A}_1)$ such that $\|y_n - y\|_0 \to 0$. By the completeness of $(\mathcal{B}_2, \tau_2)$, there exists $z \in \mathcal{B}_2$ such that y_n τ_2-converges to z. Thus $y = z \in \mathcal{B}_2 \cap \phi(\mathcal{A}_1)$; and $(\mathcal{B}_2 \cap \phi(\mathcal{A}_1), \tau)$ is a Frechet algebra. Now $\phi(\mathcal{B}_1) \subset \mathcal{B}_2 \cap \phi(\mathcal{A}_1)$ and $\phi(\mathcal{B}_1)$ is dense in the C^*-algebra $\phi(\mathcal{A}_1)$; and so $\mathcal{B}_2 \cap \phi(\mathcal{A}_1)$ is dense in $\phi(\mathcal{A}_1)$ and is a Frechet (D^*_∞)-subalgebra of $\phi(\mathcal{A}_1)$. Now with the relative topology, $\mathcal{B}_1 / \ker(\phi|_{\mathcal{B}_1})$ is a Frechet D^*_∞ subalgebra of the quotient C^*-algebra $\mathcal{A}_1 / \ker\phi$. The remaining conclusion follows in the light of natural isomorphisms.

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