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The boundedness of multilinear Calderón–Zygmund operators on Hardy spaces

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Abstract. In this paper, we study the boundedness of the multilinear Calderón–Zygmund operators on products of Hardy spaces.

Keywords. Multilinear Calderón–Zygmund operators; Hardy spaces; atomic decomposition; molecular decomposition.

1. Introduction

The study of multilinear singular integral operators has recently received increasing attention. The class of multilinear Calderón–Zygmund operators was introduced and first investigated by Coifman and Meyer in [3–5] and was systematically studied by Grafakos and Torres in [10]. In [11], the authors proved that the multilinear Calderón–Zygmund operators were bounded from products of Hardy spaces to Lebesgue space and they pointed out that some further cancellation conditions must be found in order to get the boundedness from products of Hardy spaces to Hardy spaces. In this paper, we give some cancellation conditions that can imply the boundedness of multilinear Calderón–Zygmund operators on products of Hardy spaces.

We first recall the definition of Hardy space $H^p(\mathbb{R}^d)$. We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of all Schwartz functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ its dual space, the set of all tempered distributions on $\mathbb{R}^d$. Similarly we denote by $\mathcal{D}(\mathbb{R}^d)$ the set of all C^∞ functions with compact support on $\mathbb{R}^d$ and by $\mathcal{D}'(\mathbb{R}^d)$ the set of all distributions on $\mathbb{R}^d$.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ with supp $\varphi \subset B(0,1)$ and

$$\int_0^\infty \left|\hat{\varphi}(t\xi)\right|^2 \frac{\mathrm{d}t}{t} = 1 \tag{1}$$

for $\xi \in \mathbb{R}^d$ and $\xi \neq 0$, where $B(0,1) = \{x \in \mathbb{R}^d : |x| = 1\}$. Let $\varphi_t(x) = \frac{1}{t^d}\varphi(\frac{x}{t})$ for $t > 0$. The Littlewood–Paley g-function of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$g(f)(x) = \left(\int_0^\infty |\varphi_t * f(x)|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}. \tag{2}$$

Let $0 < p \leq 1$. Then the Hardy space $H^p(\mathbb{R}^d)$ is defined by (cf. Theorem 7.28 of [8] or [6])

$$H^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : g(f) \in L^p(\mathbb{R}^d)\}$$

and $\|f\|_{H^p(\mathbb{R}^d)} = \|g(f)\|_{L^p(\mathbb{R}^d)}$.

Let $0 < p \leq 1$ and $1 < q \leq \infty$. We call a function $a(x)$ is a $H^{p,q}$ atom associated to a cube Q in $\mathbb{R}^d$, if

(i) $\operatorname{supp} a \subset Q$,
(ii) $\|a\|_{L^q} \leq |Q|^{\frac{1}{q}-\frac{1}{p}}$,
(iii) for all multi-index γ with $0 \leq |\gamma| \leq [d(\frac{1}{p} - 1)]$, we have $\int_{\mathbb{R}^d} a(x)x^\gamma \,\mathrm{d}x = 0$.

The atomic decomposition of H^p can be stated as follows (cf. [2, 13]).

PROPOSITION 1.1

Let $0 < p \leq 1$, $1 < q \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then $f \in H^p(\mathbb{R}^d)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^{p,q}$ atoms, $\sum_j |\lambda_j|^p < \infty$, and the sum converges in $H^p(\mathbb{R}^d)$ norm. Moreover,

$$\|f\|_{H^p} \sim \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{\frac{1}{p}}\right\},$$

where the infimum is taken over all atomic decompositions of f into $H^{p,q}$ atoms.

Hardy space H^p also admits molecular decomposition similar to atomic decomposition. For completeness, we give the definition of molecular decomposition (cf. [14]).

Let $0 < p \leq 1 \leq q \leq \infty$, $p < q$, $s \geq s_0 = [d(\frac{1}{p} - 1)]$, $\epsilon > \max\{\frac{s}{d}, \frac{1}{p} - 1\}$ and $a = 1 - \frac{1}{p} + \epsilon$, $b = 1 - \frac{1}{q} + \epsilon$. A function $M \in L^q(\mathbb{R}^d)$ is called a (p, q, s, ϵ) molecule with the center at x_0, if

(1) $|x|^{db} M(x) \in L^q(\mathbb{R}^d)$,
(2) $\mathcal{N}_q(M) := \|M\|_q^{a/b} \|(\cdot - x_0)M(\cdot)\|_q^{1-a/b} < \infty$,
(3) $\int_{\mathbb{R}^d} M(x)x^\beta \mathrm{d}x = 0,\ 0 \leq |\beta| \leq s$.

The molecular decomposition of H^p can be stated as follows:

PROPOSITION 1.2

Let $0 < p \leq 1 \leq q \leq \infty$, $p < q$, $s \geq s_0 = [d(\frac{1}{p} - 1)]$, $\epsilon > \max\{\frac{s}{d}, \frac{1}{p} - 1\} a = 1 - \frac{1}{p} + \epsilon$ and $b = 1 - \frac{1}{q} + \epsilon$. For $f \in \mathcal{S}'(\mathbb{R}^d)$, $f \in H^p(\mathbb{R}^d)$ if and only if f can be written as $f = \sum_j \lambda_j M_j$, where M_j are (p, q, s, ϵ) molecules, $\sum_j |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{H^p} \sim \inf\left\{\left(\sum_j |\lambda_j|\right)^{\frac{1}{p}}\right\},$$

where the infimum is taken over all decompositions $f = \sum \lambda_j M_j$, where M_j are (p, q, s, ϵ) molecules.

Remark 1.3. Molecular decomposition is very useful in the proof of boundedness of singular integral operators T on Hardy spaces. However, Bownik proved that we cannot get the boundedness on Hardy space from T maps atom to molecule in [1]. In order to get our result, we solve this problem by showing that the kernel of T is in the dual space of the Hardy–Campanato space (cf. [9]). It is easy to know that the kernel of multilinear Calderón–Zygmund operator given by (3)–(4) is in Campanato space associated to every variable. Therefore, T is a linearly continuous functional. Then, for any atomic decomposition of $f_j \in H^{p_j}$, $1 \le j \le m$, as a sum of H^{p_j,q_j} atoms, $f_j = \sum_{k=1}^{\infty} \lambda_{j,k} a_{j,k}$,

$$\begin{aligned} T(f_1,\ldots,f_m)(x) &= T(\lim_{N_1\to\infty}\sum_{k_1=1}^{N_1}\lambda_{1,k_1}a_{1,k_1},\ldots,\lim_{N_m\to\infty}\sum_{k_m=1}^{N_m}\lambda_{m,k_m}a_{m,k_m}) \\ &= \lim_{N_1\to\infty}\cdots\lim_{N_m\to\infty}\sum_{k_1=1}^{N_1}\cdots\sum_{k_m=1}^{N_m}\lambda_{1,k_1} \\ &\quad \cdots\lambda_{m,k_m}T(a_{1,k_1},\ldots,a_{m,k_m})(x) \\ &= \sum_{k_1=1}^{\infty}\cdots\sum_{k_m=1}^{\infty}\lambda_{1,k_1}\cdots\lambda_{m,k_m}T(a_{1,k_1},\ldots,a_{m,k_m})(x). \end{aligned}$$

In the following, we give some notations about the multilinear Calderón–Zygmund operators. Let $K(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^d)^{m+1}$ which satisfies the size estimate

$$|K(x, y_1, \ldots, y_m)| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{md}} \tag{3}$$

for some $A > 0$ and all $(x, y_1, \ldots, y_m) \in (\mathbb{R}^d)^{m+1}$ with $x \ne y_j$ for some j. Furthermore, we assume that it satisfies smoothness estimates

$$|\partial_x^{\alpha_0}\cdots\partial_{y_m}^{\alpha_m} K(x, y_1, \ldots, y_m)| \le \frac{A_\alpha}{(|x-y_1| + \cdots + |x-y_m|)^{md+|\alpha|}}, \quad \text{for all } |\alpha| \le N, \tag{4}$$

where $N = [d(1/p - 1)] + 1$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$ and $|\alpha| = \sum_{i=0}^{m} \alpha_i$.

In this paper, we study m-linear operators $T : \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ for which there exists a function K satisfying (3)–(4) which is defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^d)^{m+1}$ such that

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^d)^m} K(x, y_1, \ldots, y_m) f_1(y_1)\cdots f_m(y_m)\mathrm{d}y_1\cdots\mathrm{d}y_m, \tag{5}$$

whenever $f_1, \ldots, f_m \in \mathcal{D}(\mathbb{R}^d)$ and $x \notin \bigcap_{j=1}^{m} \operatorname{supp} f_j$. We will say that T is an m-linear operator with the Calderón–Zygmund kernel K. We consider a class of multilinear Calderón–Zygmund operators, whose kernels satisfy some cancellation conditions similar to those introduced by Fefferman and Stein in [7] and, Han and Yang in [12]. A function ϕ is said to be a normalized bump function if ϕ is supported in the unit ball and $|\partial_x^\alpha \phi(x)| \le 1$

for $\alpha \in \mathbb{Z}_+^d$ and $0 \le |\alpha| \le 1$. Then cancellation conditions are given as follows: for all normalized bump functions ϕ on $\mathbb{R}^d$ and for all $\delta > 0$,

$$\left| \int_{\mathbb{R}^d} K(x-y, y_1, \ldots, y_m)\phi(\delta y)\mathrm{d}y \right| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{(m-1)d}} \tag{6}$$

and

$$\left| \int_{\mathbb{R}^d} \partial_x^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K(x-y, y_1, \ldots, y_m)\phi(\delta y)\mathrm{d}y \right| \le \frac{A_\alpha}{(|x-y_1| + \cdots + |x-y_m|)^{(m-1)d+|\alpha|}} \tag{7}$$

with $|\alpha| \le [d(1/p-1)] + 1$.

The main result of this paper is the following theorem:

Theorem 1.4. *Let $1 < q_1, \ldots, q_m, q < \infty$ be fixed indices satisfying*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$$

and $0 < p_1, \ldots, p_m, p \le 1$ be real numbers satisfying

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Suppose that K satisfies (3), (4), (6) and (7). Let T be related to K as in (5) and assume that T admits an extension that maps $L^{q_1}(\mathbb{R}^d) \times \cdots \times L^{q_m}(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$. Then T can be extended to a bounded operator from $H^{p_1}(\mathbb{R}^d) \times \cdots \times H^{p_m}(\mathbb{R}^d)$ to $H^p(\mathbb{R}^d)$.

In what follows, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line.

2. The proof of the main result

In this section, we give the proof of Theorem 1.4. Our proof is motivated by Han and Yang [12].

Let φ be the function defined by (1) and K be the kernel in Theorem 1.4. For $t > 0$, we denote

$$K_t(x, y_1, \ldots, y_m) = \int_{\mathbb{R}^d} \varphi_t(x-y)K(y, y_1, \ldots, y_m)\mathrm{d}y. \tag{8}$$

Then, we have the following lemma.

Lemma 2.1. *Let K_t be defined by (8) and $t > 0$. Then*

(1) $$|K_t(x, y_1, \ldots, y_m)| \le C \frac{t}{(t + |x-y_1| + \cdots + |x-y_m|)^{d+1}} \times \frac{1}{(|x-y_1| + \cdots + |x-y_m|)^{(m-1)d}};$$

(2)
$$\begin{aligned}&|\partial_x^{\alpha_0}\cdots\partial_{y_m}^{\alpha_m}K_t(x,y_1,\cdots,y_m)|\\ &\quad\le C\frac{t}{(t+|x-y_1|+\cdots+|x-y_m|)^{d+1}}\\ &\qquad\times\frac{1}{(|x-y_1|+\cdots+|x-y_m|)^{(m-1)d+|\alpha|}},\\ &\qquad \mathit{for}\ |\alpha|\le[d(1/p-1)]+1\}.\end{aligned}$$

Proof. Without loss of generality, we can assume that φ in (2) is a normalized bump function.

(1) We consider two cases:

Case 1. $|x-y_j|\le 2t$ for all $1\le j\le m$. By (6), we have

$$\begin{aligned}|K_t(x,y_1,\dots,y_m)| &= t^{-d}\left|\int_{\mathbb{R}^d}\varphi\left(\frac{x-y}{t}\right)K(y,y_1,\dots,y_m)\mathrm{d}y\right|\\ &\le C\frac{t}{(t+|x-y_1|+\cdots+|x-y_m|)^{d+1}}\\ &\quad\times\frac{1}{(|x-y_1|+\cdots+|x-y_m|)^{(m-1)d}}.\end{aligned}$$

This completes the proof of Case 1.

Case 2. There exists some $1\le j\le m$ such that $|x-y_j|>2t$. Without loss of generality, we can assume that $|x-y_1|>2t$. Then, by the vanishing condition of φ and condition (4), we have

$$\begin{aligned}|K_t(x,y_1,\dots,y_m)| &= \left|\int_{\mathbb{R}^d}\varphi_t(x-y)(K(y,y_1,\dots,y_m)\right.\\ &\qquad\left.-K(x,y_1,\dots,y_m))\mathrm{d}y\right|\\ &\le C\int_{\mathbb{R}^d}|\varphi_t(x-y)|\frac{A_1t}{(|x-y_1|+\cdots+|x-y_m|)^{md+1}}\mathrm{d}y\\ &\le C\frac{t}{(|x-y_1|+\cdots+|x-y_m|)^{md+1}}\int_{\mathbb{R}^d}|\varphi_t(x-y)|\mathrm{d}y\\ &\le C\frac{t}{(t+|x-y_1|+\cdots+|x-y_m|)^{d+1}}\\ &\quad\times\frac{1}{(|x-y_1|+\cdots+|x-y_m|)^{(m-1)d}}.\end{aligned}$$

This gives the proof of Case 2.

(2)

Case 1. $|x-y_j|\le 2t$ for all $1\le j\le m$. By condition (7), we can get

$$\begin{aligned}&|\partial_x^{\alpha_0}\cdots\partial_{y_m}^{\alpha_m}K_t(x,y_1,\dots,y_m)|\\ &\quad\le t^{-d}\int_{\mathbb{R}^d}\left|\varphi\left(\frac{y}{t}\right)\right|\,|\partial_x^{\alpha_0}\cdots\partial_{y_m}^{\alpha_m}K(x-y,y_1,\dots,y_m)|\mathrm{d}y\\ &\quad\le C\frac{t}{(t+|x-y_1|+\cdots+|x-y_m|)^{d+1}}\frac{1}{(|x-y_1|+\cdots+|x-y_m|)^{(m-1)d+|\alpha|}}.\end{aligned}$$

Case 2. There exists some $1 \le j \le m$ such that $|x - y_j| > 2t$. The proof of this case is quite similar to Case 2 of (1), and so we omit it.

Let H be the Hilbert space defined by

$$H = \{f : (0, \infty) \to \mathbb{R} \ \text{ is measurable}, \ |f|_H < \infty\},$$

where

$$|f|_H = \left(\int_0^\infty |f(t)|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

For $p > 0$, we also define

$$L^p_H(\mathbb{R}^d) = \{f_t(x) : \mathbb{R}^d \to H \quad \text{and} \quad \|\{f_t\}_{t>0}\|_{L^p_H(\mathbb{R}^d)} < \infty\},$$

where

$$\|\{f_t\}_{t>0}\|_{L^p_H(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f_t(x)|^p_H \,\mathrm{d}x \right)^{1/p}.$$

Using the above notations, we have the following lemma.

Lemma 2.2. *Let K_t be defined by* (10) *and $t > 0$. Then there exists some $\epsilon > 0$ such that*

(1) $$|K_t(x, y_1, \ldots, y_m)|_H \le \frac{C}{(|x - y_1| + \cdots + |x - y_m|)^{md}};$$

(2) $$|\partial_x^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K_t(x, y_1, \ldots, y_m)|_H \le \frac{C}{(|x - y_1| + \cdots + |x - y_m|)^{md+|\alpha|}}, \qquad \textit{for } |\alpha| \le [d(1/p - 1)] + 1.$$

Proof. The proofs of (1) and (2) are similar, and hence we give the proof of (2).

By (2) of Lemma 2.1, we get

$$\begin{aligned}
&|\partial_x^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K_t(x, y_1, \ldots, y_m)|_H \\
&\le C \left(\int_0^\infty \frac{t^2}{(t + |x - y_1| + \cdots + |x - y_m|)^{2d+2}} \frac{\mathrm{d}t}{t} \right)^{1/2} \\
&\quad \times \frac{1}{(|x - y_1| + \cdots + |x - y_m|)^{(m-1)d+|\alpha|}} \\
&= C \left(\int_0^{|x-y_1|+\cdots+|x-y_m|} \frac{t^2}{(t + |x - y_1| + \cdots + |x - y_m|)^{2d+2}} \frac{\mathrm{d}t}{t} \right. \\
&\quad \left. + \int_{|x-y_1|+\cdots+|x-y_m|}^\infty \frac{t^2}{(t + |x - y_1| + \cdots + |x - y_m|)^{2d+2}} \frac{\mathrm{d}t}{t} \right)^{1/2} \\
&\quad \times \frac{1}{(|x - y_1| + \cdots + |x - y_m|)^{(m-1)d+|\alpha|}} \\
&\le C \frac{1}{(|x - y_1| + \cdots + |x - y_m|)^{md+|\alpha|}}.
\end{aligned}$$

Similar to the proof of Theorem 1.1 in [11], we can prove the following lemma. In fact, we just need to replace the Euclidean norm by $|\cdot|_H$.

Lemma 2.3. *Let* $1 < q_1, \dots, q_m, q < \infty$ *be fixed indices satisfying*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$$

and let $0 < p_1, \dots, p_m, p \leq 1$ *be real numbers satisfying*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Suppose that K_t *satisfies Lemma* 2.2. *Let* T_t *be related to* K_t *as in* (5) *and assume that* T_t *admits an extension that maps* $L^{q_1}(\mathbb{R}^d) \times \cdots \times L^{q_m}(\mathbb{R}^d)$ *into* $L^q_H(\mathbb{R}^d)$. *Then* T_t *can be extended to a bounded operator from* $H^{p_1}(\mathbb{R}^d) \times \cdots \times H^{p_m}(\mathbb{R}^d)$ *into* $L^p_H(\mathbb{R}^d)$.

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. Let

$$T_t(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^d)^m} \left(\int_{\mathbb{R}^d} \varphi_t(x-y) K(y, y_1, \dots, y_m) \mathrm{d}y \right) \times f_1(y_1), \dots, f_m(y_m) \mathrm{d}y_1, \dots, \mathrm{d}y_m.$$

By the L^p-bounedness of Littlewood–Paley g-function, we know that T_t is bounded from $L^{q_1}(\mathbb{R}^d) \times \cdots \times L^{q_m}(\mathbb{R}^d)$ to $L^q_H(\mathbb{R}^d)$. Then, Theorem 1.4 follows from Lemmas 2.2 and 2.3. □

Finally, we give a necessary and sufficient cancellation condition that can imply the boundedness of mutilinear Calderón–Zygmund singular integral operators on the products of Hardy spaces. The main tool we use is the molecular theory (cf. [14]).

PROPOSITION 2.4

Let $1 < q_1, \dots, q_m, q < \infty$ *be fixed indices satisfying*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$$

and let $0 < p_1, \dots, p_m, p \leq 1$ *be real numbers satisfying*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Suppose that K *satisfies*

$$|\partial_x^{\alpha_0} \cdots \partial_{y_m}^{\alpha_m} K(x, y_1, \dots, y_m)| \leq \frac{A_\alpha}{(|x-y_1| + \cdots + |x-y_m|)^{md+|\alpha|}}, \quad \text{for all } |\alpha| \leq N, \tag{9}$$

with $N = [d(1/p - 1)]$. *Let* T *be related to* K *as in* (5) *and assume that* T *admits an extension that maps* $L^{q_1}(\mathbb{R}^d) \times \cdots \times L^{q_m}(\mathbb{R}^d)$ *to* $L^q(\mathbb{R}^d)$. *Then* T *can be extended to a bounded operator from* $H^{p_1}(\mathbb{R}^d) \times \cdots \times H^{p_m}(\mathbb{R}^d)$ *to* $H^p(\mathbb{R}^d)$ *if and only if*

$$\int_{\mathbb{R}^d} T(a_{1,k_1}, \dots, a_{m,k_m})(x) x^s \mathrm{d}x = 0 \tag{10}$$

holds, where a_{j,k_j} are H^{p_j,q_j} atoms and $|s| \le [d(1/p-1)]$.

Proof. It is easy to know that (10) holds when T can be extended to a bounded operator from $H^{p_1}(\mathbb{R}^d)\times\cdots\times H^{p_m}(\mathbb{R}^d)$ to $H^p(\mathbb{R}^d)$.

For the reverse, writing each $f_j \in H^{p_j},\ 1\le j\le m$ as a sum of H^{p_j,q_j} atoms, $f_j=\sum_k \lambda_{j,k}a_{j,k}$. Then by Remark 1.3, we have

$$T(f_1,\ldots,f_m)(x)=\sum_{k_1}\cdots\sum_{k_m}\lambda_{1,k_1}\cdots\lambda_{m,k_m}T(a_{1,k_1},\ldots,a_{m,k_m})(x). \quad (11)$$

Therefore, it is sufficient to prove that $G(x):=T(a_{1,k_1},\ldots,a_{m,k_m})(x)$ are (p,q,s,ϵ) molecules for some $\epsilon>0$. For a cube Q, let Q^* denote the cube with the same center and $2\sqrt{d}$ its side length, i.e. $l(Q^*)=2\sqrt{d}\,l(Q)$. Without loss of generality, we can assume that the side length of the cube Q_{1,k_1} is the smallest among the side lengths of the cubes $Q_{1,k_1},\ldots,Q_{m,k_m}$. Let c_{j,k_j} be the center of the cube Q_{j,k_j}. Then

$$\begin{aligned}\|G\|_q &\le C\,\|a_{1,k_1}\|_{q_1}\cdots\|a_{m,k_m}\|_{q_m}\\ &\le C\,|Q_{1,k_1}|^{1/q_1-1/p_1}\cdots|Q_{m,k_m}|^{1/q_m-1/p_m}\\ &\le C\,|Q_{1,k_1}|^{1/q-1/p}. \end{aligned} \quad (12)$$

Writing

$$\begin{aligned}&\int_{\mathbb{R}^d}|x-c_{1,k_1}|^{dbq}|G(x)|^q\mathrm{d}x\\ &=\int_{Q^*_{1,k_1}}|x-c_{1,k_1}|^{dbq}|G(x)|^q\mathrm{d}x+\int_{(Q^*_{1,k_1})^c}|x-c_{1,k_1}|^{dbq}|G(x)|^q\mathrm{d}x\\ &=I_1+I_2.\end{aligned}$$

For I_1, we have

$$I_1\le|Q_{1,k_1}|^{bq}\|G\|_q^q\le C\,|Q_{1,k_1}|^{(b+1/q-1/p)q}. \quad (13)$$

In [11], the authors proved the following pointwise estimate (cf. (9) of [11]),

$$|G(x)|\le C\prod_{j=1}^m\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/dm}}{(|x-c_{j,k_j}|+l(Q_{j,k_j}))^{d+(N+1)/m}} \quad (14)$$

for all x which belongs to the complement of at least one Q^*_{j,k_j}.

By (13), for $x\in(Q^*_{1,k_1})^c$, we get

$$\begin{aligned}|G(x)| &\le C\frac{|Q_{1,k_1}|^{1-1/p_1+(N+1)/dm}}{(|x-c_{1,k_1}|+l(Q_{1,k_1}))^{d+(N+1)/m}}\cdot\prod_{j=2}^m|Q_{j,k_j}|^{-1/p_j}\\ &\le C\frac{|Q_{1,k_1}|^{1-1/p+(N+1)/dm}}{(|x-c_{1,k_1}|)^{d+(N+1)/m}}.\end{aligned}$$

We choose ϵ such that $\epsilon < \frac{N+1}{dm}$, then

$$|I_2| = C\,|Q_{1,k_1}|^{(1-1/p+(N+1)/dm)q} \int_{(Q^*_{1,k_1})^c} \frac{1}{(|x-c_{1,k_1}|)^{(d+\frac{N+1}{m})q-dbq}}\mathrm{d}x$$
$$\leq C\,|Q_{1,k_1}|^{(b+1/q-1/p)q}.$$

Hence,

$$\left(\int_{\mathbb{R}^n} |x-C_{1,k_1}|^{nbq}|G(x)|^q \mathrm{d}x\right)^{1/q} \leq C\,|Q_{1,k_1}|^{(b+1/q-1/p)}. \tag{15}$$

By (12) and (15), we know that

$$\|G\|_q^{a/b}\|(\cdot - x_0)G(\cdot)\|_q^{1-a/b}$$
$$\leq C\,|Q_{1,k_1}|^{(1/q-1/p)(a/b)}|Q_{1,k_1}|^{(b+1/q-1/p)(1-a/b)}$$
$$= C\,|Q_{1,k_1}|^{b+1/q-1/p-a} = C. \tag{16}$$

Proposition 2.4 follows from (10) and (16).

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