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A note on a kind of character sums over the short interval

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Abstract. Let p be a prime, χ denote the Dirichlet character modulo p and $L(p) = \{a \in \mathbb{Z}^+ | (a, p) = 1, a\bar{a} \equiv 1 (\bmod p), |a - \bar{a}| \leq H\}$. We study the distribution of elements in the set $L(p)$ in character over the short interval. In this paper, we use the analytic method and show the distribution property of

$$\sum_{\substack{n\leq N\\ n\in L(p)}} \chi(n),$$

and give a non-trivial estimate.

Keywords. Character sum; short interval; inverse of integers; estimate.

1. Introduction

Let $q \geq 3$ be an integer and χ the Dirichlet character modulo q. For any real number $x \geq 1$,

$$\sum_{n\leq x} \chi(n).$$

Perhaps one of the most famous results is Pólya's inequality [8]. That is when χ is the primitive character modulo q,

$$\sum_{n\leq x} \chi(n) < q^{\frac{1}{2}} \log q.$$

In fact, the result can be extended to the nonprincipal character χ modulo q [4]. Then Burgess [2,3] improved the result of [8] as follows:

$$\sum_{n\leq x} \chi(n) \ll H^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+o(1)},$$

where H is any positive integer, $r = 1, 2, 3$ for any integer $q \geq 3$ and arbitrary positive integer r if q is cube-free.

For any fixed integer $H > 0$, define the set

$$L(q) = \{a \in Z^+ | (a, q) = 1, a\bar{a} \equiv 1 (\bmod\ q), \quad |a - \bar{a}| \leq H\},$$

for any positive integer $q \geq 3$, χ the Dirichlet character modulo q, define the sums as follows:

$$\sum_{\substack{n \leq q \\ n \in L(q)}} \chi(n).$$

In fact, when χ is the principal character modulo q, Zhang [11] obtained

$$\sum_{\substack{n \leq q \\ n \in L(q)}}{}' 1 = \frac{H}{q}\left(2 - \frac{H}{q}\right)\phi(q) + O(q^{\frac{1}{2}} d^2(q) \log^3 q),$$

where $0 < H \leq q$ is a constant, $\phi(q)$ is the Euler function and $d(q)$ is the divisor function, $\sum'$ denotes the summation over the integers which are coprime to q. For further results, we refer to [10].

In this paper, we generalize the sums and find some internal rules and relationships of the Dirichlet character between an integer and its inverse over the short interval. That is, *for* $p \geq 3$ *is a prime*, χ denotes the non-principal Dirichlet character modulo p, for any positive integer $N < p$, we study the sums of the elements in the set $L(p)$ with character over the short interval,

$$\sum_{\substack{n \leq N \\ n \in L(p)}} \chi(n).$$

More precisely, we prove the following:

Theorem. *Let $p \geq 3$ be a prime and χ denote the non-principal Dirichlet character modulo p. We know $\bar{a}$ is the integer inverse such that $a\bar{a} \equiv 1 (\bmod\ p)$. Let $L(p) = \{1 \leq a \leq p | (a, p) = 1, |a - \bar{a}| \leq H\}$ be a set in which the integer and the integer inverse satisfy certain conditions ($3 \leq H \leq p$ is any fixed integer). Then for any prime $p \geq 3$ and $3 \leq H \leq p$, we have the estimate*

$$\sum_{\substack{n \leq N \\ n \in L(p)}} \chi(n) \ll p^{\frac{1}{2}+\epsilon},$$

where ϵ is any positive real number.

Remark. This is a non-trivial estimate for $N \gg p^{\frac{1}{2}}$. However, for any integer $q \geq 3$, whether there is the estimate of

$$\sum_{\substack{n \leq N \\ n \in L(q)}} \chi(n)$$

is an open problem.

2. Some lemmas

To complete the proof of the above theorem, we need the following lemmas. First, we quote a familiar result on a kind of exponential sums.

Lemma 1. *For any integer* $K \geq 1$ *and* $0 < \alpha < 1$, *we have*

$$\left|\sum_{n=1}^{K} e(\alpha n)\right| \leq \min\left(K, \frac{1}{2\langle\alpha\rangle}\right),$$

where $\langle\alpha\rangle = \min(\{\alpha\}, 1 - \{\alpha\})$, $\{\alpha\}$ *is the decimal part of* α.

Proof. See ref. [1].

Lemma 2. *Let* $q \geq 3$ *be an integer, and* χ *denote the Dirichlet character. The generalized Kloosterman sum is defined by*

$$S(m, n, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma + n\bar{a}}{q}\right),$$

where a *and* $\bar{a}$ *are such that the congruence equation* $a\bar{a} \equiv 1 (\bmod\ q)$. *Then for any positive integers* m *and* n, *we have the estimate*

$$S(m, n, \chi; q) \ll (m, n, q)^{\frac{1}{2}} q^{\frac{1}{2}} d(q),$$

where (m, n, q) *is the greatest common divisor of* m, n *and* q, $d(q)$ *is the divisor function.*

Proof. See ref. [6,9].

Lemma 3. *Let* $q \geq 3$ *be an integer, and* χ *denote the Dirichlet character. The Gauss sum is defined by*

$$G(n, \chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{na}{q}\right).$$

Then by the principal Dirichlet character χ_0 *modulo* q, *we have the identity*

$$G(n, \chi_0) = \mu\left(\frac{q}{(n, q)}\right) \phi(q) \phi^{-1}\left(\frac{q}{(n, q)}\right),$$

where $\mu(n)$ *is the Möbius function,* $\phi(q)$ *is Euler function, and* (n, q) *is the greatest common divisor of* n *and* q.

Proof. See ref. [7].

Lemma 4. *For* $p \geq 3$ *a prime, any non-principal Dirichlet character* χ *modulo* p *and any positive integers* b *and* l, *we have the estimate*

$$\sum_{\chi' \bmod p} G(b, \chi') G(l, \chi'\chi) \ll (b, l, p)^{\frac{1}{2}} p^{\frac{3}{2}+\epsilon}, \tag{1}$$

where the definition of (b, l, p) is the same as in Lemma 2, ϵ is any fixed positive real number.

On the other hand, for the principal Dirichlet character χ_0, any non-principal Dirichlet character χ and any positive integers b and l, we also have

$$G(b, \chi_0)G(l, \chi) \ll (b, p)p^{1+\epsilon}. \tag{2}$$

Proof. Firstly we prove (1). According to the orthogonality of character sums, we have

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \phi(q), & n \equiv 1(\bmod q) \\ 0, & n \not\equiv 1(\bmod q), \end{cases}$$

and hence we have

$$\begin{aligned}
&\sum_{\chi' \bmod p} G(b, \chi')G(l, \chi'\chi) \\
&\quad = \sum_{\chi' \bmod p} \sum_{s=1}^{p} \chi'(s) e\left(\frac{bs}{p}\right) \sum_{t=1}^{p} \chi'\chi(t) e\left(\frac{lt}{p}\right) \\
&\quad = \sum_{s=1}^{p}{}' \sum_{t=1}^{p}{}' \chi(t) e\left(\frac{bs+lt}{p}\right) \sum_{\chi' \bmod p} \chi'(s)\chi'(t) \\
&\quad = \phi(p) \sum_{\substack{s=1 \\ st\equiv 1(\bmod p)}}^{p-1} \sum_{t=1}^{p-1} \chi(t) e\left(\frac{bs+lt}{p}\right) \\
&\quad = (p-1) \sum_{t=1}^{p-1}{}' \chi(t) e\left(\frac{b\bar{t}+lt}{p}\right) \\
&\quad \ll (b, l, p)^{\frac{1}{2}} p^{\frac{3}{2}+\epsilon},
\end{aligned} \tag{3}$$

where we have used Lemma 2 for the last step.

Now we show (2). From Lemma 3 and the definition of Gauss sum, we have

$$\begin{aligned}
G(b, \chi_0)G(l, \chi) &= \mu\left(\frac{p}{(b, p)}\right)\phi(p)\phi^{-1}\left(\frac{p}{(b, p)}\right) \sum_{t=1}^{p} \chi(t) e\left(\frac{lt}{p}\right) \\
&\ll \phi(p)\frac{(b, p)d(p)}{p} p \\
&\ll (b, p)p^{1+\epsilon},
\end{aligned} \tag{4}$$

where we have used $\phi(q) \gg \frac{q}{d(q)}$ (see ref. [5]), and $d(q)$ is the divisor function. Combining (3) and (4), we can get Lemma 4.

Lemma 5. *For any integer $q \geq 3$, $\frac{q}{2} \leq N < q$, and any positive integer $a \geq 1$, we have*

$$\sum_{l=1}^{q-1} \left| \sum_{n \leq N} e\left(\frac{n(a-l)}{q}\right) \right| \ll q \log q.$$

Proof. From Lemma 1, when $a \not\equiv 0 (\mathrm{mod}\ q)$, for $1 \leq l \leq q-1$, there must be one and only one l such that $a - l \equiv 0 (\mathrm{mod}\ q)$, so we get

$$\begin{aligned}
&\sum_{l=1}^{q-1} \left| \sum_{n \leq N} e\left(\frac{n(a-l)}{q}\right) \right| \\
&= \sum_{\substack{l=1 \\ l \not\equiv a (\mathrm{mod}\ q)}}^{q-1} \left| \sum_{n \leq N} e\left(\frac{n(a-l)}{q}\right) \right| + \left| \sum_{n \leq N} 1 \right| \\
&\leq \sum_{\substack{l=1 \\ l \not\equiv a (\mathrm{mod}\ q)}}^{q-1} \left| \min\left\{ N, \frac{1}{2 < \frac{a-l}{q} >} \right\} \right| + N \\
&\leq \sum_{\substack{l=1 \\ l \not\equiv a (\mathrm{mod}\ q)}}^{q-1} \left| \frac{1}{2 < \frac{a-l}{q} >} \right| + N \\
&= \sum_{l=1}^{q-1} \left| \frac{1}{2 < \frac{l}{q} >} \right| - \left| \frac{1}{2 < \frac{a}{q} >} \right| + N \\
&\leq 2 \sum_{1 \leq l \leq \left[\frac{q}{2}\right]} \frac{1}{\frac{2l}{q}} - \left| \frac{1}{2 < \frac{a}{q} >} \right| + N \\
&= q \sum_{1 \leq l \leq \left[\frac{q}{2}\right]} \frac{1}{l} - \left| \frac{1}{2 < \frac{a}{q} >} \right| + N \\
&\ll q \log q, \qquad (5)
\end{aligned}$$

when $a \equiv 0 (\mathrm{mod}\ q)$. From Lemma 1, we also get

$$\begin{aligned}
&\sum_{l=1}^{q-1} \left| \sum_{n \leq N} e\left(\frac{n(a-l)}{q}\right) \right| \\
&= \sum_{l=1}^{q-1} \left| \sum_{n \leq N} e\left(\frac{nl}{q}\right) \right| \\
&\leq \sum_{l=1}^{q-1} \left| \min\left\{ N, \frac{1}{2 < \frac{l}{q} >} \right\} \right| \\
&\ll q \log q. \qquad (6)
\end{aligned}$$

Therefore, combining (5) and (6), for any positive integer $a \geq 1$, we have the estimate

$$\sum_{l=1}^{q-1} \left| \sum_{n \leq N} e\left(\frac{n(a-l)}{q}\right) \right| \ll q \log q.$$

This proves Lemma 5.

Lemma 6. *Let $p \geq 3$ be a prime, and χ denote the Dirichlet character modulo p, for any positive integers a, b and $\frac{q}{2} \leq N < q$. We define*

$$S(a, b, N, \chi; q) = \sum_{n \leq N} \chi(n) e\left(\frac{an + b\bar{n}}{q}\right),$$

where $\bar{n}$ is the integer inverse of n. Then for any positive integers a, b and $(b, p) = 1$, we have

$$S(a, b, N, \chi; p) \leq p^{\frac{1}{2}+\epsilon}.$$

Proof. From the definition of $S(a, b, N, \chi; p)$ and the orthogonality of character sums, we have

$$\begin{aligned}
&S(a, b, N, \chi; p) \\
&\quad = \sum_{\substack{m=1 \\ mn \equiv 1 (\text{mod } p)}}^{p} \sum_{n \leq N} \chi(n) e\left(\frac{an + bm}{p}\right) \\
&\quad = \frac{1}{\phi(p)} \sum_{m=1}^{p} \sum_{n \leq N} \chi(n) e\left(\frac{an + bm}{p}\right) \sum_{\chi' \text{ mod } p} \chi'(m) \chi'(n) \\
&\quad = \frac{1}{p-1} \sum_{\chi' \text{ mod } p} \left(\sum_{m=1}^{p} \chi'(m) e\left(\frac{bm}{p}\right) \right) \left(\sum_{n \leq N} \chi' \chi(n) e\left(\frac{an}{p}\right) \right) \\
&\quad = \frac{1}{p-1} \sum_{\chi' \neq \chi_0} \left(\sum_{m=1}^{p} \chi'(m) e\left(\frac{bm}{p}\right) \right) \left(\sum_{n \leq N} \chi' \chi(n) e\left(\frac{an}{p}\right) \right) \\
&\qquad + \frac{1}{p-1} \left(\sum_{m=1}^{p-1} e\left(\frac{bm}{p}\right) \right) \left(\sum_{n \leq N} \chi(n) e\left(\frac{an}{p}\right) \right) \\
&\quad = S_1 + S_2.
\end{aligned} \tag{7}$$

Now we will estimate both S_1 and S_2 respectively. Firstly, we shall estimate S_1. From the identity, for any Dirichlet character $\chi \neq \chi_0$ modulo q,

$$\chi(a) = \frac{1}{q} \sum_{k=1}^{q-1} G(k, \chi) e\left(-\frac{ak}{q}\right).$$

Hence according to Lemmas 4 and 5, we have

$$\begin{aligned}
S_1 &= \frac{1}{p-1}\sum_{\chi'\neq\chi_0}\left(\sum_{m=1}^{p}\frac{1}{p}\sum_{k=1}^{p-1}G(k,\chi')e\left(-\frac{mk}{p}\right)e\left(\frac{bm}{p}\right)\right)\\
&\quad\times\left(\sum_{n\le N}\frac{1}{p}\sum_{l=1}^{p-1}G(l,\chi'\chi)e\left(-\frac{nl}{p}\right)e\left(\frac{an}{p}\right)\right)\\
&= \frac{1}{p^2(p-1)}\sum_{\chi'\neq\chi_0}\left(\sum_{k=1}^{p-1}G(k,\chi')\sum_{m=1}^{p}e\left(\frac{m(b-k)}{p}\right)\right)\\
&\quad\times\left(\sum_{l=1}^{p-1}G(l,\chi'\chi)\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\right)\\
&= \frac{p}{p^2(p-1)}\sum_{\chi'\neq\chi_0}\sum_{\substack{k=1\\ p\mid(b-k)}}^{p-1}G(k,\chi')\sum_{l=1}^{p-1}G(l,\chi'\chi)\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\\
&= \frac{1}{p(p-1)}\sum_{\chi'\neq\chi_0}G(b,\chi')\sum_{l=1}^{p-1}G(l,\chi'\chi)\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\\
&= \frac{1}{p(p-1)}\sum_{l=1}^{p-1}\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\sum_{\chi'\neq\chi_0}G(b,\chi')G(l,\chi'\chi)\\
&= \frac{1}{p(p-1)}\sum_{l=1}^{p-1}\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\sum_{\chi' \bmod p}G(b,\chi')G(l,\chi'\chi)+\\
&\quad+\frac{1}{p(p-1)}\sum_{l=1}^{p-1}\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)G(b,\chi_0)G(l,\chi)\\
&\le \frac{1}{p(p-1)}\sum_{l=1}^{p-1}\left|\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\right|\left|\sum_{\chi' \bmod p}G(b,\chi')G(l,\chi'\chi)\right|+\\
&\quad+\frac{1}{p(p-1)}\sum_{l=1}^{p-1}\left|\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\right|\left|G(b,\chi_0)G(l,\chi)\right|\\
&\le \frac{1}{p(p-1)}\sum_{l=1}^{p-1}\left|\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\right|(b,l,p)^{\frac{1}{2}}p^{\frac{3}{2}+\epsilon}\\
&\quad+\frac{1}{p(p-1)}\sum_{l=1}^{p-1}\left|\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\right|(b,p)p^{1+\epsilon}\\
&\ll p^{-\frac{1}{2}+\epsilon}\sum_{l=1}^{p-1}\left|\sum_{n\le N}e\left(\frac{n(a-l)}{p}\right)\right|\\
&\ll p^{\frac{1}{2}+\epsilon}. \qquad (8)
\end{aligned}$$

Now we estimate S_2. For $(b, p) = 1$, from the identity

$$\sum_{m=1}^{p} e\left(\frac{bm}{p}\right) = 0,$$

we have

$$\begin{aligned} |S_2| &= \left| \frac{1}{p-1} \left(\sum_{m=1}^{p-1} e\left(\frac{bm}{p}\right) \right) \left(\sum_{n\le N} \chi(n) e\left(\frac{an}{p}\right) \right) \right| \\ &= \left| \frac{1}{p-1} \left(\sum_{m=1}^{p} e\left(\frac{bm}{p}\right) - 1 \right) \left(\sum_{n\le N} \chi(n) e\left(\frac{an}{p}\right) \right) \right| \\ &\le \frac{1}{p-1} \left| \sum_{n\le N} \chi(n) e\left(\frac{an}{p}\right) \right| \\ &\ll \frac{N}{p} \\ &\ll 1. \end{aligned} \tag{9}$$

Therefore, from eqs (7), (8) and (9), we have

$$S(a, b, N, \chi; p) \ll p^{\frac{1}{2}+\epsilon},$$

where ϵ is any positive real number.

3. Proof of the theorem

In this section, we shall complete the proof of the theorem. We note that p is a prime number and hence $(m, p) = 1$ for all integers m with $1 \le m \le p-1$. From Lemmas 1 and 6, we have

$$\begin{aligned} \sum_{\substack{n\le N \\ n\in L(p)}} \chi(n) &= \sum_{\substack{n\le N \\ |n-\bar{n}|\le H}} \chi(n) = \sum_{-H\le t\le H} \sum_{\substack{n\le N \\ n-\bar{n}\equiv t (\bmod p)}} \chi(n) \\ &= \frac{1}{p} \sum_{m\le p} \sum_{-H\le t\le H} e\left(\frac{-mt}{p}\right) \sum_{n\le N} \chi(n) e\left(\frac{m(n-\bar{n})}{p}\right) \\ &= \frac{1}{p} \sum_{-H\le t\le H} \sum_{n\le N} \chi(n) + \frac{1}{p} \sum_{m\le p-1} \sum_{-H\le t\le H} e\left(\frac{-mt}{p}\right) \\ &\quad \times \sum_{n\le N} \chi(n) e\left(\frac{m(n-\bar{n})}{p}\right) \end{aligned}$$

$$
\begin{aligned}
&\ll Hp^{-\frac{1}{2}}\log p + \frac{1}{p}\sum_{m\le p-1}\left|\sum_{0\le t\le H} e\left(\frac{-mt}{p}\right) + \sum_{1\le t\le H} e\left(\frac{mt}{p}\right)\right| \\
&\quad \times \left|\sum_{n\le N}\chi(n)e\left(\frac{m(n-\bar{n})}{p}\right)\right| \\
&\ll p^{\frac{1}{2}}\log p + \frac{1}{p}\sum_{m\le p-1}\min\left(H, \frac{1}{2||\frac{m}{p}||}\right)p^{\frac{1}{2}+\epsilon} \\
&\ll p^{\frac{1}{2}}\log p + p^{-\frac{1}{2}+\epsilon}\left(\sum_{m\le\frac{p}{2H}} H + \sum_{\frac{p}{2H}<m\le p-1}\frac{p}{2m}\right) \\
&\ll p^{\frac{1}{2}}\log p + p^{-\frac{1}{2}+\epsilon}\left(\frac{p}{2} + \frac{p}{2}\log 2H\right) \\
&\ll p^{\frac{1}{2}+\epsilon},
\end{aligned}
$$

where we have used the familiar estimate $\sum_{n\le N}\chi(n) \ll p^{\frac{1}{2}}\log p$ in the first inequality and $3 \le H \le p$ in the second inequality. This completes the proof of the theorem.

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