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On the number of isomorphism classes of transversals

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Abstract. In this paper we prove that there does not exist a subgroup H of a finite group G such that the number of isomorphism classes of normalized right transversals of H in G is four.

Keywords. Group torsion; right loop; normalized right transversal.

1. Introduction and statement of the main result

Let G be a finite group and H a subgroup of G. Let S be a normalized right transversal (NRT) of H in G, that is, S is a subset of G obtained by choosing one and only one element from each right coset of H in G and $1 \in S$. Then S has an induced binary operation $\circ$ given by $\{x \circ y\} = Hxy \cap S$, with respect to which S is a right loop with identity 1, that is, a right quasigroup with both-sided identity (see Proposition 2.2, p. 42 of [14], [10]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see Theorem 3.4, p. 76 of [10]). Let $\langle S \rangle$ be the subgroup of G generated by S and H_S be the subgroup $\langle S \rangle \cap H$. Then $H_S = \langle \{xy(x \circ y)^{-1} | x, y \in S\} \rangle$ and $H_S S = \langle S \rangle$ (see [10]).

Identifying S with the set $H \backslash G$ of all right cosets of H in G, we get a transitive permutation representation $\chi_S : G \to \mathrm{Sym}(S)$ defined by $\{\chi_S(g)(x)\} = Hxg \cap S$, $g \in G$, $x \in S$. The kernal $\ker \chi_S$ of this action is $\mathrm{Core}_G(H)$, the core of H in G.

Let $G_S = \chi_S(H_S)$. This group is known as the *group torsion* of the right loop S (see Definition 3.1, p. 75 of [10]). The group G_S depends only on the right loop structure $\circ$ on S and not on the subgroup H. Since χ_S is injective on S and if we identify S with $\chi_S(S)$, then $\chi_S(\langle S \rangle) = G_S S$ which also depends only on the right loop S and S is an NRT of G_S in $G_S S$. One can also verify that $\ker(\chi_S|_{H_S S} : H_S S \to G_S S) = \ker(\chi_S|_{H_S} : H_S \to G_S) = \mathrm{Core}_{H_S S}(H_S)$ and $\chi_S|_S =$ the identity map on S. If H is a corefree subgroup of G, then there exists an NRT T of H in G which generates G (see [3]). In this case, $G = H_T T \cong G_T T$ and $H = H_T \cong G_T$. Also $(S, \circ)$ is a group if and only if G_S trivial.

Let $\mathcal{T}(G, H)$ denote the set of all normalized right transversals (NRTs) of H in G. We say that S and $T \in \mathcal{T}(G, H)$ are isomorphic (denoted by $S \cong T$), if their induced right loop structures are isomorphic. Let $\mathcal{I}(G, H)$ denote the set of isomorphism classes of NRTs of H in G.

In the main theorem, p. 643 of [11], it is shown that $|\mathcal{I}((G,H))| = 1$ if and only if $H \trianglelefteq G$. It is obtained in Theorem, p. 1718 of [6] that there is no pair (G,H) such that $|\mathcal{I}(G,H)| = 2$. It is easy to observe that if H is a non-normal subgroup of G of index 3, then $|\mathcal{I}(G,H)| = 3$. The converse of this statement is proved in Theorem A, p. 2025 of [7]. Also, it is shown in Theorem 3.7, p. 2693 of [13] that if T_n denotes the number of non-isomorphic right loops of order n, then $|\mathcal{I}(\mathrm{Sym}(n), \mathrm{Sym}(n-1))| = T_n$, where $\mathrm{Sym}(m)$ denotes the symmetric group on m symbols. Moreover, if there is any pair (G,H) such that the index $[G:H]$ of H in G is n and if $|\mathcal{I}(G,H)| = T_n$, then there is a surjective homomorphism $\psi : G \to \mathrm{Sym}(n)$ such that $\psi(H) = \mathrm{Sym}(n-1)$ and $\psi^{-1}(\mathrm{Sym}(n-1)) = H$ (see Proposition 3.8, p. 2694 of [13]).

Let $\mathrm{Aut}_H G$ denote the group of all automorphisms of G taking H onto H. The group $\mathrm{Aut}_H G$ acts on each isomorphism class in $\mathcal{T}(G,H)$. Thus the number of non-isomorphic right loops is at most the number of orbits of the action of $\mathrm{Aut}_{\mathrm{Sym}(n-1)}\mathrm{Sym}(n)$ on $\mathcal{T}(\mathrm{Sym}(n), \mathrm{Sym}(n-1))$. Clearly $\mathrm{Core}_{\mathrm{Sym}(n)}(\mathrm{Sym}(n-1)) = \{1\}$. For $n \neq 6$, $\mathrm{Aut\,Sym}(n) = \mathrm{Inn}(\mathrm{Sym}(n)) \cong \mathrm{Sym}(n)$ (see Proposition 2.18, p. 300 of [15]) and $\mathrm{Aut\,Sym}(6) = \mathrm{Inn}(\mathrm{Sym}(6)) \rtimes C_2$ (see Proposition 2.19, p. 300 of [15]), where C_n denotes the cyclic group of order n. It can be checked that $\mathrm{Aut}_{\mathrm{Sym}(n-1)}\mathrm{Sym}(n) \cong \mathrm{Sym}(n-1) \leq \mathrm{Inn}(\mathrm{Sym}(n))$ for all n. It follows from the proof of Theorem 3.7, p. 2693 of [13] that binary operations of S and T define an element $\alpha \in \mathrm{Sym}(n)$ such that $\alpha(1) = 1$ and $\alpha S \alpha^{-1} = T$. Which means that the number of orbits of the action of $\mathrm{Aut}_{\mathrm{Sym}(n-1)}\mathrm{Sym}(n)$ on $\mathcal{T}(\mathrm{Sym}(n), \mathrm{Sym}(n-1))$ is precisely the number of isomorphism classes in $\mathcal{T}(\mathrm{Sym}(n), \mathrm{Sym}(n-1))$. The same is true for the pair $(\mathrm{Alt}(n), \mathrm{Alt}(n-1))$, where $\mathrm{Alt}(m)$ denotes the alternating group of degree m (since $\mathrm{Aut}(\mathrm{Alt}(n)) \cong \mathrm{Aut}(\mathrm{Sym}(n))$, $\mathrm{Aut}_{\mathrm{Alt}(n-1)}\mathrm{Alt}(n) \cong \mathrm{Sym}(n-1) \leq \mathrm{Inn}(\mathrm{Sym}(n))$).

Using GAP [16], we have calculated the number of orbits of the action by conjugation of $\mathrm{Sym}(n-1)$ on $\mathcal{T}(\mathrm{Sym}(n), \mathrm{Sym}(n-1))$ for $n = 4$ and 5. These are 44 and 14022 respectively. In [8], an explicit formula for the number of orbits of the conjugation action of $\mathrm{Sym}(n-1)$ on $\mathcal{T}(\mathrm{Sym}(n), \mathrm{Sym}(n-1))$ has been obtained. If H has a nontrivial core, then the number of $\mathrm{Aut}_H G$-orbits in $\mathcal{T}(G,H)$ may be different from the number $|\mathcal{I}(G,H)|$. For example, let $G_1 = \mathrm{Sym}(4)$ and $H_1 = \langle\{(1,3),(1,2,3,4)\}\rangle \cong D_8$, where D_{2n} denotes the dihedral group of order $2n$. Then NRTs $\{I,(3,4),(2,3)\}$, $\{I,(3,4),(2,3,4)\}$, $\{I,(3,4),(1,2,3,4)\}$ and $\{I,(2,4,3),(2,3,4)\}$ to H_1 in G_1, where I is the identity permutation, and lie in different orbits of $\mathrm{Aut}_{H_1} G_1$ (as the set of orders of group elements in any two NRTs are not the same). However, since H_1 is a non-normal subgroup of G_1 of index 3, $|\mathcal{I}(G_1, H_1)| = 3$.

Let $N = \mathrm{Core}_G(H)$. Clearly $L \mapsto \nu(L) = \{Nx \mid x \in L\}$, where ν is the quotient map from G to G/N, and is a surjective map from $\mathcal{T}(G,H)$ to $\mathcal{T}(G/N, H/N)$ such that the corresponding NRTs are isomorphic.

Let X denote the set of all pairs (G,H), where G is a finite group and H a subgroup of G. In view of the above discussion, it seems an interesting problem to find the image set and the inverse image set of the map $\varphi : X \to \mathbb{N}$ defined by $\varphi((G,H)) = |\mathcal{I}(G,H)|$. As argued in the above paragraphs, $1, 3, 44, 14022 \in \mathrm{Image}(\varphi)$. Also $5, 6 \in \mathrm{Image}(\varphi)$ (see Lemmas 2.5 and 2.7). Using GAP [16] calculations, we get that $7, 14, 20 \in \mathrm{Image}(\varphi)$.

In this paper, we prove that $4 \notin \mathrm{Image}(\varphi)$, that is, we prove the following theorem:

Theorem 1.1 (Main Theorem). *Let G be finite group and H be a subgroup of G. Then $|\mathcal{I}(G,H)| \neq 4$.*

The proof of Theorem 1.1 is based by the method of contradiction and essentially uses the same techniques of [11]. Assuming the falsity of the result, we can find a pair (G, H), to be called as a *minimal counterexample* such that

(i) $|G|$ is minimal,
(ii) the index $[G : H]$ is minimal, and
(iii) $|I(G, H)| = 4$.

We will study various properties of a minimal counterexample and come to the case of a finite non-abelian simple group. With the knowledge of the order of automorphism groups of finite non-abelian simple groups, we will derive a contradiction. Unfortunately, we do not have an alternate proof of the main theorem where the use of the classification of finite simple groups could be avoided. However in [9], we now have a short proof of Main Theorem, p. 643 of [11] where the classification of finite simple groups could be avoided.

2. Properties of a minimal counterexample

PROPOSITION 2.1

Let (G, H) be a minimal counterexample. Then

(i) $\mathrm{Core}_G(H) = \{1\}$,
(ii) *if $S \in \mathcal{T}(G, H)$ such that $\langle S\rangle = G$, then there exists an isomorphism $f : G \to G_S S$ which takes H onto G_S and fixes S elementwise*,
(iii) *if $S \in \mathcal{T}(G, H)$ such that $\langle S\rangle \neq G$, then $H_S = \{1\}$*.

Proof.

(i) Let $N = \mathrm{Core}_G(H)$. Assume that $N \neq \{1\}$. Since the quotient map $\nu : G \to G/N$ induces a surjective correspondence between $\mathcal{T}(G, H)$ and $\mathcal{T}(G/N, H/N)$ such that the corresponding right loops are isomorphic, $|\mathcal{I}(G/N, H/N)| < 4$, for (G, H) is a minimal counterexample.

If $|\mathcal{I}(G/N, H/N)| = 1$, then $H/N \trianglelefteq G/N$ (see Main Theorem, p. 643 of [11]) and so $H \trianglelefteq G$, a contradiction. By Theorem, p. 1718 of [6], $|\mathcal{I}(G/N, H/N)| \neq 2$. If $|\mathcal{I}(G/N, H/N)| = 3$, then $[G : H] = [G/N : N/H] = 3$ (see Theorem A, p. 2025 of [7]). But in this case, $|\mathcal{I}(G, H)| = 3$, which is a contradiction.
(ii) Let $S \in \mathcal{T}(G, H)$ be such that $\langle S\rangle = G$. Then $H_S = H$. Let $\chi_S : H_S S \to G_S S$ be the surjective homomorphism defined as in the second paragraph of § 1. Then $\ker \chi_S = \{1\}$ (by (i)). Hence χ_S is an isomorphism which takes H onto G_S and fixes S elementwise.
(iii) Let $S \in \mathcal{T}(G, H)$. Assume that $H_S S = \langle S\rangle \neq G$. Since $\mathcal{T}(\langle S\rangle, H_S) \subseteq \mathcal{T}(G, H)$ and (G, H) is a minimal counterexample, $|\mathcal{I}(\langle S\rangle, H_S)| < 4$. By Theorem, p. 1718 of [6], $|\mathcal{I}(H_S S, H_S)| \neq 2$ and by Theorem A, p. 2025 of [7], $|\mathcal{I}(H_S S, H_S)| \neq 3$. Hence $|\mathcal{I}(\langle S\rangle, H_S)| = 1$. Thus, by the Main Theorem, p. 643 of [11], $H_S \trianglelefteq \langle S\rangle$. Let $\chi_S : H_S S \to G_S S$ be the map defined as in the second paragraph of § 1. Then $H_S \subseteq \ker \chi_S \subseteq \mathrm{Core}_G(H) = \{1\}$. Hence $H_S = \{1\}$, that is S is a subgroup of G. □

PROPOSITION 2.2

Let H be a corefree subgroup of a finite group G. Let $S \in \mathcal{T}(G, H)$ be such that $\langle S\rangle = G$. Then $\mathrm{Aut}_H G$ acts transitively on the set $\{T \in \mathcal{T}(G, H) | T \cong S\}$.

Proof. The proof follows from the first paragraph of the proof of Proposition 2.7, p. 652 of [11]. □

PROPOSITION 2.3

Let (G, H) be a minimal counterexample. Let N be a proper $\mathrm{Aut}_H G$-invariant subgroup of G containing H properly. Then there exists $S \in \mathcal{T}(G, H)$ such that $S \neq TL$ for any $T \in \mathcal{T}(N, H)$ and $L \in \mathcal{T}(G, N)$.

Proof. Assume that each $S \in \mathcal{T}(G, H)$ can be written as $S = TL$, for some $T \in \mathcal{T}(N, H)$ and $L \in \mathcal{T}(G, N)$. Then $|\mathcal{T}(G, H)| \leq |\mathcal{T}(N, H)|\,|\mathcal{T}(G, N)|$. Keeping the same lines as in the proof of Lemma 2.5, p. 1720 of [6], we observe that $|H| = 2$, $N \cong C_4$ and $|\mathcal{I}(G, N)| \leq 3$. If $|\mathcal{I}(G, N)| = 1$, then $N \trianglelefteq G$ (see Main Theorem, p. 643 of [11]). But then this implies $H \trianglelefteq G$, a contradiction. Also, by Theorem, p. 1718 of [6], $|\mathcal{I}(G, N)| \neq 2$.

Thus $|\mathcal{I}(G, N)| = 3$. Then $[G : N] = 3$ (see Theorem A, p. 2025 of [7]). This means that $|G| = 12$ and G contains a cyclic subgroup of order 4. By the classification of non-abelian groups of order 12, the only choice for G is $G \cong C_3 \rtimes C_4$. But $C_3 \rtimes C_4$ has a unique subgroup of order 2, hence normal in G. This is again a contradiction. □

COROLLARY 2.4

Let (G, H) be a minimal counterexample. Let N be a proper $\mathrm{Aut}_H G$-invariant subgroup of G containing H properly. Let $K \in \mathcal{T}(N, H)$. Then there exists $S \in \mathcal{T}(G, H)$ containing K such that $S \neq KL$ for any $L \in \mathcal{T}(G, N)$.

Lemma 2.5. *Let $G = D_8$ and H be a non-normal subgroup of G of order* 2. *Then* $|\mathcal{I}(G, H)| = 6$.

Proof. Let $H = \{1, x\}$. Let $y \in G$ be of order 4 and $N = \langle y \rangle$. Then $G = \langle x, y \rangle$ with $xyx = y^3$ and $\mathrm{Aut}_H G = \{I, i_x\}$, where i_x denotes the inner automorphism of G determined by x. Let $\epsilon : N \to H$ be a function with $\epsilon(1) = 1$. Let $S_\epsilon = \{\epsilon(y^i)y^i \mid 1 \leq i \leq 4\} \in \mathcal{T}(G, H)$. Note that $xS_\epsilon x^{-1} = S_\epsilon$ means that $\epsilon(y^i)y^i \in S_\epsilon$ if and only if $\epsilon(y^i)y^{4-i} \in S_\epsilon$. This implies that $S_1 = N = \{I, y, y^2, y^3\}$, $S_2 = \{I, xy, y^2, xy^3\}$, $S_3 = \{I, y, xy^2, y^3\}$ and $S_4 = \{I, xy, xy^2, xy^3\}$ are the fixed point of the action of $\mathrm{Aut}_H G$ on $\mathcal{T}(G, H)$. Since $|\mathcal{T}(G, H)| = 8$, there are two orbits of length 2. Let $S_5 = \{I, xy, y^2, y^3\}$, $S_6 = \{I, xy, xy^2, y^3\}$. Then the NRTs S_5 and S_6 are in the distinct $\mathrm{Aut}_H G$ orbits which are not singletons.

One observes that $S_1 \cong C_4$, $S_2 \cong C_2 \times C_2$ and $\langle S_i \rangle = G$ $(3 \leq i \leq 6)$. Then $S_1 \ncong S_2$. Further, if $S_i \cong S_j$ for $3 \leq i \neq j \leq 6$, then by Proposition 2.2 $xS_ix^{-1} = S_j$, which is a contradiction (for S_i and S_j lie in different $\mathrm{Aut}_H G$-orbits). □

Remark 2.6.

(i) By the same argument as above, we find that $|\mathcal{I}(G, H)| = 20$, where $G = D_{12}$ and H is a non-normal subgroup of G of order 2 (see also Remark 2.7, p. 2028 of [7]). In [8], a formula for the number of orbits of the action of $\mathrm{Aut}_H G$ on $\mathcal{T}(G, H)$ has been obtained.
(ii) As argued in the proof of Lemma 2.5, we see That if H is a corefree subgroup of a finite group G, then the NRTs from different orbits of the action of $\mathrm{Aut}_H G$ on $\mathcal{T}(G, H)$ which generate the group G represent pairwise non-isomorphic NRTs.

Lemma 2.7. *Let $G = \mathrm{Alt}(4)$ and H be a subgroup of G of order* 2. *Then* $|\mathcal{I}(G, H)| = 5$.

Proof. Since H is a subgroup of $G = \text{Alt}(4)$ of order 2, there is a unique Sylow 2-subgroup P of Sym(4) such that $H = Z(P)$, the center of P. Since $\text{Aut}(\text{Alt}(4)) \cong \text{Inn}(\text{Sym}(4)) \cong \text{Sym}(4)$, $\text{Aut}_H G \cong N_{\text{Sym}(4)}(H)(= P)$, the normalizer of H in Sym(4). As there is no subgroup of Alt(4) of index 2, $\langle S \rangle = \text{Alt}(4)$ for all $S \in \mathcal{T}(G, H)$. By Remark 2.6(ii), $|\mathcal{I}(\text{Alt}(4), H)|$ is precisely the number of orbits of the conjugation action of P on $\mathcal{T}(\text{Alt}(4), H)$.

We may assume that $H = \{I, x = (1,2)(3,4)\}$ (as any two elements of order 2 in G are conjugate). Let $P = \langle (1,2), (1,3,2,4) \rangle$. Then $\text{Aut}_H(\text{Alt}(4)) = \{i_g | g \in P\}$, where for $g \in P$, i_g denotes the conjugation of Alt(4) by g.

Let $T = \{I, y = (1,3)(2,4)\}$, $L = \{I, z = (1,2,3), z^{-1} = (1,3,2)\}$ and $S = TL$. Then $S \in \mathcal{T}(G, H)$. Let $S_1 = S = \{I, y, z, z^{-1}, yz^{-1}, yz\}$, $S_2 = \{I, y, z, z^{-1}, yz^{-1}, xyz\}$, $S_3 = \{I, y, z, z^{-1}, xyz^{-1}, xyz\}$, $S_4 = \{I, y, z, z^{-1}, xyz^{-1}, yz\}$ and $S_5 = \{I, y, z, xz^{-1}, yz^{-1}, yz\}$. Let $i \in \{1, \ldots, 5\}$. We note that, if $g \in P$ such that $gS_ig^{-1} = S_i$, then $gyg^{-1} = y$ and so $g = x$. Since $xzx^{-1} = yz$ and $xz^{-1}x^{-1} = xyz^{-1}$, it follows that S_1, S_2, S_3, S_4 and S_5 lie in distinct $\text{Aut}_H G$-orbits with orbit length 8, 8, 8, 4 and 4 respectively. □

Lemma 2.8. *Let (G, H) be a minimal counterexample and N be a proper* $\text{Aut}_H G$-*invariant subgroup of G. Let $K \in \mathcal{T}(N, H)$ which is a subgroup of N. Then $[G : N] \neq 2$.*

Proof. If possible, assume that $[G : N] = 2$. Further, assume that $[N : H] = 2$. Since $\text{Core}_G(H) = \{1\}$ (Proposition 2.1(i)), we can identify G with a subgroup of Sym(4). The only possibility for the pair (G, N) we are left with is G, a Sylow 2-subgroup of Sym(4), $N \cong C_2 \times C_2$. Hence $G \cong D_8$. By Lemma 2.5, $|\mathcal{I}(G, H)| = 6$, a contradiction. Thus $[N : H] > 2$. Let $L = \{1, l\} \in \mathcal{T}(G, N)$, $k_2, k_3 (\neq k_2) \in K \setminus \{1\}$ and $U = K \setminus \{k_3\} \cup \{hk_3\}$, where $h \in H \setminus \{1\}$. We note that U is not a subgroup of N, for $k_2, k_3k_2^{-1} \in U$ but $k_3 = (k_3k_2^{-1})k_2 \notin U$. Let $S_1 = KL$ and $S_2 = UL$. As $U = N \cap S_2$ is not a subgroup of N, S_2 is not a subgroup of G. Let $S_3 = (S_1 \setminus \{k_3l\}) \cup \{hk_3l\}$, $S_4 = (S_1 \setminus \{k_2l, k_3l\}) \cup \{hk_2l, hk_3l\}$ and $S_5 = S_2 \setminus \{hk_3l\} \cup \{k_3l\}$. Observe that S_i $(3 \leq i \leq 5)$ are not subgroups of G, for $hk_3 = (hk_3l)(l^{-1}) \notin S_i$ $(i = 3, 4)$ and $k_3 = (k_3l)(l^{-1}) \notin S_5$.

We also claim that $S_i \neq T'L'$ $(3 \leq i \leq 5)$ for any $T' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$. If possible, suppose that $S_3 = T'L'$ for some $T' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$. Then $T' = S_3 \cap N = K$. Since $[G : N] = 2$, either $hk_3l \in L'$ or $kl \in L'$ for some $k \in K \setminus \{k_3\}$. Further, $N = HK$ is a subgroup of G, $hk_3 = k_3'h'$ for some $k_3' \in K$ and $h' \in H$. Also $h' \neq 1$, for $K \in \mathcal{T}(N, H)$. Assume that $hk_3l \in L'$. Then ${k_3'}^{-1}(hk_3l) = h'l \in KL' = S_3$. This is a contradiction, for $l \in S_3$ and $h' \neq 1$. Thus $kl \in L'$ for some $k \in K \setminus \{k_3\}$. Since $hk_3l \in S_3$, we have $hk_3l = k'(kl)$ for some $k' \in K$. This implies that $hk_3 \in K$, a contradiction. Similarly $S_4 \neq T'L'$ for any $T' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$. We again claim the same for S_5. If possible, suppose that $S_5 = T'L'$ for some $T' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$. As argued above $T' = U$. Since $[G : N] = 2$, $kl \in L'$ for some $k \in K$. If $k = 1$, then $(hk_3)l \in UL' = S_5$, a contradiction. Thus $k \neq 1$. This means that $k_3k \in U$, which implies that $k_3kl \in S_5$. Hence $(hk_3)kl = h(k_3kl)$ can not be in $UL' = S_5$, a contradiction.

We now show that S_i $(1 \leq i \leq 5)$ are pairwise non-isomorphic NRTs of H in G. Since $S_i \neq T'L'$ $(3 \leq i \leq 5)$ for any $T' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$, $S_1 \ncong S_i$ $(3 \leq i \leq 5)$ and $S_2 \ncong S_i$ $(3 \leq i \leq 5)$. Further, since U is not a subgroup of N, $S_1 \ncong S_2$, $S_3 \ncong S_5$ and $S_4 \ncong S_5$ (as $K \subseteq S_i$ for $i \in \{1, 3, 4\}$ and N is an $\text{Aut}_H G$-invariant subgroup of G). Next, assume that $S_3 \cong S_4$. Then, by Proposition 2.2 there exists $f \in \text{Aut}_H G$ such that

$f(S_3) = S_4$. Hence $f(K) = K$ (for N is an $\mathrm{Aut}_H G$-invariant subgroup of G). Assume that $f(l) = kl$ for some $k \in K$. Then $f(k'l) = f(k')kl \in Kl$ for all $k' \in K$. In this case, either hk_2l or hk_3l can not be an image of any element in S_4, a contradiction. Hence $f(l) = hk_2l$ or $f(l) = hk_3l$. Suppose that $f(l) = hk_2l$. Since $N = HK$ is a subgroup of G and $K \in \mathcal{T}(N, H)$, $hk_2 = k_2'h'$ for some $k_2' \in K \setminus \{1\}$ and $h' \in H \setminus \{1\}$. Let $k_2'' \in K$ such that $f(k_2'') = k_2'^{-1}$. This implies that $f(k_2''l) = k_2'^{-1}(hk_2l) = h'l$, which is a contradiction for $l \in S_4$. Similarly $f(l) \neq hk_3l$. Hence $S_3 \ncong S_4$. □

Lemma 2.9. *Let G be a finite group. Let H be a non-normal, abelian, corefree subgroup of G and N be a normal subgroup of G containing H such that* $[N : H] = 2$. *Then*

(i) *if* $[G : N] = 2$, *then* $G \cong D_8$ *and* $|H| = 2$,
(ii) *if* $[G : N] = 3$, *then* $G \cong \mathrm{Alt}(4) \times C_2$ *and* $H \cong C_2 \times C_2$ *or* $G \cong \mathrm{Alt}(4)$ *and* $H \cong C_2$.

Proof.

(i) Assume that $[G : N] = 2$. Then as argued in the first few lines of Lemma 2.8, $G \cong D_8$, $N \cong C_2 \times C_2$ and H is a non-normal subgroup of G of order 2.
(ii) Assume that $[G : N] = 3$. We can identify G with a subgroup of Sym(6). Since the order of an abelian subgroup of Sym(6) is at most 9 (Theorem 1, p. 70 of [1]), $|H| \leq 9$. Further, since Sym(6) has no subgroup of order 54, $|H| \neq 9$. Assume that $|H| = 8$. Then $|N| = 16$. Hence N is a Sylow 2-subgroup of Sym(6). Since $G \subseteq N_{\mathrm{Sym}(6)}(N)$ (the normalizer of N in Sym(6)) and a Sylow 2-subgroup of Sym(6) is self-normalizing (Corollary 1, p. 123 of [17]), $N = G$, a contradiction. Further $|H| \neq 7$, for Sym(6) does not contain a 7-cycle.

Next, assume that $|H| = 6$. Then $|N| = 12$. By classification of groups of order 12, $N \cong D_{12}$. Since $N \trianglelefteq G$ and N contains a unique cyclic subgroup of order 6, $H \trianglelefteq G$, a contradiction. Next, assume that $|H| = 5$. Then $N \cong D_{10}$. Hence, in this case also $H \trianglelefteq G$. Similar argument shows that $|H| \neq 3$.

Next, assume that $|H| = 4$. Then $|N| = 8$. Assume that $H \cong C_4$. Since Sym(6) does not contain an 8-cycle, $N \ncong C_8$. Suppose that $N \cong (C_4 \times C_2)$. Then, by 84(ii), p. 102 of [2] $G \cong N \times C_3$. This is a contradiction, for the order of an abelian subgroup of Sym(6) is at most 9 (Theorem 1, p. 70 of [1]). Assume that $N \cong D_8$. Then as argued in the above paragraph $H \trianglelefteq G$, a contradiction. Let $N \cong Q_8$ (the quaternion group of order 8). By 84(iv), p. 103 of [2], either $G \cong (Q_8 \times C_3)$ or $G \cong (Q_8 \rtimes C_3)$. If $G \cong (Q_8 \times C_3)$, then $H \trianglelefteq G$ (5.3.7 (Dedekind, Baer), p. 143 of [12]), a contradiction. Hence $G \cong (Q_8 \rtimes C_3)$. But in this case $|\mathrm{Core}_G(H)| = 2$, a contradiction. Thus, $H \cong C_2 \times C_2$. This implies $N \cong D_8$ or $N \cong C_2 \times C_4$ or $N \cong C_2 \times C_2 \times C_2$. Also if $N \cong D_8$, then $H \trianglelefteq G$ (since N contains a unique non-cyclic subgroup of order 4). This is a contradiction. If $N \cong C_2 \times C_4$, then $G \cong (C_2 \times C_4) \times C_3$ (84(ii), p. 102 of [2]), a contradiction. Thus $N \cong C_2 \times C_2 \times C_2$. By 84(iii), p.102 of [2], $G \cong \mathrm{Alt}(4) \times C_2$.

Lastly, assume that $|H| = 2$. Then either $N \cong C_4$ or $N \cong C_2 \times C_2$. If $N \cong C_4$, then by classification of non-abelian groups of order 12, $G \cong C_4 \rtimes C_3$. But in this case $H \trianglelefteq G$, a contradiction. Hence $N \cong C_2 \times C_2$. Since $N \trianglelefteq G$, $G \cong \mathrm{Alt}(4)$. □

Lemma 2.10. *Let* $G \cong \mathrm{Alt}(4) \times C_2$ *and H be a corefree subgroup of G of index* 6. *Then* $|\mathcal{I}(G, H)| > 4$.

Proof. Since $\mathrm{Core}_G(H) = \{1\}$, we can identify G with a subgroup of Sym(6). Thus, there exist subgroups K and L of Sym(6) such that $K \cong \mathrm{Alt}(4)$, $L \cong C_2$, $K \cap L = \{1\}$,

$G = KL$, $K \trianglelefteq G$ and $L \trianglelefteq G$. Further, since $\mathrm{Core}_G(H) = \{1\}$, $H \cap L = \{1\}$ and $|H \cap K| = 2$. Thus $HK = G$ and so $\mathcal{T}(K, K \cap H) \subseteq \mathcal{T}(G, H)$. By Lemma 2.7, $|\mathcal{I}(K, K \cap H)| = 5$. Thus $|\mathcal{I}(G, H)| > 4$. □

Lemma 2.11. *Let (G, H) be a minimal counterexample. Let N be an $\mathrm{Aut}_H G$-invariant subgroup of G such that $[N : H] = 2$. Assume that there exists $T \in \mathcal{T}(N, H)$ which is a subgroup of N. Then $[G : N] \in \{2, 3\}$.*

Proof. If possible, assume that $[G : N] \geq 4$. Let $T = \{1, x\} \in \mathcal{T}(N, H)$ which is a subgroup of N. Let $L = \{1, l_2, l_3, l_4, ..., l_r\} \in \mathcal{T}(G, N)$ and $h \in H \setminus \{1\}$. Consider $S = TL$, $S_1 = (S \setminus \{x\}) \cup \{hx\}$, $S_2 = (S \setminus \{l_r\}) \cup \{hl_r\}$, $S_3 = (S \setminus \{l_{r-1}, l_r\}) \cup \{hl_{r-1}, hl_r\}$ and $S_4 = (S \setminus \{l_{r-2}, l_{r-1}, l_r\}) \cup \{hl_{r-2}, hl_{r-1}, hl_r\}$. As argued in the proof of Lemma 2.12, p. 2030 of [7], S and S_i $(1 \leq i \leq 3)$ are non-isomorphic NRTs.

As argued in the second paragraph of the proof of Lemma 2.12, p. 2030 of [7], we can show that S_4 is not a subgroup of G and $S_4 \neq T'L'$ for any $T' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$. This shows that $S_4 \ncong S$. Now, we show that $S_4 \ncong S_i$ $(1 \leq i \leq 3)$. Assume that $S_4 \cong S_3$. Then by Proposition 2.2, there exists $f \in \mathrm{Aut}_H G$ such that $f(S_3) = S_4$. Since N and H are $\mathrm{Aut}_H G$-invariant, $f(x) = x$. Further, since $[G : N] \geq 4$, there exist $i \in \{0, 1\}$ and $k \in \{2 \cdots r - 3\}$ such that $f(x^i l_k) = hl_j$ for some $j \in \{r-2, r-1, r\}$. Hence $f(x^{i+1} l_k) = f(x)hl_j = xhl_j \notin S_4$, a contradiction. Similarly, $S_4 \ncong S_i$ $(i = 1, 2)$. Thus $|\mathcal{I}(G, H)| > 4$, a contradiction. □

PROPOSITION 2.12

Let (G, H) be a minimal counterexample. Let N be an $\mathrm{Aut}_H G$-invariant subgroup of G such that $H \subsetneq N \subsetneq G$. Then $\langle S \rangle = G$ for all $S \in \mathcal{T}(G, H)$.

Proof. On the contrary, assume that there exists $S' \in \mathcal{T}(G, H)$ such that $\langle S' \rangle \neq G$. Then S' is a subgroup of G (Proposition 2.1(iii)). Thus $K = S' \cap N \in \mathcal{T}(N, H)$ is a subgroup of N. Further, assume that all the members of $\mathcal{T}(N, H)$ are subgroups of N. This implies that $N \cong H \rtimes C_2$ and H is abelian (Lemma 2.4, p. 1719 of [6]). By Lemma 2.11, $[G : N] = 2$ or $[G : N] = 3$.

Assume that $[G : N] = 2$. By Lemma 2.9(i), $G \cong D_8$ and $H \cong C_2$. But in this case, $|I(G, H)| = 6$ (Lemma 2.5), a contradiction. Thus $[G : N] = 3$. Hence G is isomorphic to a subgroup of Sym(6). By Lemma 2.9(ii), the choices for the pair (G, H) in this case are $G \cong \mathrm{Alt}(4) \times C_2$, $H \cong C_2 \times C_2$ or $G \cong \mathrm{Alt}(4)$, $H \cong C_2$. By Lemmas 2.10 and 2.7, $|\mathcal{I}(G, H)| > 4$, again a contradiction. Thus, there exists $U \in \mathcal{T}(N, H)$ which is not a subgroup of G.

Let $L_1 \in \mathcal{T}(G, N)$, $S_1 = S'$ and $S_2 = UL_1$. By Corollary 2.4, there exists $S_3 \in \mathcal{T}(G, H)$ such that $U \subseteq S_3$ and $S_3 \neq UL$ for any $L \in \mathcal{T}(G, N)$. Also, let $S_4 = KL_1$, $S_4' = (S_4 \setminus \{l\}) \cup \{hl\}$, where $h \in H \setminus \{1\}$ and $l \in L_1 \setminus \{1\}$. Let $S_5 = (UL_1 \setminus U) \cup K$. Let $S_4{}^{\epsilon}$ denote S_4 if it is not subgroup of G, otherwise it is S_4'. As argued in paragraphs three and four of the proof of Lemma 2.16, p. 2032 of [7], S_1, S_2, S_3 and S_4^{ϵ} are pairwise non-isomorphic NRTs and each of S_2, S_3 and S_4^{ϵ} generates G. We show that S_5 is not isomorphic to S_i $(1 \leq i \leq 4)$.

The NRT S_5 is not a subgroup of G, for if $u \in U \setminus K$, then $l, ul \in S_5$, but $u = (ul)l^{-1} \notin S_5$.

Next, assume that $S_5 = K'L'$, for some $K' \in \mathcal{T}(N, H)$ and $L' \in \mathcal{T}(G, N)$. Then $K' = N \cap S_5 = K$. Fix $u \in U \setminus K$. If possible, assume that $ku \in U$ for all $k \in K \setminus U$. Then $ku \in U \setminus K$ for all $k \in K \setminus U$ (since K is a subgroup of G). Thus the map $k \mapsto ku$ is a bijection from $K \setminus U$ to $U \setminus K$. But, this is a contradiction, for $u \in U \setminus K$ is not an image under this map. Thus, there exists $k \in K \setminus U$ such that $ku \notin U$. Fix such a $k \in K \setminus U$.

Since $k'l, u'l \in S_5$ ($k' \in K \cap U, u' \in U \setminus K$) are in the same right coset of N in G and $L' \subseteq S_5$, either $k'l \in L'$ for some $k' \in K \cap U$ or $u'l \in L'$ for some $u' \in U \setminus K$. Suppose that $u'l \in L'$ for some $u' \in U \setminus K$. Then as argued in the above paragraph, there exists $k' \in K \setminus U$ such that $k'u' \notin U$. This implies that $k'(u'l) \notin S_5$, which is a contradiction. Thus $k'l \in L'$ for some $k' \in K \cap U$. But in this case, for any $u' \in U \setminus K$, $u'l \in S_5$ can not be written as a product of a member of K and a member of L'. This is again a contradiction. Thus $S_5 \neq K'L'$ for any $K' \in \mathcal{T}(N, H)$ and for any $L' \in \mathcal{T}(G, N)$. Hence S_5 is neither isomorphic to S_2 nor isomorphic to S_4 (if $S_4^{\epsilon} = S_4$). If possible assume that $S_5 \cong S_3$. Then by Proposition 2.2 there exists $f \in \mathrm{Aut}_H G$ such that $f(S_5) = S_3$. Since N is an $\mathrm{Aut}_H G$-invariant subgroup of G, $f(K) = U$. This is a contradiction (for U is not a subgroup of G).

Finally, assume that $S_4^{\epsilon} = S_4'$ and $S_5 \cong S_4'$. Since $\langle S_4' \rangle = \langle S_5 \rangle = G$, by Proposition 2.2 there exists $f \in \mathrm{Aut}_H G$ such that $f(S_4') = S_5$. As N is $\mathrm{Aut}_H G$-invariant, $f(K) = f(S_4' \cap N) = S_5 \cap N = K$. Since S_4 is a subgroup of G, $S_5' = (S_5 \setminus \{f(hl)\}) \cup \{f(l)\}$ is also a subgroup of G. We claim that $f(hl) \neq l$. Suppose that $f(hl) = l$. Then $f(l) = h_1 l$, where $h_1 = f(h)^{-1} \in H$. Since $k, ul \in S_5'$, but $k(ul) \notin S_5'$, a contradiction (for S_5' is a subgroup of G). Thus $f(hl) \neq l$. Since $f(hl) \notin K$, there exists $u_1 \in U$ and $l_1 \in L_1$ such that $f(hl) = u_1 l_1$. By Lemma 2.8, $|L_1| \geq 3$. Let $l_2 \in L_1 \setminus \{1, l_1\}$. Then $k, ul_2 \in S_5'$, but $k(ul_2) \notin S_5'$, a contradiction. □

PROPOSITION 2.13

Let (G, H) be a minimal counterexample and $S \in \mathcal{T}(G, H)$. Let N be an $\mathrm{Aut}_H G$-invariant subgroup of G such that $H \subsetneq N \subsetneq G$. Then S is indecomposable.

Proof. If possible, suppose that S is decomposable and $S = S_1 \times S_2 \times \cdots \times S_n$ ($n \geq 2$) is a Remak–Krull–Schmidt decomposition of S (see Theorem 1.11, p. 648 of [11]). By Proposition 2.12 and Remark 2.4, p. 650 of [11], we may identify (G, H) with $(G_{S_1} S_1 \times G_{S_2} S_2 \times \cdots \times G_{S_n} S_n, G_{S_1} \times G_{S_2} \times \cdots \times G_{S_n})$. We claim that $|\mathcal{I}(G_{S_i} S_i, G_{S_i})| \leq 4$ ($1 \leq i \leq n$). If possible, assume that there exists k ($1 \leq k \leq n$) such that $|\mathcal{I}(G_{S_k} S_k, G_{S_k})| > 4$. Let $T_1, T_2, T_3, T_4, T_5 \in \mathcal{T}(G_{S_k} S_k, G_{S_k})$ be pairwise non-isomorphic NRTs. Then $L_i = S_1 \times \cdots \times S_{k-1} \times T_i \times S_{k+1} \times \cdots \times S_n$ ($1 \leq i \leq 5$) are pairwise non-isomorphic NRTs of H in G by Remak–Krull–Schmidt theorem (see Theorem 1.11, p. 648 of [11]), a contradiction.

Since (G, H) is a minimal counterexample, $|\mathcal{I}(G_{S_k} S_k, G_{S_k})| \leq 3$ for all $k \in \{1 \cdots n\}$. Further, since by Theorem, p. 1718 of [6], $|\mathcal{I}(G_{S_k} S_k, G_{S_k})| \neq 2$ for all $k \in \{1 \cdots n\}$, either $|\mathcal{I}(G_{S_k} S_k, G, _{S_k})| = 1$ or $|\mathcal{I}(G_{S_k} S_k, G_{S_k})| = 3$. In either case, we get $T_k \in \mathcal{T}(G_{S_k} S_k, G_{S_k})$ which is a group. Let $T = T_1 \times T_2 \times \cdots \times T_n$. Then $T \in \mathcal{T}(G, H)$ and is a group. Since $\langle T \rangle = G$ (Proposition 2.12), by Proposition 2.1(ii), $H = G_T = \{1\}$, a contradiction. □

3. Proof of the theorem

In this section, we study some more properties of a minimal counterexample and reduce it to the case of a finite non-abelian simple group. Then we apply the classification of finite simple groups (the knowledge of the order of automorphism groups of finite non-abelian simple groups) to complete the proof of the main theorem.

PROPOSITION 3.1

Let (G, H) be a minimal counterexample. Then G is indecomposable.

Proof. If possible, suppose that G is decomposable. Let G_1 and G_2 be nontrivial proper normal subgroups of G such that $G = G_1G_2$ and $G_1 \cap G_2 = \{1\}$. Let $\pi_i : G \to G_i$ $(i = 1, 2)$ be projections. Let $\pi_i(H) = U_i$ $(i = 1, 2)$. The restriction $\pi_i|_H$ of π_i to H induces isomorphism $\sigma_i : H/(H \cap G_1)(H \cap G_2) \to (U_i/(H \cap G_i))$ $(i = 1, 2)$. This gives an isomorphism $\theta = \sigma_2 \circ \sigma_1^{-1}$ from $U_1/(H \cap G_1)$ to $U_2/(H \cap G_2)$ given by $\theta(\pi_1(h)(H \cap G_1)) = \pi_2(h)(H \cap G_2)$, $h \in H$. Also

$$H = \{u_1u_2 \in U_1U_2 | \theta(u_1(H \cap G_1)) = u_2(H \cap G_2)\}.$$

Since $\mathrm{Core}_G(H) = \{1\}$ (Proposition 2.1(i)), $H \cap G_i \neq G_i$ for $i = 1, 2$. Suppose that $H \cap G_1 = U_1$. Then the isomorphism θ implies $H \cap G_2 = U_2$. Let $S_i \in \mathcal{T}(G_i, U_i)$ $(i = 1, 2)$. Now as argued in the second paragraph of the proof of Proposition 2.6, p. 650 of [11], we get an $S = S_1S_2 \in \mathcal{T}(G, H)$ which is decomposable. If there exists an $\mathrm{Aut}_H G$-invariant subgroup N of G such that $H \subsetneqq N \subsetneqq G$, then this is a contradiction (Proposition 2.13). Thus, there does not exist any $\mathrm{Aut}_H G$-invariant subgroup N of G such that $H \subsetneqq N \subsetneqq G$.

Next, we prove that there is a member of $\mathcal{T}(G, H)$ which is not a subgroup of G and is decomposable. Assume that each member of $\mathcal{T}(G_i, U_i)$ $(i = 1, 2)$ is a subgroup of G_i. Then by Lemma 2.4, p. 1719 of [6], $|S_1| = |S_2| = 2$. Hence $|S| = 4$. Since $\mathrm{Core}_G(H) = \{1\}$, we can identify G with a subgroup of Sym(4). By subgroups structure of Sym(4), there is no non-abelian decomposable subgroup of Sym(4). This is a contradiction. Therefore, there exist $S_i' \in \mathcal{T}(G_i, U_i)$ $(i = 1, 2)$ such that atleast one of them is not a subgroup of G. This implies that $S' = S_1'S_2' \in \mathcal{T}(G, H)$ is decomposable which is not a subgroup of G.

Let $S' = S_1 \times S_2 \times \cdots \times S_n$ $(n \geq 2)$ be a Remak–Krull–Schmidt decomposition of S (see Theorem 1.11, p. 648 of [11]). Since $\mathrm{Core}_G(H) = \{1\}$, $(G, H) = H_{S'}S' \cong G_{S'}S'$ (Proposition 2.1). By Proposition Remark 2.4, p. 650 of [11] we identify (G, H) with $(G_{S_1}S_1 \times G_{S_2}S_2 \times \cdots \times G_{S_n}S_n, G_{S_1} \times G_{S_2} \times \cdots \times G_{S_n})$. Also since $\mathrm{Core}_G(H) = \{1\}$, $\mathrm{Core}_{H_{S_i}S_i}(H_{S_i}) = \{1\}$ $(1 \leq i \leq n)$. By the similar arguments as in the first paragraph of the proof of Proposition 2.13, $|\mathcal{I}(G_{S_i}S_i, G_{S_i})| \leq 4$ for each i $(1 \leq i \leq n)$. Since (G, H) is a minimal counterexample, either $|\mathcal{I}(G_{S_k}S_k, G_{S_k})| = 1$ or $|\mathcal{I}(G_{S_k}S_k, G_{S_k})| = 3$ for all $k \in \{1 \cdots n\}$ (for $|\mathcal{I}(G_{S_k}S_k, G_{S_k})| \neq 2$ by Theorem, p. 1718 of [6]). Since $H_{S'}S' = G$, there exists $1 \leq i \leq n$ such that $G_{S_i} \neq \{1\}$. Without any loss, we may assume that $i = 1$. Then by Theorem A, p. 2025 of [7], $|G_{S_1}| = 2$, $|S_1| = 3$ and $G_{S_1}S_1 \cong \mathrm{Sym}(3)$. Further, assume that $G_{S_k} \neq \{1\}$ for some $k > 1$. We may assume that $k = 2$. Then as argued above $G_{S_2}S_2 \cong \mathrm{Sym}(3)$ and $|G_{S_2}| = 2$. Thus $G_{S'}S' \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times G_{S_3}S_3 \times \cdots \times G_{S_n}S_n$. Let $T_l \in \mathcal{T}(G_{S_1}S_1, G_{S_1})$ $(1 \leq l \leq 3)$ be pairwise non-isomorphic NRTs and T_l' $(1 \leq l \leq 3)$ be those in $\mathcal{T}(G_{S_2}S_2, G_{S_2})$ such that T_1 and T_1' are subgroups of $G_{S_1}S_1$ and $G_{S_2}S_2$ respectively. Let $K = S_3 \times \cdots \times S_n$. Let $S_{rq} = T_r \times T_q' \times K \in \mathcal{T}(G, H)$ $(1 \leq r, q \leq 3)$. Then by Theorem 1.11, p. 648 of [11], S_{11}, S_{12}, S_{13}, S_{23} and S_{33} are pairwise non-isomorphic NRTs. This is a contradiction (for $|\mathcal{I}(G, H)| = 4$). Therefore, $G_{S_j} = \{1\}$ for $2 \leq j \leq n$. This means that S_j $(2 \leq j \leq n)$ is a subgroup of G.

Assume that S_j $(2 \leq j \leq n)$ are perfect. Hence, the solvable radical R of G is isomorphic to $G_{S_1}S_1$ and $N = HR = HS_1$ is an $\mathrm{Aut}_H G$-invariant proper subgroup of G containing H properly. This is again a contradiction. Thus, there exists r $(2 \leq r \leq n)$

such that S_r is not a perfect group. Now, consider the commutator $[G, G]$ of the group G. Since $[G, G] \subsetneqq T_1 \times S_2 \times \cdots \times S_n$ (for $[S_r, S_r] \neq S_r$), $[G, G] \notin \mathcal{T}(G, H)$. This implies that $H[G, G]$ is an $\mathrm{Aut}_H G$-invariant proper subgroup of G containing H properly. This is a contradiction.

Thus, we may now assume that $H \cap G_i \neq U_i$ $(i = 1, 2)$. Suppose that $U_1 = G_1$ and $U_2 = G_2$. Then as argued in the third paragraph of the proof of Proposition 2.6, p. 650 of [11] replacing Corollary 2.3 of [11] by Proposition 2.1(i), we get $G_2 \in \mathcal{T}(G, H)$ with $H \cong G_1 \cong G_2$. If there exists an $\mathrm{Aut}_H G$-invariant subgroup N such that $H \subsetneqq N \subsetneqq G$, then this is a contradiction (Proposition 2.12). Thus, there does not exist any $\mathrm{Aut}_H G$-invariant subgroup N of G such that $H \subsetneqq N \subsetneqq G$. Now, assume that G_2 is not characteristically simple. Let K be a non-trivial proper characteristic subgroup of G_2. Then $H \subsetneqq HK \subsetneqq G$. Since HK is $\mathrm{Aut}_H G$-invariant subgroup of G, this is a contradiction. Hence G_2 is characteristically simple. Since $G_2 \cong H$, H is characteristically simple. By 3.3.15 of [12], G_2 is a direct product of isomorphic finite simple groups. Assume that G_2 is an elementary abelian p-group. This implies that G is a p-group. Then the center $Z(G)$ of the group G is non-trivial. Note that $Z(G) \notin \mathcal{T}(G, H)$ (for otherwise $G \cong H \times Z(G)$). Since $HZ(G)$ is an $\mathrm{Aut}_H G$-invariant proper subgroup of G containing H properly, we get a contradiction. Thus G_2 is non-abelian. Let $H = H_1 \times \cdots \times H_n$ and $G_2 = L_1 \times \cdots \times L_n$, where all H_i and L_j $(1 \leq i, j \leq n)$ are isomorphic to a fixed non-abelian simple group (see 3.3.15 of [12]). Since G_2 is a direct factor G, each direct factor of G_2 is a normal subgroup of G. Hence $H_i L_j$ $(1 \leq i, j \leq n)$ is a subgroup of G. By Theorem 1 of [5], $H_i L_j \cong H_i \times L_j$ $(1 \leq i, j \leq n)$. This implies that $G = H \times G_2$. This is a contradiction.

Thus, we may assume that $H \cap G_i \neq U_i$ $(i = 1, 2)$ and $U_1 \neq G_1$. Let $S_i \in \mathcal{T}(G_i, U_i)$ $(i = 1, 2)$ and $T \in \mathcal{T}(U_2, H \cap G_2)$. Then by the same argument as in the last paragraph of the proof of Proposition 2.6, p. 650 of [11], we get that $T \in \mathcal{T}(U_1U_2, H)$ and $S = S_1(TS_2) \in \mathcal{T}(G, H)$ which is decomposable. If there exists an $\mathrm{Aut}_H G$-invariant subgroup N of G such that $H \subsetneqq N \subsetneqq G$, then this is a contradiction (Proposition 2.13). Thus, there does not exist any $\mathrm{Aut}_H G$-invariant subgroup N of G such that $H \subsetneqq N \subsetneqq G$.

Assume that $U_2 = G_2$. By similar arguments as in the third paragraph of the above proof, we get a member of $\mathcal{T}(G, H)$ which is not a subgroup of G and is decomposable. Next, assume that $U_2 \neq G_2$. Further assume that each member of $\mathcal{T}(G_1, U_1)$ and $\mathcal{T}(G_2, U_2)$ are subgroups of G_1 and G_2 respectively. Then by Lemma 2.4, p. 1719 of [6], $G_i = U_i \rtimes C_2$ $(i = 1, 2)$ with U_i abelian subgroup of G_i. Then U_1U_2 is an abelian subgroup of G. Since $H \subsetneqq U_1U_2$, $H \subsetneqq N_G(H)$. This is again a contradiction for $N_G(H)$ is an $\mathrm{Aut}_H G$-invariant subgroup of G. Therefore, there exists $S_1' \in \mathcal{T}(G_1, U_1)$ or $S_2' \in \mathcal{T}(G_2, U_2)$ which is not a subgroup of G. This implies that $S' = S_1'(TS_2') \in \mathcal{T}(G, H)$ is decomposable which is not a subgroup of G. Thus whether $U_2 = G_2$ or $U_2 \neq G_2$, we always get an $S' \in \mathcal{T}(G, H)$ which is not a subgroup of G and is decomposable. But this gives a contradiction as argued in fourth paragraph of the above proof. □

Lemma 3.2. *Let (G, H) be a minimal counterexample. Let N be an* $\mathrm{Aut}_H G$*-invariant proper subgroup of G containing H properly. Then*

(i) $\mathrm{Aut}_H G$ *has at most two orbits in* $\mathcal{T}(N, H)$,
(ii) $H \trianglelefteq N$, *and*
(iii) *there are no* $K_1, K_2 \in \mathcal{T}(N, H)$ *which are in distinct* $\mathrm{Aut}_H G$*-orbits such that* K_1 *is a subgroup of* N *but* K_2 *is not a subgroup of* N.

Proof.

(i) Assume that $\mathrm{Aut}_H G$ has more than two orbits in $\mathcal{T}(N, H)$. Let K_2, K_2 and K_3 be in distinct $\mathrm{Aut}_H G$-orbits in $\mathcal{T}(N, H)$. Fix $L \in \mathcal{T}(G, N)$. Then $K_1 L$, $K_2 L$, $K_3 L \in \mathcal{T}(G, H)$. By Corollary 2.4, there exist S_1, S_2 and S_3 in $\mathcal{T}(G, H)$ containing K_1, K_2 and K_3 respectively such that $S_i \neq K_i L'$, $(1 \leq i \leq 3)$ for any $L' \in \mathcal{T}(G, N)$.

Since there is no $f \in \mathrm{Aut}_H G$ such that $f(K_i) = K_j$ $1 \leq i \neq j \leq 3$ and $\langle S \rangle = G$ for all $S \in \mathcal{T}(G, H)$ (Proposition 2.12), thus S_1, S_2, S_3, $K_1 L$, $K_2 L$ and $K_3 L$ are pairwise non-isomorphic NRTs in $\mathcal{T}(G, H)$, a contradiction.

(ii) Suppose that H is not normal in N. Then by Main Theorem, p.643 of [11] and Theorem, p. 1718 of [6], $|\mathcal{I}(N, H)| > 2$. Let K_1, K_2 and K_3 be pairwise non-isomorphic NRTs in $\mathcal{T}(N, H)$. As argued in (i), we get $|\mathcal{I}(G, H)| > 4$, a contradiction. Thus $H \trianglelefteq N$.

(iii) Let $K_1, K_2 \in \mathcal{T}(N, H)$ be in distinct $\mathrm{Aut}_H G$-orbits. If possible, suppose that K_1 is a subgroup of N but K_2 is not a subgroup of N. Let $L = \{1, l_2, \ldots, l_r\} \in \mathcal{T}(G, N)$. By Lemma 2.8, $|L| \geq 3$. Let $h \in H \setminus \{1\}$. Let $S_1 = K_1 L$, $S_2 = K_2 L$, $S_3 = (S_1 \setminus \{l_r\}) \cup \{hl_r\}$, $S_4 = (S_1 \setminus \{l_{r-1}, l_r\}) \cup \{hl_{r-1}, hl_r\}$. By Corollary 2.8, there exists $S_5 \in \mathcal{T}(G, H)$ containing K_2 such that $S_5 \neq K_2 L'$ for any $L' \in \mathcal{T}(G, N)$. By Proposition 2.12, $\langle S_i \rangle = G$ for all i $(1 \leq i \leq 5)$.

We claim that S_i $(1 \leq i \leq 5)$ are pairwise non-isomorphic NRTs in $\mathcal{T}(G, H)$. Since N is an $\mathrm{Aut}_H G$-invariant subgroup of G, $S_i \ncong S_j$ $(i = 1, 3, 4;\ j = 2, 5)$ (for otherwise by Proposition 2.2, $f(K_1) = K_2$ for some $f \in \mathrm{Aut}_H G$). Assume that $S_1 \cong S_3$. By Proposition 2.2 there exists $f \in \mathrm{Aut}_H G$ such that $f(S_1) = S_3$. As N is an $\mathrm{Aut}_H G$-invariant subgroup of G, $f(K_1) = K_1$. If possible, suppose that $f(l) = hl_r$ for some $l \in L$. Let $k \in K_1 \setminus \{1\}$ and $f(k) = k_1$. Then $f(kl) = k_1(hl_r)$. Since $N = HK_1$ is a subgroup of G and $K_1 \in \mathcal{T}(N, H)$, $k_1 h = h' k_1'$ for some $k_1' \in K_1 \setminus \{1\}$ and $h' \in H \setminus \{1\}$. This implies that $f(kl) = h'(k_1' l_r) \in S_3$. This is a contradiction, for $k_1' l_r \in S_3$. Therefore $f(k' l) = hl_r$ for some $k' \in K_1 \setminus \{1\}$. This implies that $f(l) = f(k')^{-1} hl_r \in S_3$. As argued above, this gives a contradiction. Hence $S_1 \ncong S_3$. Similar arguments prove that $S_1 \ncong S_4$ and $S_3 \ncong S_4$. Now, assume that $S_2 \cong S_5$. By Proposition 2.2, there exists $f \in \mathrm{Aut}_H G$ such that $f(S_2) = S_5$. Since N is $\mathrm{Aut}_H G$-invariant subgroup of G, $f(K_2) = K_2$ and so $S_5 = K_2 f(L)$, a contradiction to the choice of S_5. □

PROPOSITION 3.3

Let (G, H) be a minimal counterexample. Then G has no proper characteristic subgroup U such that $G = UH$.

Proof. If possible, assume that G has a proper characteristic subgroup U such that $G = UH$. Then since $\mathcal{T}(U, U \cap H) \subseteq \mathcal{T}(UH, H) = \mathcal{T}(G, H)$ and $|U| < |G|$, by minimality of the pair (G, H), $|I(U, U \cap H)| \leq 3$. If $|\mathcal{I}(U, U \cap H)| = 3$, then $[G : H] = [U : U \cap H] = 3$ (see Theorem A, p. 2025 of [7]). But in this case $|\mathcal{I}(G, H)| = 3$, a contradiction. Therefore, $|\mathcal{I}(U, U \cap H)| = 1$ (since $|\mathcal{I}(U, U \cap H)| \neq 2$ by Theorem, p. 1718 of [6]). Hence by Main Theorem, p. 643 of [11], $U \cap H \trianglelefteq U$. Since $H \cap U$ is also normal in H, $H \cap U \trianglelefteq UH = G$. Since $\mathrm{Core}_G(H) = \{1\}$ (see Proposition 2.1), $U \cap H = \{1\}$. This implies that $U \in \mathcal{T}(G, H)$. The above arguements show that any proper characteristic subgroup U of G is an NRT of H in G, if $G = UH$. Let $n = [G : H]$. Since $\mathrm{Core}_G(H) = \{1\}$ (see Proposition 2.1), we may identify G with a subgroup of $\mathrm{Sym}(n)$. Since $|\mathcal{I}(G, H)| = 4$, $n \geq 4$.

Now, if possible assume that $n = 4$. By the subgroup structure of Sym(4), we have $(G, H) = (D_8, C_2)$ or $(G, H) = (\mathrm{Alt}(4), \mathrm{Alt}(3))$ or $(G, H) = (\mathrm{Sym}(4), \mathrm{Sym}(3))$. By Lemma 2.5, $(G, H) \neq (D_8, C_2)$. Assume that $(G, H) = (\mathrm{Sym}(4), \mathrm{Sym}(3))$. We may assume that $H = \langle (12), (123) \rangle$. Let $x = (12)$ and $y = (1234)$. Then $S = \{I, xy, y^2, y^3\} \in \mathcal{T}(G, H)$ such that $H_S = \{I, x\}$. Hence $G_S S \cong H_S S \cong D_8$. Since $\mathcal{T}(H_S S, H_S) \subseteq \mathcal{T}(G, H)$, $|\mathcal{I}(G, H)| = 6$ (by Lemma 2.5). Hence $|\mathcal{I}(G, H)| \geq 6$. In fact $|\mathcal{I}(G, H)| = 44$ (see the discussion in the seventh paragraph of the Introduction). This is a contradiction. Hence $(G, H) = (\mathrm{Alt}(4), \mathrm{Alt}(3))$. Let x be a non-trivial element of H. Let $S = \{I, a, b, c\}$ be the unique subgroup of order 4 of G. Let $S_1 = \{I, xa, b, c\}$, $S_2 = \{I, xa, xb, c\}$, $S_3 = \{I, xa, xb, xc\}$ and $S_4 = \{I, xa, x^2b, c\}$. We note that each S_i $(1 \leq i \leq 4)$ generates G. Since S is a characteristic subgroup of G, S, S_i $(1 \leq i \leq 4)$ are pairwise non-isomorphic NRTs of H in G. This is a contradiction. Thus $n \geq 5$.

Again, if possible assume that $n = 5$. Since Sym(5) contains no charcteristic subgroup of order 5, $G \ncong \mathrm{Sym}(5)$. Also since Alt(5) is simple, $G \ncong \mathrm{Alt}(5)$. By Lemma 2.9 A, p. 59 of [4], $|G| \leq 24$. Since G is transitive on 5 symbols, by Table 2.1, p. 60 of [4], $G \cong \mathrm{Aff}(1, 5)$ or $G \cong D_{10}$, where $\mathrm{Aff}(n, q)$ denotes the n-dimensional affine group over the field containing q elements. Assume that $G \cong \mathrm{Aff}(1, 5)$. Then $H \cong C_4$. In this case, G has a characteristic subgroup U' of order 10. Obviously $U' \notin \mathcal{T}(G, H)$. This is a contradiction. Hence, $G \cong D_{10}$. Then, there exist $x, y \in G$ of orders 2 and 5 respectively such that $H = \{1, x\}$ and $U = \{1, y, y^2, y^3, y^4\}$. Let $f \in \mathrm{Aut}(G)$ defined by $f(x^i y^j) = x^i y^{2j}$. Then $\mathrm{Aut}_H(G) = \langle f \rangle \cong C_4$. Let $S_1 = \{1, xy, xy^2, xy^3, xy^4\}$. Then $\{U\}$ and $\{S_1\}$ are $\mathrm{Aut}_H G$-orbits under the action of $\mathrm{Aut}_H G$ on $\mathcal{T}(G, H)$. Observe that each $S \in \mathcal{T}(G, H) \backslash \{U, S_1\}$ generates G and $|\mathcal{T}(G, H) \backslash \{U, S_1\}| = 14$. Since $|\mathrm{Aut}_H G| = 4$, there are atleast 4 $\mathrm{Aut}_H G$-orbits other than $\{U\}$ and $\{S_1\}$. Thus $|\mathcal{I}(G, H)| \geq 6$. This is again a contradiction. Hence, $[G : H] \geq 6$.

Let $h \in H \setminus \{1\}$. Let u_2, u_3, u_4, u_5 and u_6 be distinct non-trivial elements of U such that $u_2 u_3 = u_4$. Let $S_1 = (U \setminus \{u_2\}) \cup \{hu_2\}$, $S_2 = (U \setminus \{u_2, u_5\}) \cup \{hu_2, hu_5\}$, $S_3 = (U \setminus \{u_2, u_5, u_6\}) \cup \{hu_2, hu_5, hu_6\}$ and $S_4 = \{1\} \cup \{hu : u \in (U \setminus \{1\})\}$. Since $hu_2, u_3 \in S_i$ and $hu_2 u_3 \notin S_i$ $(i = 1, 2, 3)$, S_i is not a subgroup. Also S_4 is not a subgroup, for $hu_2, hu_3 \in S_4$ but $hu_2(hu_3)^{-1} = hu_2 u_3^{-1} h^{-1} \in U$. Therefore, by Proposition 2.1, $\langle S_i \rangle = G$ $(1 \leq i \leq 4)$. Hence $U \ncong S_i$ $(1 \leq i \leq 4)$. Since H and U are $\mathrm{Aut}_H G$-invariant subgroups of G, S_i $(1 \leq i \leq 4)$ are pairwise non-isomorphic NRTs of H in G. Therefore, $|\mathcal{I}(G, H)| > 4$. This is a contradiction. □

PROPOSITION 3.4

Let (G, H) be a minimal counterexample. Then G is characteristically simple.

Proof. If possible, assume that U is a nontrivial proper characteristic subgroup of G. By Proposition 3.3, $G \neq UH$. Let $N = UH$. Then $H \subsetneqq N \subsetneqq G$ is an $\mathrm{Aut}_H G$-invariant subgroup.

Assume that $H \nsubseteq U$. Then $\mathcal{T}(U, U \cap H) \subsetneqq \mathcal{T}(UH, H)$. Since U and H are $\mathrm{Aut}_H G$-invariant, by Lemma 3.2(i), $\mathcal{T}(U, U \cap H)$ and $\mathcal{T}(UH, H) \setminus \mathcal{T}(U, U \cap H)$ are $\mathrm{Aut}_H G$-orbits. Let $T_1 \in \mathcal{T}(U, U \cap H)$ and $T_2 \in \mathcal{T}(UH, H) \setminus \mathcal{T}(U, U \cap H)$. By Lemma 3.2(iii), either both T_1 and T_2 are subgroups of G or both are not subgroups of G.

Assume that both T_1 and T_2 are subgroups of G. Thus each member of $\mathcal{T}(UH, H)$ is a subgroup of UH. Hence by Lemma 2.4, p. 1719 of [6], $[UH : H] = 2$. By Lemma 2.11, $[G : UH] = 2$ or $[G : UH] = 3$.

Assume that $[G : UH] = 2$. Then $[G : H] = 4$. As $\mathrm{Core}_G(H) = \{1\}$ (Proposition 2.1(i)), G is isomorphic to a subgroup of Sym(4). By Lemma 2.9(i), $G \cong D_8$ and $|H| = 2$. But in this case $|\mathcal{I}(G, H)| = 6$ (Lemma 2.5), a contradiction.

Thus $[G : UH] = 3$. Then by Lemma 2.9(ii), $G \cong \mathrm{Alt}(4)$ and $|H| = 2$ or $G \cong \mathrm{Alt}(4) \times C_2$ and $H \cong C_2 \times C_2$. But for both choices $|\mathcal{I}(G, H)| > 4$ (Lemmas 2.7 and 2.10), a contradiction.

Thus T_1 and T_2 are not subgroups of G. Further, assume that $[U : U \cap H] > 2$. Let $h \in H \setminus U$. Let $u_2, u_3 \in T_1$ be distinct nontrivial elements. Let $T_2' = (T_1 \setminus \{u_2\}) \cup \{hu_2\}$ and $T_3 = (T_1 \setminus \{u_2, u_3\}) \cup \{hu_2, hu_3\}$. Then T_2' and T_3 are in the same $\mathrm{Aut}_H G$-orbit. This is a contradiction for H and U are $\mathrm{Aut}_H G$-invariant and $h \notin U$. Thus $[U : U \cap H] = 2$. If possible, assume that $[G : HU] = 2$. Then G will be isomorphic to a subgroup of Sym(4). The only possibility we have in this case is $G \cong D_8$ and H is a non-normal subgroup of order 2. But, then $|\mathcal{I}(G, H)| = 6$ (Lemma 2.5). This is a contradiction. Thus $[G : HU] > 2$.

Let $h \in H \setminus U$. Let $T = \{1, u\} \in \mathcal{T}(U, U \cap H)$ and $L = \{1, l_2, \ldots, l_{r-1}, l_r\} \in \mathcal{T}(G, HU)$. Then $T_1 = \{1, hu\} \in \mathcal{T}(UH, H) \setminus \mathcal{T}(U, U \cap H)$. Consider $S_1 = TL$, $S_2 = T_1 L$, $S_3 = (S_2 \setminus (L \setminus \{1\})) \cup \{hl | l \in L \setminus \{1\}\}$, $S_4 = (S_1 \setminus \{l_r\}) \cup \{hl_r\}$ and $S_5 = (S_1 \setminus \{l_{r-1}, l_r\}) \cup \{hl_{r-1}, hl_r\}$. We claim that $S_i \neq T'L'$ $(3 \leq i \leq 5)$ for any $T' \in \mathcal{T}(UH, H)$ and $L' \in \mathcal{T}(G, UH)$. If possible, suppose that $S_3 = T'L'$ for some $T' \in \mathcal{T}(UH, H)$ and $L' \in \mathcal{T}(G, UH)$. Then $T_1 = S_3 \cap UH = T'$. Since $hl_r, hul_r \in S_3$ are in the same right coset of HU in G, therefore $hl_r \in L'$ or $hul_r \in L'$. Assume that $hl_r \in L'$. Then $huhl_r \in S_3$. Since HU is a subgroup of G and $[U : U \cap H] = 2$, $uh = h'u$, for some $h' \in H \setminus U$. This implies that $huhl_r = hh'ul_r \in S_3$, a contradiction. Thus $hul_r \in L'$. Then $hu(hul_r) \in S_3$, a contradiction (for $huhu \neq h$ and $huhu \neq hu$).

Now, assume that $S_4 = T'L'$ for some $T' \in \mathcal{T}(UH, H)$ and $L' \in \mathcal{T}(G, UH)$. Then $T = S_4 \cap UH = T'$. Since $hl_r, ul_r \in S_4$ are in the same right coset of HU in G, therefore $hl_r \in L'$ or $ul_r \in L'$. Assume that $hl_r \in L'$. Then $u(hl_r) \in S_4$. As argued in the above paragraph, $u(hl_r) = h'ul_r \in S_4$ for some $h' \in H \setminus U$, a contradiction. Thus $ul_r \in L'$. Then $u^2 l_r \in S_4$, which is again a contradiction (for $[U : U \cap H] = 2$, $u^2 \in U \cap H$). Similarly, $S_5 \neq T'L'$ for any $T' \in \mathcal{T}(UH, H)$ and $L' \in \mathcal{T}(G, UH)$.

We now claim that S_i $(1 \leq i \leq 5)$ are pairwise non-isomorphic NRTs in $\mathcal{T}(G, H)$. Since $S_k \neq T'L'$ $(3 \leq k \leq 5)$ for any $T' \in \mathcal{T}(UH, H)$ and $L' \in \mathcal{T}(G, UH)$, $S_j \ncong S_k$ for $1 \leq j \leq 2$, $3 \leq k \leq 5$. Assume that $S_1 \cong S_2$. By Proposition 2.2, there exists $f \in \mathrm{Aut}_H G$ such that $f(S_1) = S_2$. Since U and H are $\mathrm{Aut}_H G$-invariant subgroups of G, $hu = f(u) \in U$, a contradiction. Thus, $S_1 \ncong S_2$. Similarly, $S_3 \ncong S_4$ and $S_3 \ncong S_5$.

Next, assume that $S_4 \cong S_5$. Then there exists $f \in \mathrm{Aut}_H G$ such that $f(S_4) = S_5$. Since U is an $\mathrm{Aut}_H G$-invariant subgroup of G, $f(u) = u$. Now there exist $i \in \{0, 1\}$ and $k \in \{2, \ldots, r-1\}$ such that $f(u^i l_k) = hl_j$ for some $j \in \{r-1, r\}$. Assume that $i = 0$. Then as argued earlier, there exists $h' \in H \setminus U$ such that $f(ul_k) = uhl_j = h'ul_j \in S_5$, a contradiction. Therefore, $f(ul_k) = hl_j$ for some $j \in \{r-1, r\}$. Then again there exists $h' \in H \setminus U$ such that $f(l_k) = u^{-1}hl_j = h'ul_j \in S_5$, a contradiction. Hence $S_4 \ncong S_5$. Thus each nontrivial characteristic subgroup U contains H.

Let N be the smallest characteristic subgroup of G containing H. Then N is a direct product of isomorphic simple groups (3.3.15 of [12]). Thus there exists $K \in \mathcal{T}(N, H)$ which is a subgroup of N. By Lemma 3.2(i), (iii) all $K \in \mathcal{T}(N, H)$ are subgroups of N. Thus by Lemma 2.4, p. 1719 of [6], $[N : H] = 2$. Hence N is an elementary abelian 2-subgroup of G containing H. By Lemma 2.11, $[G : N] = 2$ or $[G : N] = 3$. Now, as argued in the third and fourth paragraphs, we get a contradiction. □

COROLLARY 3.5

Let (G, H) be a minimal counterexample. Then G is simple.

Proof. Since G is indecomposable (Proposition 3.1) and characteristically simple (Proposition 3.4), by 3.3.15 of [12], G is a simple group. □

PROPOSITION 3.6

Let (G, H) be a minimal counterexample. Let $S \in \mathcal{T}(G, H)$ be such that $H_S = H$. Let $\mathcal{A} = \{L \in \mathcal{T}(G, H) | L \cong S\}$. Then $|\mathcal{A}| < \frac{m^{n-1}}{8}$, where m and n are the order and the index of H in G respectively.

Proof. Suppose that $|\mathcal{A}| \geq \frac{m^{n-1}}{8}$. By Corollary 3.5, G is a non-abelian simple group. Since $\langle S \rangle = G$, by Proposition 2.1(iii), $G \cong G_S S \subseteq \mathrm{Sym}(S \setminus \{1\})S$ and hence $|G| \leq n!$. Since a non-abelian simple group has order at least 60, $n \geq 5$. By Proposition 2.2, $\mathrm{Aut}_H G$ acts transitively on $\mathcal{A}$. So,

$$|\mathrm{Aut}(G)| \geq |\mathrm{Aut}_H G| \geq |\mathcal{A}| \geq \frac{m^{n-1}}{8}.$$

We show that $\frac{m^{n-1}}{8} > m^2 n^2 = |G|^2$. If $n \in \{5, 6, 7, \ldots, 13\}$, then the fact that $|G| \geq 60$ implies that $\frac{m^{n-1}}{8} > m^2 n^2 = |G|^2$. Suppose that $n \geq 14$. For showing $\frac{m^{n-1}}{8} > m^2 n^2$ or $\frac{m^{n-3}}{8} > n^2$, it is sufficient to show that $\frac{2^{n-3}}{8} > n^2$ or $2^{n-6} > n^2$, that is $2^{\frac{n-6}{2}} > n$. Now, one can prove it easily by using the induction. Thus $|\mathrm{Aut}(G)| > |G|^2$. This is a contradiction to Lemma 3.4, p. 643 of [11]. This proves the proposition. □

Proof of the Main Theorem. Let $\mathcal{B} = \{S \in \mathcal{T}(G, H) | H_S = \{1\}\}$ and $\mathcal{B}' = \{L \in \mathcal{T}(G, H) | H_L = H\}$. Note that $\mathcal{T}(G, H) = \mathcal{B}' \cup \mathcal{B}$ and $\mathcal{B}' \cap \mathcal{B} = \emptyset$. We will know that $|\mathcal{B}| < \frac{3}{8} m^{n-1}$.

If $\mathcal{B} = \emptyset$, there is nothing to prove. So assume that $\mathcal{B} \neq \emptyset$. Observe that no member of $\mathcal{B}$ is isomorphic to a member of $\mathcal{B}'$. Let $S \in \mathcal{B}$. Since $H_S = \{1\}$, S is a subgroup of G (Proposition 2.1). Further, since $\mathrm{Core}_G(H) = \{1\}$ and $|\mathcal{I}(G, H)| = 4$, $|S| \geq 4$. Let x_1, x_2 and x_3 be distinct non-trivial elements of S such that $x_3 = x_1 x_2$. Let $h \in H \setminus \{1\}$. Let $S' = S \setminus \{x_3\} \cup \{hx_3\}$. Then $H_{S'} \neq \{1\}$. Hence $\langle S' \rangle = G$ (Proposition 2.1(iii)) and so $S' \in \mathcal{B}'$. By a similar argument as in the second paragraph of the proof of Proposition 3.5, p. 2035 of [7], $S \mapsto S'$ defines an injective map from $\mathcal{B}$ to $\mathcal{B}'$.

Let $\mathcal{A}_i$ $(1 \leq i \leq 4)$ be isomorphism classes in $\mathcal{T}(G, H)$. By Proposition 2.1, $\mathcal{B}' \neq \emptyset$. Now, $m^{n-1} = |\mathcal{T}(G, H)| = |\mathcal{B}'| + |\mathcal{B}| \leq 2|\mathcal{B}'| < 2(\frac{4}{8} m^{n-1})$. This is a contradiction. This completes the proof of the theorem. □

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