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Asymptotic distribution of products of sums of independent random variables

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Abstract. In the paper we consider the asymptotic distribution of products of weighted sums of independent random variables.

Keywords. Asymptotic distribution; products of sums.

1. Introduction

Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed (i.i.d.) positive random variables. It is well known that the products of partial sums of i.i.d. positive, square integrable random variables (r.v.) are asymptotically log-normal. This fact is an immediate consequence of the classical central limit theorem (CLT). This point, according to the authors, was first argued by Arnold and Villaseñor [1], who considered the limiting properties of the sum of records. They obtained the following version of the CLT for a sequence of i.i.d. exponential r.v.'s $(X_n)_{n\geq 1}$ with the mean equal to one:

$$\frac{\sum_{k=1}^{n} \log S_k - n \log n + n}{\sqrt{2n}} \xrightarrow{d} \Phi, \qquad \text{as } n \to \infty,$$

where $S_k = \sum_{j=1}^{k} X_j$, $1 \leq k \leq n$, and Φ is a standard normal r.v. Rempała and Wesołowski [12] have noted that this limit behavior of a product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following.

Theorem RW. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu$, $\operatorname{Var} X_1 = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$. Then*

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{d} \mathrm{e}^{\sqrt{2}\Phi}.$$

Recently, Gonchigdanzan and Rempała [3] discussed an almost sure limit theorem for the product of the partial sums of i.i.d. positive random variables as follows.

Theorem GR. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu > 0$, $\operatorname{Var} X_1 = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \leq x\right) = F(x), \quad a.s.,$$

where $F(\cdot)$ is the distribution function of the r.v. $\mathrm{e}^{\sqrt{2}\Phi}$.

For further discussions of the CLT, the authors refer to [5,8,11]. Zhang and Huang [13] obtained the invariance principle of the product of sums of random variables. It is perhaps worthy to notice that by the strong law of large numbers and the property of the geometric mean it follows directly that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!}\right)^{1/n} \xrightarrow{\text{a.s.}} \mu \tag{1.1}$$

if only existence of the first moment is assumed. Very recently, Miao [6,7] obtained CLT and ASCLT for the product of some general partial sums. Miao and Qian [10], Miao and Mu [9] got the moderate deviation of product of partial sums.

In the present paper we are interested in the asymptotic distribution of products of weighted sums of independent random variables, which is a simple and interesting model.

2. Main results

Theorem 2.1. *Let $(X_{k,n})_{1\leq k\leq n,\, n\geq 1}$ be a triangular array of i.i.d. positive random variables with identical expectation $\mu = \mathbb{E}(X_{1,1}) > 0$ and variance $\sigma^2 = \operatorname{Var}(X_{1,1})$. Denote the coefficient of variation $\gamma = \sigma/\mu$ and $S_{k,n} = a_{1,n}X_{1,n} + a_{2,n}X_{2,n} + \cdots + a_{k,n}X_{k,n}$ for all $1 \leq k \leq n$ where $\{a_{k,n}\}_{1\leq k\leq n, n\geq 1}$ is a triangular array of positive real numbers with $\sum_{k=1}^{n} a_{k,n} = 1$ for all $n \geq 1$. Assume that $\mathbb{E}|X_{1,1}|^{2+\delta} < \infty$ for some $0 < \delta < 1$ and*

$$\frac{S_{n,n}}{\mu} - 1 \to 0 \;\; a.e. \tag{2.1}$$

Denote $A_n = \sum_{k=1}^{n} \sum_{i=1}^{k} a_{i,k}^2$ and assume that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{i,k}^2\right)^{1+\delta/2}}{A_n^{\delta/2}} = 0. \tag{2.2}$$

Then we have

$$\left(\mathrm{e}^{\frac{\gamma^2}{2}A_n} \prod_{k=1}^{n} (S_{k,k}/\mu)\right)^{\frac{1}{\gamma\sqrt{A_n}}} \xrightarrow{d} \mathrm{e}^{\Phi},$$

where Φ is a standard normal random variable.

Remark 2.2. Since $\sum_{i=1}^{k} a_{i,k} = 1$, then from the inequality

$$1 = \left(\sum_{i=1}^{k} a_{i,k}\right)^2 \leq k\left(\sum_{i=1}^{k} a_{i,k}^2\right) \quad \text{for } k \geq 1,$$

we have

$$\sum_{k=1}^{n}\sum_{i=1}^{k} a_{i,k}^2 \geq \sum_{k=1}^{n}\frac{1}{k} \to \infty.$$

That is to say $A_n \to \infty$.

Remark 2.3. Let us consider the special case $a_{i,k} = 1/k$ for all $1 \leq i \leq k$. Then the conditions (2.1) and (2.2) are satisfied. It is easy to check that $A_n \sim \log n$ and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{i,k}^2\right)^{1+\delta/2}}{A_n^{\delta/2}} = 0.$$

Furthermore, by Markov's inequality, Rosenthal's inequality (see Lemma 3.1) and c_r-inequality, for any $r > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}\left(\left|\frac{S_{n,n}}{\mu} - 1\right| > r\right) \\
&\quad\leq \sum_{n=1}^{\infty}\mathbb{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{i,n} - \mu)\right| > r\mu\right) \\
&\quad\leq \sum_{n=1}^{\infty}\frac{1}{(nr\mu)^{2+\delta}}\mathbb{E}\left|\sum_{i=1}^{n}(X_{i,n} - \mu)\right|^{2+\delta} \\
&\quad\leq \frac{C_{2+\delta}}{(r\mu)^{2+\delta}}\sum_{n=1}^{\infty}\frac{1}{n^{2+\delta}}\left[\left(\sum_{i=1}^{n}\mathbb{E}(X_{i,n} - \mu)^2\right)^{1+\delta/2} + \sum_{i=1}^{n}\mathbb{E}|X_{i,n} - \mu|^{2+\delta}\right] \\
&\quad\leq \frac{2C_{2+\delta}}{(r\mu)^{2+\delta}}\sum_{n=1}^{\infty}\frac{1}{n^{1+\delta/2}}\mathbb{E}|X_{1,1} - \mu|^{2+\delta} < \infty
\end{aligned}$$

which implies (2.1).

Throughout the paper, let C be a positive constant which might not be the same in each of its appearances.

3. Several lemmas

Before proving Theorem 2.1, we need to state some lemmas.

Lemma 3.1 (*Rosenthal's inequality*) [4]. *If* $\{X_k, k \geq 1\}$ *is a sequence of independent random variables with* $\mathbb{E}X_k = 0$, *then for any* $r \geq 2$,

$$\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^r \leq c_r \left[\left(\sum_{k=1}^{n} \mathbb{E}X_k^2\right)^{r/2} + \sum_{k=1}^{n} \mathbb{E}|X_k|^r\right],$$

where c_r *is a positive constant, which depends on* r.

Lemma 3.2. *Under the assumptions of Theorem* 2.1, *for* $k \geq 1$ *and* $i = 1, 2, \ldots, k$, *let* $Y_{i,k} = (X_{i,k} - \mu)/\mu$. *Then we have*

$$\mathbb{E}\left|\sum_{i=1}^{k} a_{i,k} Y_{i,k}\right|^{2+\delta} \leq C\left(\sum_{i=1}^{k} a_{i,k}^2\right)^{1+\delta/2}.$$

Proof. By Rosenthal's inequality, we have

$$\begin{aligned}\mathbb{E}\left|\sum_{i=1}^{k} a_{i,k} Y_{i,k}\right|^{2+\delta} &\leq \frac{c_{2+\delta}}{\mu^{2+\delta}}\left[\left(\sum_{i=1}^{k} a_{i,k}^2 \mathbb{E}|X_{i,k}|^2\right)^{1+\delta/2} + \sum_{i=1}^{k} a_{i,k}^{2+\delta} \mathbb{E}|X_{i,k}|^{2+\delta}\right] \\ &\leq C\left(\sum_{i=1}^{k} a_{i,k}^2\right)^{1+\delta/2},\end{aligned}$$

where we used the inequality

$$\sum_{i=1}^{k} a_{i,k}^{2+\delta} \leq \left(\sum_{i=1}^{k} a_{i,k}^2\right)^{1+\delta/2}.$$

□

Lemma 3.3. *Under the assumptions of Theorem* 2.1, *we have*

$$\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{n} \left(\frac{S_{k,k}}{\mu} - 1\right) \xrightarrow{d} \Phi. \tag{3.1}$$

Proof. Let $Y_{i,k} = (X_{i,k} - \mu)/\sigma$, $i = 1, 2, \ldots, k$, $k \geq 1$. Then (3.1) becomes

$$\frac{1}{\sqrt{A_n}} \sum_{k=1}^{n} \sum_{i=1}^{k} a_{i,k} Y_{i,k} \xrightarrow{d} \Phi.$$

Now define

$$Z_{k,n} = \frac{1}{\sqrt{A_n}} \sum_{i=1}^{k} a_{i,k} Y_{i,k},$$

then

$$\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{n} \left(\frac{S_{k,k}}{\mu} - 1\right) = \sum_{k=1}^{n} Z_{k,n}.$$

It is easy to check that

$$\mathbb{E}Z_{k,n} = 0, \quad \mathrm{Var}(Z_{k,n}) = \frac{1}{A_n}\sum_{i=1}^{k} a_{i,k}^2, \quad k = 1, \ldots, n$$

and

$$\mathrm{Var}\left(\sum_{k=1}^{n} Z_{k,n}\right) = \frac{1}{A_n}\sum_{k=1}^{n}\sum_{i=1}^{k} a_{i,k}^2 = 1.$$

In order to complete the proof we need to check that the Lindeberg condition (see p. 530 in [2]) is satisfied for the triangular array $\{Z_{k,n}\}$. For any $r > 0$, from Lemma 3.2, we have

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{E}(Z_{k,n}^2 I(|Z_{k,n}| > r)) &\leq \frac{C}{A_n^{1+\delta/2}}\sum_{k=1}^{n}\mathbb{E}\left(\sum_{i=1}^{k} a_{i,k}Y_{i,k}\right)^{2+\delta} \\ &\leq \frac{C}{A_n^{1+\delta/2}}\sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{i,k}^2\right)^{1+\delta/2} \\ &\leq \frac{C}{A_n^{1+\delta/2}}\max_{1\leq k\leq n}\left(\sum_{i=1}^{k} a_{i,k}^2\right)^{\delta/2}\sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_{i,k}^2\right) \to 0,\end{aligned}$$

where we used the fact that $\max_{1\leq k\leq n}\left(\sum_{i=1}^{k} a_{i,k}^2\right) \leq 1$. Hence the Lindeberg condition holds. □

Lemma 3.4. *Under the assumptions of Theorem* 2.1, *for* $k \geq 1$ *and* $i = 1, 2, \ldots, k$, *let* $Y_{i,k} = (X_{i,k} - \mu)/\mu$. *Then we have*

$$\frac{1}{\sqrt{A_n}}\sum_{k=1}^{n}\left[\left(\sum_{i=1}^{k} a_{i,k}Y_{i,k}\right)^2 - \gamma^2\sum_{i=1}^{k} a_{i,k}^2\right] \xrightarrow{\mathbb{P}} 0 \tag{3.2}$$

and

$$\frac{1}{\sqrt{A_n}}\sum_{k=1}^{n}\left|\sum_{i=1}^{k} a_{i,k}Y_{i,k}\right|^3 \xrightarrow{\mathbb{P}} 0. \tag{3.3}$$

Proof. Denote

$$U_k = \left(\sum_{i=1}^{k} a_{i,k}Y_{i,k}\right)^2 - \gamma^2\sum_{i=1}^{k} a_{i,k}^2,$$

then we have $\mathbb{E}U_k = 0$. Let $U_k' = U_k I(|U_k| \le A_n)$. By Lemma 3.2, we get

$$\begin{aligned}
&\mathbb{P}\left(U_k \neq U_k',\ \text{for some } 1 \le k \le n\right) \\
&\quad \le \sum_{k=1}^{n} \mathbb{P}(|U_k| > A_n) \\
&\quad \le A_n^{-(1+\delta/2)} \sum_{k=1}^{n} \mathbb{E}|U_k|^{1+\delta/2} \\
&\quad \le C A_n^{-(1+\delta/2)} \sum_{k=1}^{n} \left(\mathbb{E}\left| \sum_{i=1}^{k} a_{i,k} Y_{i,k} \right|^{2+\delta} + \left(\sum_{i=1}^{k} a_{i,k}^2 \right)^{1+\delta/2} \right) \\
&\quad \le C A_n^{-(1+\delta/2)} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{i,k}^2 \right)^{1+\delta/2} \\
&\quad \le C A_n^{-(1+\delta/2)} \max_{1 \le k \le n} \left(\sum_{i=1}^{k} a_{i,k}^2 \right)^{\delta/2} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{i,k}^2 \right) \to 0. \qquad (3.4)
\end{aligned}$$

Furthermore, for any $r > 0$, by using Lemma 3.2 again, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\sqrt{A_n}} \left| \sum_{k=1}^{n} U_k' - \mathbb{E}U_k' \right| > r \right) \\
&\quad \le \frac{C}{A_n} \sum_{k=1}^{n} \mathbb{E}(U_k' - \mathbb{E}U_k')^2 \le \frac{C}{A_n} \sum_{k=1}^{n} \mathbb{E}(U_k')^2 \\
&\quad \le \frac{C}{A_n^{\delta/2}} \sum_{k=1}^{n} \mathbb{E}|U_k|^{1+\delta/2} \\
&\quad \le \frac{C}{A_n^{\delta/2}} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{i,k}^2 \right)^{1+\delta/2} \to 0, \qquad (3.5)
\end{aligned}$$

where we used the condition (2.2). Note that $\mathbb{E}U_k = 0$. Then we have

$$\left| \sum_{k=1}^{n} \mathbb{E}U_k' \right| = \left| \sum_{k=1}^{n} \mathbb{E}U_k I(|U_k| > A_n) \right| \le \frac{1}{A_n^{\delta/2}} \sum_{k=1}^{n} \mathbb{E}\,|U_k|^{1+\delta/2} \to 0. \qquad (3.6)$$

Hence (3.4)–(3.6) implies (3.2). Next we prove (3.3). Let

$$V_k = \sum_{i=1}^{k} a_{i,k} Y_{i,k} \qquad \text{and} \qquad V_k' = V_k I(|V_k| \le \sqrt{A_n}).$$

By a similar proof of (3.2),

$$\begin{aligned}
&\mathbb{P}\left(V_k \neq V_k', \text{ for some } 1 \le k \le n\right) \\
&\quad \le \sum_{k=1}^{n} \mathbb{P}(|V_k| > \sqrt{A_n}) \le A_n^{-(1+\delta/2)} \sum_{k=1}^{n} \mathbb{E}|V_k|^{2+\delta} \\
&\quad \le C A_n^{-(1+\delta/2)} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{i,k}^2 \right)^{1+\delta/2} \to 0 \qquad (3.7)
\end{aligned}$$

and

$$\frac{1}{\sqrt{A_n}} \sum_{k=1}^{n} \mathbb{E}\left|V_k'\right|^3 \le \frac{1}{A_n^{\delta/2}} \sum_{k=1}^{n} \mathbb{E}\left|V_k'\right|^{2+\delta} \le \frac{C}{A_n^{\delta/2}} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_{i,k}^2 \right)^{1+\delta/2} \to 0. \qquad (3.8)$$

Then (3.3) can be obtained easily. □

4. Proof of Theorem 2.1

Here we will use the delta-method expansion to prove our results. Denote $C_k = S_{k,k}/\mu$. By the condition (2.1), for any $\delta > 0$, there exists a number R such that for any $r > R$,

$$\mathbb{P}\left(\sup_{k \ge r} |C_k - 1| > \delta \right) \le \delta.$$

Consequently, there exist two sequences $\{\delta_m\} \downarrow 0$ ($\delta_1 = 1/2$) and $(R_m) \uparrow \infty$ such that

$$\mathbb{P}\left(\sup_{k \ge R_m} |C_k - 1| > \delta_m \right) \le \delta_m.$$

Taking now any real x and any m, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{n} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) \le x \right) \\
&\quad = \mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{n} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) \le x, \sup_{k \ge R_m} |C_k - 1| \ge \delta_m \right) \\
&\qquad + \mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{n} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) \le x, \sup_{k \ge R_m} |C_k - 1| < \delta_m \right) \\
&\quad := A_{mn} + B_{mn},
\end{aligned}$$

where $A_{mn} \le \delta_m$. Next we will control the term B_{mn}. By the following equality for the logarithm:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3},$$

where $\theta \in (0, 1)$ depends on $x \in (-1, 1)$, we have

$$\begin{aligned}
B_{mn} &= \mathbb{P}\left\{ \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) + \frac{1}{\gamma\sqrt{A_n}} \right. \\
&\quad \left. \times \sum_{k=R_m+1}^{n} \left(\log[1 + (C_k - 1)] + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) \le x,\ \sup_{k \ge R_m} |C_k - 1| < \delta_m \right\} \\
&= \mathbb{P}\left\{ \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} (C_k - 1) \right. \\
&\quad - \frac{1}{2\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} \left[(C_k - 1)^2 - \gamma^2 \sum_{i=1}^{k} a_{i,k}^2 \right] \\
&\quad \left. + \frac{1}{3\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k - 1)^3}{(1 + \theta_k (C_k - 1))^3} \le x,\ \sup_{k \ge R_m} |C_k - 1| < \delta_m \right\} \\
&= \mathbb{P}\left\{ \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} (C_k - 1) \right. \\
&\quad - \frac{1}{2\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} \left[(C_k - 1)^2 - \gamma^2 \sum_{i=1}^{k} a_{i,k}^2 \right] \\
&\quad \left. + \left[\frac{1}{3\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k - 1)^3}{(1 + \theta_k (C_k - 1))^3} \right] I \left(\sup_{k \ge R_m} |C_k - 1| < \delta_m \right) \le x \right\} \\
&\quad - \mathbb{P}\left\{ \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \left(\log C_k + \frac{\gamma^2}{2} \sum_{i=1}^{k} a_{i,k}^2 \right) + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} (C_k - 1) \right. \\
&\quad \left. - \frac{1}{2\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} \left[(C_k - 1)^2 - \gamma^2 \sum_{i=1}^{k} a_{i,k}^2 \right] \le x,\ \sup_{k \ge R_m} |C_k - 1| \ge \delta_m \right\} \\
&:= D_{mn} - F_{mn},
\end{aligned}$$

where $\theta_k, k = 1, \ldots, n$ are $(0, 1)$-valued random variables and $F_{mn} \le \delta_m$. To estimate the term D_{mn}, we rewrite it as

$$\begin{aligned}
D_{mn} &= \mathbb{P}\left\{ \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \left[\log C_k - (C_k - 1) + \frac{(C_k - 1)^2}{2} \right] \right. \\
&\quad + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{n} (C_k - 1) - \frac{1}{2\gamma\sqrt{A_n}} \sum_{k=1}^{n} \left[(C_k - 1)^2 - \gamma^2 \sum_{i=1}^{k} a_{i,k}^2 \right] \\
&\quad + \left[\frac{1}{3\gamma\sqrt{A_n}} \sum_{k=R_m+1}^{n} \frac{(C_k - 1)^3}{(1 + \theta_k (C_k - 1))^3} \right] \\
&\quad \left. \times I \left(\sup_{k \ge R_m} |C_k - 1| < \delta_m \right) \le x \right\}.
\end{aligned}$$

For any fixed m,

$$\frac{1}{\gamma\sqrt{A_n}}\sum_{k=1}^{R_m}\left[\log C_k-(C_k-1)+\frac{(C_k-1)^2}{2}\right]\xrightarrow{\mathbb{P}}0$$

as $n\to\infty$. From Lemma 3.4, we get

$$\frac{1}{2\gamma\sqrt{A_n}}\sum_{k=1}^{n}\left[(C_k-1)^2-\gamma^2\sum_{i=1}^{k}a_{i,k}^2\right]\xrightarrow{\mathbb{P}}0.$$

Furthermore, note the following elementary inequality: for $|x|<1/2$ and any $\theta\in(0,1)$ it follows that $|x|^3/|1+\theta x|^3\le 8|x|^3$. Then by Lemma 3.4,

$$\frac{1}{3\gamma\sqrt{A_n}}\left[\sum_{k=R_m+1}^{n}\frac{|C_k-1|^3}{|1+\theta_k(C_k-1)|^3}\right]I\left(\sup_{k\ge R_m}|C_k-1|\le\delta_m\right)$$
$$\le\frac{8}{3\gamma\sqrt{A_n}}\sum_{k=1}^{n}|C_k-1|^3\xrightarrow{\mathbb{P}}0.$$

Since by Lemma 3.3, it follows that

$$\frac{1}{\gamma\sqrt{A_n}}\sum_{k=1}^{n}(C_k-1)\xrightarrow{d}\Phi$$

which implies from the above discussion that

$$D_{mn}\to\tilde{\Phi}(x),$$

where $\tilde{\Phi}$ denotes the standard normal distribution function. Finally, by observing that

$$\Delta_n:=\mathbb{P}\left(\log\left(\mathrm{e}^{\frac{\gamma^2}{2}A_n}\prod_{k=1}^{n}(S_{kk}/\mu)\right)^{\frac{1}{\gamma\sqrt{A_n}}}\le x\right)$$
$$=\mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}}\sum_{k=1}^{n}\left(\log C_k+\frac{\gamma^2}{2}\sum_{i=1}^{k}a_{i,k}^2\right)\le x\right)$$
$$=A_{mn}+D_{mn}-F_{mn},$$

we have

$$D_{mn}-F_{mn}\le\Delta_n\le A_{mn}+D_{mn}.$$

Since $A_{mn}\le\delta_m\to0$ and $F_{mn}\le\delta_m\to0$ as $m\to\infty$, and $\lim\limits_{n\to\infty}D_{mn}\to\tilde{\Phi}(x)$ for every m, the desired result can be obtained.

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