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Porosity of free boundaries in the obstacle problem for quasilinear elliptic equations

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Abstract. In this paper, we establish growth rate of solutions near free boundaries in the identical zero obstacle problem for quasilinear elliptic equations. As a result, we obtain porosity of free boundaries, which is naturally an extension of the previous works by Karp *et al.* (*J. Diff. Equ.* **164** (2000) 110–117) for p-Laplacian equations, and by Zheng and Zhang (*J. Shaanxi Normal Univ.* **40(2)** (2012) 11–13, 18) for p-Laplacian type equations.

Keywords. Obstacle problem; free boundary; porosity; quasilinear elliptic equations.

1. Introduction

In this paper we consider the obstacle problem for elliptic degenerate and singular equations of p-Laplacian type associated with the operator

$$Au = \operatorname{div} a(x,u,\nabla u) \quad \text{in } \mathcal{D}'(\Omega),$$

where Ω is an open bounded domain of $\mathbb{R}^N (N \geq 2)$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies natural growth conditions. Given functions $g,\ \psi \in W^{1,p}(\Omega)(1 < p < \infty)$, define

$$K_{g,\psi} = \{v \in W^{1,p}(\Omega);\ v - g \in W_0^{1,p}(\Omega),\ v \geq \psi,\ \text{a.e. in } \Omega\},$$

which is nonempty provided $(\psi - g)^+ \in W_0^{1,p}(\Omega)$.

A function u in $K_{g,\psi}$ is a solution to the obstacle problem

$$\operatorname{div} a(x,u,\nabla u) = b(x,u,\nabla u) \ \text{ in } \{u > \psi\}, \tag{1.1}$$

if

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(v-u) + b(x,u,\nabla u)(v-u)\mathrm{d}x \geq 0, \quad \forall\, v \in K_{g,\psi},$$

where $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies natural growth conditions.

By standard arguments (see [5, 6, 8–10, 14–18] for instance), we know that any bounded solution u to (1.1) is $C^{1,\beta}(\Omega)$ for some $\beta \in (0, 1)$. But there is only little information regarding free boundaries. As is well-known, regularity theory is one of the most important work in such a problem. At this aspect, an interesting work is to study porosity and finite Hausdorff measure of free boundaries, which intrigues many mathematicians among others (see [3, 4, 11, 12]). When $b(x, u, \eta) \equiv b(x)$ is a Lipschitz function, Karp *et al*. [11] considered an identically zero obstacle for p-Laplacian equations contained in the quasilinear elliptic equations associated with the operator $Au = \operatorname{div} a(x, u, \nabla u)$. The authors proved that any solution near the free boundary has an exact growth rate of order $\frac{p}{p-1}$. Thus the free boundary is porous and therefore its Hausdorff dimension is less than N [2, 11, 12, 20], and hence it is of Lebesgue measure zero. Under the same assumptions on b as [11], Zheng and Zhang [21] extends the results to the obstacle problem associated with the operator $Au = \operatorname{div} a(\nabla u)$. We should note that porosity and finite Hausdorff measure of free boundaries in A-obstacle problem (in Orlicz–Sobolev spaces) was established by Challal and Lyaghfouriand [3, 4]. In this paper, we are interested in dealing with the identical zero obstacle problem (1.1) for a larger class of elliptic equations. Under natural growth conditions on a and b, we establish exact growth rate for solutions of (1.1) and porosity of free boundaries. Our results are naturally extensions of [11] and [21].

In this paper, we assume that the function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is such that $a(x, \mu, 0) = 0$ for a.e. $x \in \Omega$ and any $\mu \in \mathbb{R}$, and satisfies the standard structural conditions for some positive constants γ_0, γ_1, namely

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x, \mu, \eta)\xi_i\xi_j \geq \gamma_0|\eta|^{p-2}|\xi|^2, \tag{1.2}$$

$$\sum_{i,j=1}^{N} \left|\frac{\partial a_i}{\partial \eta_j}(x, \mu, \eta)\right| \leq \gamma_1|\eta|^{p-2}, \tag{1.3}$$

$$\sum_{i,j=1}^{N} \left|\frac{\partial a_i}{\partial x_j}(x, \mu, \eta)\right| + \sum_{i,j=1}^{N} \left|\frac{\partial a_i}{\partial \mu}(x, \mu, \eta)\right| \leq \gamma_1|\eta|^{p-1}, \tag{1.4}$$

for a.e. $x \in \Omega$, all $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^N\backslash\{0\}$, and all $\xi \in \mathbb{R}^N$.

We assume that the function $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$|b(x, \mu, \eta)| \leq \gamma_1(1 + |\mu|^{p-1} + |\eta|^{p-1}), \quad b(x, \mu, \eta) \geq 0, \tag{1.5}$$

for a.e. $x \in \Omega$, all $\mu \in \mathbb{R}$, and all $\eta \in \mathbb{R}^N$.

Throughout this paper, we assume the obstacle function as $\psi \equiv 0$ and the boundary function as $g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

For the existence of a solution of (1.1), we refer to [1] and references therein with general results. Now we prove the following proposition.

PROPOSITION 1.1

Let u be a solution of (1.1). *Then*

(i) $b \geq 0$ *in* $\Omega \times \mathbb{R} \times \mathbb{R}^N \Rightarrow 0 \leq u \leq \|g\|_{L^\infty(\Omega)}$ *in* Ω.
(ii) $Au = b(x, u, \nabla u)$ *in* $\mathcal{D}'(\{u > 0\})$.
(iii) $b(x, u, \nabla u)\chi_{\{u>0\}} \leq Au \leq b(x, u, \nabla u)$ *a.e. in* Ω.

Proof. We argue as in [11] (see also [3]).

(i) Taking $v = \min(u, \|g\|_\infty) = u - (u - \|g\|_\infty)^+ \in K_g := K_{g,0}$ as a test function in (1.1), we get

$$\int_\Omega a(x,u,\nabla u)\nabla(u - \|g\|_\infty)^+ \mathrm{d}x \le -\int_\Omega b(x,u,\nabla u)(u - \|g\|_\infty)^+ \mathrm{d}x \le 0. \tag{1.6}$$

Notice that (1.2)–(1.4) imply (see [7] for instance) that

$$a(x,u,\nabla u)\nabla u \ge \frac{\gamma_0}{p-1}|\nabla u|^p. \tag{1.7}$$

We deduce from (1.6) and (1.7) that $\nabla(u - \|g\|_\infty)^+ = 0$ a.e. in Ω. It follows that $(u - \|g\|_\infty)^+ = 0$ a.e. in Ω since $u = g$ on $\partial\Omega$.

(ii) Let $\zeta \in \mathcal{D}(\{u > 0\})$, $\zeta \ge 0$. Setting $\delta = \min_{\mathrm{supp}\,\zeta} u$ and taking $u \pm \delta\frac{\zeta}{\|\zeta\|_\infty}$ as a test function in (1.1), we obtain

$$\int_{\{u>0\}} a(x,u,\nabla u)\nabla\zeta + b(x,u,\nabla u)\zeta\,\mathrm{d}x = 0.$$

Therefore div $a(x,u,\nabla u) = b(x,u,\nabla u)$ in $\mathcal{D}'(\{u > 0\})$.

(iii) Let $\zeta \in \mathcal{D}(\Omega)$, $\zeta \ge 0$. Since $u \pm \zeta \in K_g$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla\zeta + b(x,u,\nabla u)\zeta\,\mathrm{d}x \ge 0.$$

It follows that div $a(x,u,\nabla u) \le b(x,u,\nabla u)$ in $\mathcal{D}'(\Omega)$.

Now for $\varepsilon > 0$, $u - H_\varepsilon(u-\varepsilon)\zeta \in K_g$ with $H_\varepsilon(s) = \min\{1, \frac{s^+}{\varepsilon}\}$. Then we have

$$\int_\Omega H_\varepsilon(u-\varepsilon)a(x,u,\nabla u)\nabla\zeta + b(x,u,\nabla u)H_\varepsilon(u-\varepsilon)\zeta\,\mathrm{d}x \le 0.$$

Letting $\varepsilon \to 0$, we get

$$\int_\Omega a(x,u,\nabla u)\nabla\zeta + b(x,u,\nabla u)\chi_{\{u>0\}}\zeta\,\mathrm{d}x \le 0.$$

Thus div $a(x,u,\nabla u) \ge b(x,u,\nabla u)\chi_{\{u>0\}}$ in $\mathcal{D}'(\Omega)$. □

Remark 1.2. If $b \ge 0$ in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, we know from Proposition 1.1 that u is bounded in Ω. Regarding regularity, we know that for any $\Omega' \subset\subset \Omega$, $u \in C^{1,\alpha}(\overline{\Omega'})$ for some $\alpha \in (0,1)$ (see [5, 6, 9] for instance). Moreover, $\|u\|_{C^{1,\alpha}(\overline{\Omega'})}$ depends only on $p, N, \gamma_0, \gamma_1, \|g\|_{L^\infty(\Omega)}$ and Ω'. Therefore $|b(x,u,\nabla u)| \le C$ for a.e. $x \in \Omega'$, and $\|Au\|_{L^\infty(\Omega')} \le C$, with constants C depending only on $p, N, \gamma_0, \gamma_1, \|g\|_{L^\infty(\Omega)}$ and Ω'.

2. A class of functions on the unit ball

In this section, due to the local character of the results in this paper, we will restrict ourselves to the unit ball. Furthermore, according to Remark 1.2, we may consider the following local formulation of the obstacle problem.

DEFINITION 2.1

We say that a function u in $W^{1,p}(B_1)$, where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, belongs to the class $\mathcal{G}_A$ if

$$\begin{cases} \|\mathrm{div}\, a(x, u, \nabla u)\|_\infty \le 1, \\ 0 \le u \le 1, \quad \text{in } B_1, \\ u(0) = 0. \end{cases} \tag{2.1}$$

It is well-known that $u \in C^{1,\alpha}(B_1)$ for some $\alpha \in (0, 1)$. But the exact value for α is unknown. The following result gives a growth of the elements of $\mathcal{G}_A$. It is optimal for α.

Theorem 2.2. *There exists a positive constant $K = K(p, N, \gamma_0, \gamma_1)$ such that for every $u \in \mathcal{G}_A$, we have*

$$0 \le u(x) \le K|x|^{\frac{p}{p-1}}, \quad \forall\, x \in B_1.$$

In order to prove Theorem 2.2, we need the following lemma. Firstly we introduce some notations as [11]. For a nonnegative bounded function u, we define

$$S(r, u, z) = \sup_{x \in B_r(z)} u(x), \quad S(r, u) = S(r, u, 0).$$

We also define for $u \in \mathcal{G}_A$ the set

$$\mathbb{M}(u) = \{j \in \mathbb{N} |\, 2^{\frac{p}{p-1}} S(2^{-j-1}, u) \ge S(2^{-j}, u)\}.$$

Then we have the following.

Lemma 2.3. *Assume that $\mathbb{M}(u) \ne \emptyset$. Then there exists a constant $K' = K'(p, N, \gamma_0, \gamma_1)$ such that*

$$S(2^{-j-1}, u) \le K'(2^{-j})^{\frac{p}{p-1}},$$

for all $u \in \mathcal{G}_A$ and $j \in \mathbb{M}(u)$.

Proof. We argue by contradiction. Then we have

$$\forall\, k \in \mathbb{N},\ \exists\, u_k \in \mathcal{G}_A,\ \exists\, j_k \in \mathbb{M}(u) \ \text{ such that } \ S(2^{-j_k-1}, u_k) \ge k(2^{-j_k})^{\frac{p}{p-1}}. \tag{2.2}$$

It follows from (2.2) and (uniform) boundedness of u_k that $j_k \to \infty$ as $j \to \infty$. Let $\tilde{u}_k(x) = \frac{u_k(2^{-j_k} x)}{S(2^{-j_k-1}, u_k)}$ be defined in B_1. We have

$$0 \le \tilde{u}_k(x) \le \frac{S(2^{-j_k}, u_k)}{S(2^{-j_k-1}, u_k)} \le 2^{\frac{p}{p-1}} \ \text{ in } B_1, \quad \inf_{B_{\frac{1}{2}}} \tilde{u}_k(x) = 0. \tag{2.3}$$

Define for $(x, \mu, \eta) \in B_1 \times \mathbb{R} \times \mathbb{R}^N$,

$$\begin{aligned} a^k(x, \mu, \eta) \ &= \ \left(\frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}\right)^{p-1} \\ &\quad \times a\left(2^{-j_k} x, S(2^{-j_k-1}, u_k)\mu, \frac{S(2^{-j_k-1}, u_k)}{2^{-j_k}} \eta\right). \end{aligned}$$

We claim that a^k satisfies the same structural conditions as a for large k. Indeed, letting $s_k = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}$, one may verify directly that

$$\begin{aligned}\sum_{i,j=1}^N \frac{\partial a_i^k}{\partial \eta_j}(x,\mu,\eta)\xi_i\xi_j &= \sum_{i,j=1}^N s_k^{p-2}\frac{\partial a_i}{\partial \eta_j}(2^{-j_k}x, S(2^{-j_k-1}, u_k)\mu, s_k^{-1}\eta)\xi_i\xi_j \\ &\geq \gamma_0 s_k^{p-2}|s_k^{-1}\eta|^{p-2}|\xi|^2 \\ &= \gamma_0|\eta|^{p-2}|\xi|^2.\end{aligned}$$

$$\begin{aligned}\sum_{i,j=1}^N \left|\frac{\partial a_i^k}{\partial \eta_j}(x,\mu,\eta)\right| &= \sum_{i,j=1}^N s_k^{p-2}\left|\frac{\partial a_i}{\partial \eta_j}(2^{-j_k}x, S(2^{-j_k-1}, u_k)\mu, s_k^{-1}\eta)\right| \\ &\leq \gamma_1 s_k^{p-2}|s_k^{-1}\eta|^{p-2} \\ &= \gamma_1|\eta|^{p-2}.\end{aligned}$$

$$\begin{aligned}&\sum_{i,j=1}^N\left[\left|\frac{\partial a_i^k}{\partial x_j}(x,\mu,\eta)\right| + \left|\frac{\partial a_i^k}{\partial \mu}(x,\mu,\eta)\right|\right] \\ &\quad= \sum_{i,j=1}^N s_k^{p-1}2^{-j_k}\left|\frac{\partial a_i}{\partial x_j}(2^{-j_k}x, S(2^{-j_k-1}, u_k)\mu, s_k^{-1}\eta)\right| \\ &\qquad+ \sum_{i,j=1}^N s_k^{p-1}S(2^{-j_k-1}, u_k)\frac{\partial a_i}{\partial \mu}(2^{-j_k}x, S(2^{-j_k-1}, u_k)\mu, s_k^{-1}\eta) \\ &\quad\leq 2^{-j_k}\gamma_1|\eta|^{p-1} + S(2^{-j_k-1}, u_k)\gamma_1|\eta|^{p-1} \\ &\quad\leq \gamma_1|\eta|^{p-1},\end{aligned}$$

where in the last inequality we used the fact that $u_k(0) = 0$ and $u_k \in C^1(\bar{B}_{\frac{3}{4}})$, which imply that for large k, $S(2^{-j_k-1}, u_k) \leq C2^{-j_k-1} \leq \frac{\gamma_1}{2}$ with $C = C(\|u_k\|_{C^{1,\alpha}(\bar{B}_{\frac{3}{4}})}) = C(p, N, \gamma_0, \gamma_1)$. Now notice that

$$\begin{aligned}&\|\operatorname{div} a^k(x, \tilde{u}_k(x), \nabla\tilde{u}_k(x))\|_\infty \\ &\quad= s_k^{p-1}\|\operatorname{div} a(2^{-j_k}x, u_k(2^{-j_k}x), (\nabla u_k)(2^{-j_k}x))\|_\infty \\ &\quad= 2^{-j_k}s_k^{p-1}\|(Au_k)(2^{-j_k}x)\|_\infty \\ &\quad\leq 2^{-j_k}\left(\frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}\right)^{p-1} \\ &\quad\leq \frac{1}{k^{p-1}} \quad \text{by (2.2)}. \end{aligned} \tag{2.4}$$

We may use Harnack's inequality [19] to get a contradiction. Indeed, to this end, for $p < N$, one may let $\alpha = p$, $a = \frac{\gamma_1}{p-1}$, $b = e = c = d = g = 0$, $f = \frac{1}{k^{p-1}}$, $\varepsilon = 1$, $R = \frac{1}{2}$ in Theorem 5, and $k = (\|e\| + R^\varepsilon\|f\|)^{\frac{1}{\alpha}} + (R^\varepsilon\|g\|)^{\frac{1}{\alpha}}$ in Theorem 1 of [19]. As for $p \geq N$, one may choose the quantities in Theorems 6 and 9 of [19] in a similar way. It suffices to notice that $\alpha = p$, $e = g = 0$, $f = \frac{1}{k^{p-1}}$ whenever $1 < p < \infty$. Thus, combining (2.3), (2.4) with (1.5), we obtain by Harnack's inequality [19]

$$1 = \sup_{B_{\frac{1}{2}}} \tilde{u}_k \leq C \begin{cases} \left(\inf_{B_{\frac{1}{2}}} \tilde{u}_k + \left(\frac{1}{2} \left\| \frac{1}{k^{p-1}} \right\|_{L^{\frac{N}{p-1}}(B_{1/2})}\right)^{\frac{1}{p-1}}\right), & \text{when } p \leq N \\ \left(\inf_{B_{\frac{1}{2}}} \tilde{u}_k + \left(\left(\frac{1}{2}\right)^{p-N} \left\| \frac{1}{k^{p-1}} \right\|_{L^1(B_{1/2})}\right)^{\frac{1}{p-1}}\right), & \text{when } p > N \end{cases}$$

$$\leq \frac{C}{k} \to 0,$$

where C is a positive constant depending only on p, N, γ_0, γ_1. Hence this is a contradiction. □

Proof of Theorem 2.2. Arguing as [3, 11], we prove that for $K = \max\{2^{\frac{p}{p-1}}, K' 2^{\frac{2p}{p-1}}\}$ there holds

$$S(2^{-j}, u) \leq K(2^{-j-1})^{\frac{p}{p-1}} \quad \forall\, j \in \mathbb{N}. \tag{2.5}$$

Indeed, for $j = 0$, we have $S(2^{-0}, u) = S(1, u) \leq 1 \leq K(2^{-1})^{\frac{p}{p-1}}$. Let $j \geq 1$. Assume that $S(2^{-j}, u) \leq K(2^{-j-1})^{\frac{p}{p-1}}$.

(1) If $j \in \mathbb{M}(u)$, by Lemma 2.3, it follows that

$$S(2^{-j-1}, u) \leq K'(2^{-j})^{\frac{p}{p-1}} = K'(2^2)^{\frac{p}{p-1}}(2^{-j-2})^{\frac{p}{p-1}} \leq K(2^{-j-2})^{\frac{p}{p-1}}.$$

(2) If $j \notin \mathbb{M}(u)$, we infer by the definition of $\mathbb{M}(u)$ and the induction assumption that

$$S(2^{-j-1}, u) \leq 2^{-\frac{p}{p-1}} S(2^{-j}, u) \leq 2^{-\frac{p}{p-1}} K(2^{-j-1})^{\frac{p}{p-1}} = K(2^{-j-2})^{\frac{p}{p-1}}.$$

Then (2.2) holds for all $j \in \mathbb{N}$.

Let $x \in B_1$, $r = |x|$. Then there exists $j \in \mathbb{N}$ such that $2^{-j-1} \leq r \leq 2^{-j}$. We deduce from (2.5)

$$u(x) \leq \sup_{\bar{B}_r} u = S(r, u) \leq S(2^{-j}, u) \leq K(2^{-j-1})^{\frac{p}{p-1}} \leq K|x|^{\frac{p}{p-1}}.$$ □

3. Porosity of the free boundary

In what follows, we assume that there exists positive constant γ_2 such that

$$\gamma_2 \leq b(x, \mu, \eta) \leq \gamma_1(1 + |\mu|^{p-1} + |\eta|^{p-1}),$$

for a.e. $x \in \Omega$, all $\mu \in \mathbb{R}$, and all $\eta \in \mathbb{R}^N$. Moreover, we always assume that $\partial\Omega$ is of class $C^{1,\alpha}$ and $g \in W^{1,p}(\Omega) \cap C^{1,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$, which by Proposition 1.1, by standard arguments (see [9, 10, 13] for instance), guarantee that any solution u of (1.1) belongs to $C^{1,\alpha}(\bar{\Omega})$, denoted by $\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq M = M(p, N, \gamma_0, \gamma_1, \|g\|_{L^\infty(\Omega)}, \|g\|_{C^{1,\alpha}(\partial\Omega)})$.

Now we establish non-degeneracy for solutions of (1.1) that shows any solution can not grow too slowly near the free boundary.

PROPOSITION 3.1

Suppose that $u \in K_g$ *is a solution of* (1.1). *Then there exists a constant* r_0, *depending only on* p, γ_1, γ_2, *such that for each* $z \in \bar{\Omega}_+ = \overline{\{u > 0\}}$ *and* $r \in (0, r_0)$ *satisfying* $B_r(z) \subset \Omega$, *we have*

$$\sup_{B_r(z)\cap\Omega^+} u \geq C_0 r^{\frac{p}{p-1}},$$

for some positive constant C_0 *depending only on* p, γ_1, γ_2 *and* r_0.

Proof. Let $z \in \Omega_+$, $B_r(z) \subset \Omega$. Define $v(x) = C_0|x - z|^{\frac{p}{p-1}}$ in $B_r(z)$. It follows that

$$\nabla v(x) = \frac{pC_0}{p-1}|x - z|^{\frac{2-p}{p-1}}(x - z), \quad |D_{ij}v| \leq \frac{pC_0}{\underline{(p-1)^2}}|x - z|^{\frac{2-p}{p-1}}.$$

One may verify that

$$\begin{aligned}
\operatorname{div} a(x, v, \nabla v) &= \sum_{i=1}^{N}\left[\frac{\partial a_i}{\partial x_i}(x, v, \nabla v) + \frac{\partial a_i}{\partial \mu}(x, v, \nabla v)\frac{\partial v}{\partial x_i}(x)\right] \\
&\quad + \sum_{i,j=1}^{N}\frac{\partial a_i}{\partial \eta_j}(x, v, \nabla v)\frac{\partial v_j}{\partial x_i}(x) \leq \gamma_1|\nabla v|^{p-1} \\
&\quad + \gamma_1|\nabla v|^{p-1}|\nabla v| + \gamma_1|\nabla v|^{p-2}\frac{pC_0}{\underline{(p-1)^2}}|x - z|^{\frac{2-p}{p-1}} \\
&\leq \gamma_1\left(\frac{pC_0}{p-1}\right)^{p-1}\left(|x - z| + \frac{pC_0}{p-1}|x - z|^{\frac{p}{p-1}} + \frac{1}{\underline{p-1}}\right) \\
&\leq \gamma_1\left(\frac{pC_0}{p-1}\right)^{p-1}\left(r_0 + \frac{pC_0}{p-1}r_0^{\frac{p}{p-1}} + \frac{1}{\underline{p-1}}\right) \\
&\leq \gamma_2 \quad \text{for small } C_0.
\end{aligned}$$

Therefore

$$-\operatorname{div} a(x, v, \nabla v) \geq -\gamma_2 \geq -\operatorname{div} a(x, u, \nabla u) \quad \text{in } \Omega_+ \cap B_r(z).$$

Obviously, $u \leq v$ on $\partial\Omega_+ \cap B_r(z)$. If also $u \leq v$ on $\partial B_r(z) \cap \Omega_+$, then we deduce from comparison principle that

$$u \leq v \quad \text{in } \Omega_+ \cap B_r(z).$$

But $u(z) > 0 = v(z)$, which is a contradiction. So we have

$$\sup_{\partial B_r(z)\cap\Omega_+} u \geq \sup_{\partial B_r(z)\cap\Omega_+} v = C_0 r^{\frac{p}{p-1}}.$$

Now if $z \in \partial\Omega^+$, taking $z^j \in \Omega^+$ such that $z^j \to z$, one may obtain the desired result by continuity. □

The following theorem establishes porosity of the free boundary $\partial\Omega_+ \cap \Omega$ of problem (1.1). We recall that a set $\mathbb{E} \subset \mathbb{R}^N$ is called porous with porosity δ, if there is an $r_* > 0$ such that

$$\forall\, x \in \mathbb{E}, \forall\, r \in (0, r_*),\ \exists\, y \in \mathbb{R}^N \quad \text{such that} \quad B_{\delta r}(y) \subset B_r(x) \setminus \mathbb{E}.$$

A porous set has Hausdorff dimension not exceeding $N - C\delta^N$, where the C is a positive constant depending only on N. In particular, a porous set has Lebesgue measure zero.

Theorem 3.2. *Let u be a solution of (1.1). Then for every compact set $\mathbb{K} \subset \Omega$, the intersection $\partial\Omega_+ \cap \mathbb{K}$ is porous with porosity constant $\delta = \delta(p, N, \gamma_0, \gamma_1, \gamma_2, \|g\|_{L^\infty(\Omega)})$ and $0 < \delta \le 1$.*

Proof. We prove as [11]. Without loss of generality, we may assume that the compact $\mathbb{K}$ is the closed unit ball $\bar{B}_1$, and moreover that $\bar{B}_2 \subset \Omega$.

For $x \in \Omega_+ \cap \bar{B}_1$, define $d_x = \operatorname{dist}(x, \bar{B}_1 \setminus \Omega_+)$ and take $z_x \in \partial\Omega_+ \cap \bar{B}_1$ with $|x - z_x| = d_x$. We define a function in B_1 by $\tilde{u}(y) = u(z_x + y)$. Let $\tilde{M} = \max\{\|g\|_{L^\infty(\Omega)}, 2^{p-1}\gamma_1(1 + M^{p-1})\}$. Define $\tilde{a}(y, \mu, \eta) = \frac{a(z_x+y, \tilde{M}\mu, \tilde{M}\eta)}{\tilde{M}}$ and operator $\tilde{A}$ by $\tilde{A}\tilde{u} = \operatorname{div} \tilde{a}(y, \tilde{u}, \nabla\tilde{u})$. We claim that $\frac{\tilde{u}}{\tilde{M}} \in \mathscr{G}_{\tilde{A}}$. Indeed, one may verify easily that $\tilde{a}$ satisfies all structural conditions (not necessarily with the same constants as a). Furthermore, we have

$$\begin{aligned}
\tilde{A}\left(\frac{\tilde{u}}{\tilde{M}}\right) &= \frac{1}{\tilde{M}} \operatorname{div} a(z_x + y, u(z_x + y), (\nabla u)(z_x + y)) \\
&= \frac{1}{\tilde{M}} (Au)(z_x + y) \\
&\le \frac{1}{\tilde{M}} |b(z_x + y, u(z_x + y), (\nabla u)(z_x + y))| \\
&\le \frac{2^{p-1}\gamma_1(1 + M^{p-1})}{\tilde{M}} \\
&\le 1,
\end{aligned}$$

$$0 \le \frac{\tilde{u}}{\tilde{M}} \le \frac{\|g\|_{L^\infty(\Omega)}}{\tilde{M}} \le 1 \quad \text{and} \quad \frac{\tilde{u}(0)}{\tilde{M}} = \frac{u(z_x)}{\tilde{M}} = 0.$$

Therefore we infer by Theorem 2.2 that

$$u(x) = \tilde{u}(x - z_x) \le \tilde{M}K|x - z_x|^{\frac{p}{p-1}} = \tilde{M}K d_x^{\frac{p}{p-1}}. \tag{3.1}$$

Let $z \in \partial\Omega_+ \cap B_1$. Then for $0 < r < r_0 < 1$, by Proposition 3.1, there exists $x_z \in \partial B_r(z) \cap \Omega_+$ such that

$$u(x_z) \ge C_0 r^{\frac{p}{p-1}}.$$

It follows from (3.1) that

$$C_0 r^{\frac{p}{p-1}} \le u(x_z) \le \tilde{M}K d_x^{\frac{p}{p-1}}.$$

Let $\delta = (\frac{C_0}{\tilde{M}K})^{\frac{p-1}{p}}$. Then $d_{x_z} \ge \delta r$ and $0 < \delta \le 1$. Therefore

$$B_{\delta r}(x_z) \cap B_r(z) \subset \Omega_+.$$

Now choose $y \in [z, x_z]$ such that $|y - x_z| = \frac{\delta r}{2}$. Then we have

$$B_{\frac{\delta r}{2}}(y) \subset B_{\delta r}(x_z) \cap B_r(z) \subset B_r(z) \backslash \partial\Omega_+ \subset B_r(z) \backslash (\partial\Omega_+ \cap \bar{B}_1).$$

Indeed, for any $y_0 \in B_{\frac{\delta r}{2}}(y)$, we get

$$|y_0 - x_z| \leq |y_0 - y| + |y - x_z| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r.$$

Moreover, since $|y - z| = |z - x_z| - |y - x_z|$, it follows that

$$|y_0 - z| \leq |y_0 - y| + (|z - x_z| - |y - x_z|) \leq \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r.$$

This shows that $\partial\Omega_+ \cap \bar{B}_1$ is porous with the porosity contant $\frac{\delta}{2}$. □

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