Proc. Indian Acad. Sci. (Math. Sci.) Vol. 123, No. 2, May 2013, pp. 253–256.

A note on the Fuglede–Putnam theorem

FOTIOS C PALIOGIANNIS

Department of Mathematics, St. Francis College, 180 Remsen Street,
Brooklyn Heights, New York 11201, USA
E-mail: fpaliogiannis@sfc.edu

MS received 11 April 2011; revised 24 April 2012

Abstract. We prove the following generalization of the Fuglede–Puntam theorem. Let N be an unbounded normal operator in the Hilbert space, and let A be an unbounded self-adjoint operator such that $D(N) \subseteq D(A)$. Then, $AN \subseteq N^*A \Rightarrow AN^* \subseteq NA$.

Keywords. Unbounded normal operator; abelian von Neumann algebra; bounding sequence.

1. Introduction and preliminaries

In this note, we use the notion of a bounding sequence for an unbounded normal operator to prove the unbounded version and a generalization of the Fuglede–Putnam theorem [6]. This paper gives a new and much simpler proof of the rather complicated proof of Theorem 5 of [3] due to Mortad. We begin with some definitions and preliminary results.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. We denote by $\mathrm{Op}(\mathcal{H})$ the set of unbounded densely defined linear operators on $\mathcal{H}$. For $A \in \mathrm{Op}(\mathcal{H})$ we denote the domain of A by $D(A)$. An operator $A \in \mathrm{Op}(\mathcal{H})$ is called *closed* if $x_n \in D(A)$ with $x_n \to x$ and $Ax_n \to y$ implies $x \in D(A)$ and $Ax = y$. Let $A, B \in \mathrm{Op}(\mathcal{H})$, we recall that B is called an *extension* of A, denoted by $A \subseteq B$, if $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$. A closed operator $A \in \mathrm{Op}(\mathcal{H})$ is said to *commute* with $T \in B(\mathcal{H})$, if $TA \subseteq AT$, that is, if for $x \in D(A)$, we have $Tx \in D(A)$ and $TAx = ATx$. Let $\{A\}' = \{T \in B(\mathcal{H}) : TA \subseteq AT\}$. Then, it is easily seen that $\{A\}'$ is a strongly closed subalgebra of $B(\mathcal{H})$. Furthermore, $T \in \{A\}'$ if and only if $T^* \in \{A^*\}'$. Thus, $\{A\}' \cap \{A^*\}'$ is a von Neumann algebra.

DEFINITION 1

Let $A \in \mathrm{Op}(\mathcal{H})$ be closed and $\mathcal{A}$ a von Neumann algebra. The operator A is said to be affiliated with $\mathcal{A}$, denoted by $A\eta\mathcal{A}$, if $\mathcal{A}' \subseteq \{A\}'$.

Note that, if $A \in B(\mathcal{H})$, the double commutant theorem tells us that $A\eta\mathcal{A}$ if and only if $A \in \mathcal{A}$. Note also that $A\eta\mathcal{A}$ if and only if $A^*\eta\mathcal{A}$. Equivalently, $A\eta\mathcal{A}$ if and only if $\mathcal{A}' \subseteq \{A\}' \cap \{A^*\}'$ or $\left\{\{A\}' \cap \{A^*\}'\right\}' \subseteq \mathcal{A}$. We denote by $W^*(A) = \left\{\{A\}' \cap \{A^*\}'\right\}'$.

Clearly, $W^*(A)$ is the smallest von Neumann algebra with which A is affiliated and is referred to as the *von Neumann algebra generated* by A.

DEFINITION 2

Let $A \in \mathrm{Op}(\mathcal{H})$. A bounding sequence for A is a non-decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of projections on $\mathcal{H}$ such that $\bigvee_{n=1}^{\infty} F_n = I$, $F_n A \subseteq AF_n$ and $AF_n \in B(\mathcal{H})$ for all $n \in \mathbb{N}$.

Lemma 1. *Let $A \in \mathrm{Op}(\mathcal{H})$ be closed and $\mathcal{A}$ a von Neumann algebra. If $A\eta\mathcal{A}$, then $A^*A\eta\mathcal{A}$.*

Proof. Since A is closed, A^*A is self adjoint (Theorem 2.7.8 of [2]) and hence closed. Let $T \in \mathcal{A}'$. Since $\mathcal{A}' \subseteq \{A\}' \cap \{A^*\}'$, we have $TA^*A \subseteq A^*TA \subseteq A^*AT$. Hence $\mathcal{A}' \subseteq \{A^*A\}'$, and so $A^*A\eta\mathcal{A}$. □

Theorem 1. *If $\mathcal{A}$ is an abelian von Neumann algebra and $A\eta\mathcal{A}$, then there is a bounding sequence $\{F_n\}$ for A such that $F_n \in \mathcal{A}$ and $AF_n \in \mathcal{A}$ for all $n \in \mathbb{N}$.*

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the self-adjoint operator A^*A and let $F_n = E_n - E_{-n}$ for each $n \in \mathbb{N}$. Then, from the spectral theorem for unbounded self-adjoint operators (Theorem 4.3 of [4]), we have $F_n \in \{A^*A\}''$, $\bigvee_{n=1}^{\infty} F_n = I$ and $(A^*A)F_n \in B(\mathcal{H})$. Since $D(AF_n) \supseteq D(A^*AF_n) = \mathcal{H}$, the closed operator AF_n is everywhere defined, and the closed graph theorem implies that $AF_n \in B(\mathcal{H})$. Since $A^*A\eta\mathcal{A}$, it follows that $\{A^*A\}'' \subseteq \mathcal{A}'' = \mathcal{A}$. Hence, $F_n \in \mathcal{A}$. Since $\mathcal{A}$ is abelian, $F_n A \subseteq AF_n$ for all $n \in \mathbb{N}$. Moreover, if $T \in \mathcal{A}'$, then $TAF_n \subseteq ATF_n = AF_nT$. Hence $TAF_n = AF_nT$ and $AF_n \in \mathcal{A}'' = \mathcal{A}$. □

A closed operator $N \in \mathrm{Op}(\mathcal{H})$ is called *normal* if $N^*N = NN^*$. This implicitly carries the assumption $D(N) = D(N^*)$. The von Neumann algebra generated by N, $W^*(N)$ is abelian [5]. By Theorem 1, there is a bounding sequence $\{F_n\}$ for N in $W^*(N)$. Moreover, the sequence $\{F_n\}$ is also bounding for N^* and $(NF_n)^* = N^*F_n$ (Lemma 5.6.14 of [2]).

2. Results

The Fuglede–Putnam theorem in the bounded case reads as follows:

Theorem 2 (Fuglede–Putnam). *Let $M, N \in B(\mathcal{H})$ be normal and $T \in B(\mathcal{H})$. If $TN = MT$, then $TN^* = M^*T$* [1].

If $N = M$, this is Fuglede's theorem. The unbounded version of Fuglede's theorem reads as follows:

Theorem 3 (Fuglede's theorem). *Let $N \in \mathrm{Op}(\mathcal{H})$ be normal and $T \in B(\mathcal{H})$. If $TN \subseteq NT$, then $TN^* \subseteq N^*T$* [5].

Thus, if $N \in \mathrm{Op}(\mathcal{H})$ is normal, then $W^*(N) = \{N\}''$ (the double commutant of $\{N\}$).

Theorem 4 (Fuglede–Putnam). *Let $M, N \in \mathrm{Op}(\mathcal{H})$ be normal and $T \in B(\mathcal{H})$. If $TN \subseteq MT$, then $TN^* \subseteq M^*T$.*

Proof. Let $\{F_n\}$ and $\{G_m\}$ be bounding sequences for N and M, respectively. Since $TN \subseteq MT$, it follows that $TNF_n \subseteq MTF_n$. Since TNF_n is bounded everywhere, we get $TNF_n = MTF_n$. Applying G_m to this equality, we get $G_mTNF_n = G_mMTF_n \subseteq MG_mTF_n$. Again since G_mTNF_n is bounded, we get

$$G_mTNF_n = G_mMTF_n = MG_mTF_n = MG_m(G_mTF_n).$$

At the same time, $G_mTNF_n = G_mTNF_nF_n \supseteq (G_mTF_n)NF_n$. But $(G_mTF_n)NF_n$ is bounded, so $G_mTNF_n = (G_mTF_n)NF_n$. Thus $(G_mTF_n)NF_n = MG_m(G_mTF_n)$.

Since NF_n and MG_m are bounded normal operators, the bounded case implies that

$$(G_mTF_n)N^*F_n = M^*G_m(G_mTF_n).$$

Now, for $x \in D(N^*)$, we have

$$\begin{aligned} G_mTN^*F_nx &= (G_mTF_n)N^*F_nx = M^*G_m(G_mTF_n)x = M^*G_mTF_nx \\ &= G_mM^*TF_nx. \end{aligned}$$

Hence

$$G_mTN^*F_nx = G_mM^*TF_nx.$$

Since $G_m \to I$ (strongly), as $m \to \infty$, it follows that

$$TN^*F_nx = M^*TF_nx.$$

Finally, since $F_nx \to x$ as $n \to \infty$ and T is bounded, $TF_nx \to Tx$. Furthermore,

$$M^*TF_nx = TN^*F_nx = TF_nN^*x \to TN^*x.$$

Since M^* is closed, it follows that $x \in D(M^*T)$ and $M^*Tx = TN^*x$.

Thus, $TN^* \subseteq M^*T$. □

Theorem 5. *Let $N \in \mathrm{Op}(\mathcal{H})$ be a normal operator and $A \in \mathrm{Op}(\mathcal{H})$ a self-adjoint operator such that $D(N) \subseteq D(A)$. If $AN \subseteq N^*A$, then $AN^* \subseteq NA$.*

Proof. Let $\{F_n\}$ be a bounding sequence for N. First note that AF_n is closed. Since $D(N) \subseteq D(A)$, it follows that $\mathcal{H} = D(NF_n) \subseteq D(AF_n)$. Hence, by the closed graph theorem, $AF_n \in B(\mathcal{H})$.

Since $AN \subseteq N^*A$, it follows that $ANF_n \subseteq N^*AF_n$. At the same time, since $F_nN \subseteq NF_n$, we have $AF_nN \subseteq ANF_n$. Hence,

$$(AF_n)N \subseteq N^*(AF_n). \tag{1}$$

Since AF_n is bounded, the Fuglede–Putnam theorem implies that

$$(AF_n)N^* \subseteq N(AF_n). \tag{2}$$

Multiplying (2) by F_n we get $(AF_n)N^*F_n \subseteq N(AF_n)F_n = NAF_n$. Since $(AF_n)(N^*F_n)$ is bounded everywhere, $NAF_n = (AF_n)(N^*F_n)$. Furthermore, since $(AF_n)N^*F_n \subseteq AN^*F_n$, we have $AN^*F_n = (AF_n)(N^*F_n)$. Hence, for all $n \in \mathbb{N}$,

$$AN^*F_n = NAF_n. \tag{3}$$

We prove that $F_nA \subseteq AF_n$ for all $n \in \mathbb{N}$. By the normality of N and (1), (2) it follows that $(AF_n)NN^* \subseteq (AF_n)N^*N \subseteq N(AF_n)N \subseteq NN^*(AF_n)$, that is, $(AF_n)NN^* \subseteq NN^*(AF_N)$. Hence $AF_n \in \{NN^*\}'$. As $F_n \in \{NN^*\}''$, we have $F_n(AF_n) = (AF_n)F_n = AF_n$. Taking adjoints, the self-adjointness of A gives $(AF_n)^*F_n = (AF_n)^* \supseteq F_nA^* = F_nA$, and so $F_nA \subseteq (AF_n)^*F_n$. Multiplying the latter by F_n we get $F_nAF_n \subseteq (AF_n)^*F_n^2 = (AF_n)^*F_n$. Since $F_nAF_n \in B(\mathcal{H})$, we conclude that $F_nA \subseteq F_nAF_n = AF_n$, that is, $F_nA \subseteq AF_n$.

Let $x \in D(AN^*)$. Then $x \in D(N^*)$ and $N^*x \in D(A)$. Since $F_n \to I$ (strongly), we have

$$AF_nx = F_nAx \to Ax \ \text{ as } \ n \to \infty.$$

Moreover, since $F_nN^* \subseteq N^*F_n$, by (3), we have

$$NAF_nx = AN^*F_nx = AF_nN^*x = F_nAN^*x \to AN^*x.$$

The closeness of N implies that $x \in D(NA)$ and $NAx = AN^*x$.

Thus, $AN^* \subseteq NA$. □

References

[1] Conway J B, A course in Functional Analysis, Graduate Texts in Math (1985) (New York: Springer)

[2] Kadison R and Ringrose J, Fundamentals of the Theory of Operator Algebras, vol. I (1983) (New York: Academic Press)

[3] Mortad M H, An application of the Fugled-Putnam theorem to normal products of self-adjoint operators, *Proc. Am. Math. Soc.* **131(10)** (2003) 3135–3141

[4] Paliogiannis F C, The algebra of unbounded continuous functions on a Stonean space and unbounded operators, *Michigan Math. J.* **46(1)** (1999) 39–52

[5] Paliogiannis F C, On Fuglede's theorem for unbounded normal operators, *Ricerche Mat.* **LI(2)** (2002) 261–264

[6] Putnam C R, On normal operators in Hilbert space, *Am. J. Math.* **73** (1951) 357–3622