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Notes on discrete subgroups of Möbius transformations

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Abstract. Jørgensen's inequality gives a necessary condition for a nonelementary two generator subgroup of $SL(2, \mathbb{C})$ to be discrete. By embedding $SL(2, \mathbb{C})$ into $\hat{U}(1, 1; \mathbb{H})$, we obtain a new type of Jørgensen's inequality, which is in terms of the coefficients of involved isometries. We provide an example to show that this result gives an improvement over the classical Jørgensen's inequality.

Keywords. Jørgensen's inequality; quaternionic hyperbolic space; embedding.

1. Introduction

Jørgensen [7] obtained the following Jørgensen's inequality, which is important in the study of the geometry of discrete groups and the structure of 3-dimensional hyperbolic manifold.

Theorem J. *Let $g, h \in SL(2, C)$. If the two-generator subgroup $\langle f, g\rangle$ is discrete and nonelementary, then*

$$|\mathrm{tr}^2(g) - 4| + |\mathrm{tr}[g, h] - 2| \geq 1,$$

where $[g, h] = ghg^{-1}h^{-1}$ and $\mathrm{tr}(\cdot)$ is the trace function.

By dealing with the geometry of the action of the group on the boundary of complex hyperbolic space, Basmajian and Miner [1] generalized the Jørgensen's inequality to the two generator subgroup of $PU(2, 1)$ when the generators are loxodromic or boundary elliptic.

Jiang *et al.* [6] obtained a similar form of the above theorem in $SU(2, 1)$. Kim [8] and Markham [9] found analogues in quaternionic hyperbolic space of results in [6]. Recently, Cao and Parker [3] and Cao and Tan [4] generalized and improved the results in [6] to the setting of quaternionic hyperbolic n-space.

Contrast to the above generalizations to higher dimension and bigger algebraic body, there is less attempt to improve the classical Jørgensen's inequality. Recently, using the embedding of $SL(2,\mathbb{C})$ into $U(1,1,\mathbb{H})$, Cao and Tan [4] obtained some new types of Jørgensen's inequality with one element being elliptic.

In this paper, using the embedding of $SL(2,\mathbb{C})$ into $\hat{U}(1,1,\mathbb{H})$, we will make another attempt to obtain a new type of Jørgensen's inequality with one element being loxodromic.

In order to compare our results with the classical Jørgensen's inequality, we reformulate part of Theorem J as the following proposition.

PROPOSITION 1.1

Let g be a loxodromic element and h be any element in $SL(2,\mathbb{C})$ *with the forms*:

$$g = \begin{pmatrix} r\mathrm{e}^{\mathbf{i}\theta} & 0 \\ 0 & r^{-1}\mathrm{e}^{-\mathbf{i}\theta} \end{pmatrix}, \qquad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

If $\langle g, h\rangle$ *is discrete and non-elementary*, *then*

$$(r^2 + r^{-2} - 2\cos(2\theta))(1 + |bc|) \geq 1. \tag{2}$$

Our main result is the following theorem.

Theorem 1.1. *Let g be a loxodromic element and h be any element in* $SL(2,\mathbb{C})$ *with the forms* (1). *If* $\langle g, h\rangle$ *is discrete and non-elementary*, *then*

$$|abcd|^{\frac{1}{2}} \geq \frac{1 - M_g}{M_g^2}, \tag{3}$$

where

$$M_g = |r\mathrm{e}^{\mathbf{i}\theta} - 1| + |r^{-1}\mathrm{e}^{-\mathbf{i}\theta} - 1| = \sqrt{(r + r^{-1} + 2)(r + r^{-1} - 2\cos\theta)}. \tag{4}$$

The structure of the remainder of this paper is as follows: In § 2, we give the necessary background material for quaternionic hyperbolic space and a lemma for the proof of our main result. Section 3 contains the proof of Theorem 1.1 and in § 4, we offer an example to reveal the merit of our inequality.

2. Preliminaries

Let $\mathbb{H}$ denote the division ring of real quaternions. Elements of $\mathbb{H}$ have the form $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \in \mathbb{H}$, where $q_i \in \mathbb{R}$ and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Let $\bar{q} = q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}$ be the *conjugate* of q, and

$$|q| = \sqrt{\bar{q}q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

be the *modulus* of q. We define $\Re(q) = (q + \bar{q})/2$ to be the *real part* of q, and $\Im(q) = (q - \bar{q})/2$ to be the *imaginary part* of q. Also $q^{-1} = \bar{q}|q|^{-2}$ is the *inverse* of q. We remark that for a complex number c, we have $\mathbf{j}c = \bar{c}\mathbf{j}$.

Let $\mathbb{H}^{1,1}$ be the vector space of dimension 2 over $\mathbb{H}$ with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z},\ \mathbf{w}\rangle = \mathbf{w}^* J\mathbf{z} = \overline{w_1}z_1 - \overline{w_2}z_2,$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in $\mathbb{H}^{1,1}$ with entries (z_1, z_2) and (w_1, w_2) respectively, $*$ denotes the conjugate transpose and J is the Hermitian matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a *unitary transformation* g to be an automorphism $\mathbb{H}^{1,1}$, that is, a linear bijection such that $\langle g(\mathbf{z}),\ g(\mathbf{w})\rangle = \langle \mathbf{z},\ \mathbf{w}\rangle$ for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{1,1}$. We denote the group of all unitary transformations by $U(1, 1; \mathbb{H})$, this is sometimes also denoted by $\mathrm{Sp}(1, 1)$.

Following § 2 of [5], let

$$V_0 = \{\mathbf{z} \in \mathbb{H}^{1,1} - \{0\} : \langle \mathbf{z},\ \mathbf{z}\rangle = 0\}, \qquad V_- = \{\mathbf{z} \in \mathbb{H}^{1,1} : \langle \mathbf{z},\ \mathbf{z}\rangle < 0\}.$$

It is obvious that V_0 and V_- are invariant under $U(1, 1; \mathbb{H})$. We define V^s to be $V^s = V_- \cup V_0$. Let $P : V^s \to P(V^s) \subset \mathbb{H}$ be the projection map defined by

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 {z_2}^{-1}.$$

We define $\mathbb{B} = P(V_-)$, the ball model of 1-dimensional quaternionic hyperbolic space. It is easy to see that $\mathbb{B}$ can be identified with the quaternionic unit ball $\{z \in \mathbb{H} :\ |z| < 1\}$. Also the unit sphere in $\mathbb{H}$ is $\partial\mathbb{B} = P(V_0)$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(1, 1; \mathbb{H})$ then, by definition, g preserves the Hermitian form. Hence

$$\mathbf{w}^* J\mathbf{z} = \langle \mathbf{z},\ \mathbf{w}\rangle = \langle g\mathbf{z},\ g\mathbf{w}\rangle = \mathbf{w}^* g^* J g\mathbf{z}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in V. Letting $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for V we see that $J = g^* Jg$. From this we find that $g^{-1} = J^{-1} g^* J$, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}.$$

It is convenient to introduce the Cayley transform

$$C = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}. \tag{5}$$

We denote $\hat{U}(1, 1; \mathbb{H}) = CU(1, 1; \mathbb{H})C^{-1}$, which is the automorphism group of the Hermitian form

$$\langle \mathbf{z},\ \mathbf{w}\rangle = \overline{w_1}z_2 + \overline{w_2}z_1.$$

Viewed as a projective transformation, the Cayley transformation C maps $\mathbb{B}$ and its boundary $\partial\mathbb{B}$ to the Siegel domain $\Sigma_{\mathbb{H}}^1$ and its boundary $\partial\Sigma_{\mathbb{H}}^1$, respectively. The Bergman metric on $\Sigma_{\mathbb{H}}^1$ is given by the distance formula

$$\cosh^2\frac{\rho(z,w)}{2}=\frac{\langle \mathbf{z},\ \mathbf{w}\rangle\langle \mathbf{w},\ \mathbf{z}\rangle}{\langle \mathbf{z},\ \mathbf{z}\rangle\langle \mathbf{w},\ \mathbf{w}\rangle},\quad \text{where } z,w\in\Sigma_{\mathbb{H}}^1,\ \ \mathbf{z}\in P^{-1}(z),\mathbf{w}\in P^{-1}(w). \tag{6}$$

If $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\hat{U}(1,1;\mathbb{H})$, then $g^{-1}=\begin{pmatrix} \bar{d} & \bar{b}\\ \bar{c} & \bar{a}\end{pmatrix}$ and consequently,

$$a\bar{d}+b\bar{c}=1,\ a\bar{b}+b\bar{a}=0,\ c\bar{d}+d\bar{c}=0,\ \bar{d}b+\bar{b}d=0,\ \bar{c}a+\bar{a}c=0. \tag{7}$$

The quaternionic cross-ratio of four points z_1,z_2,w_1,w_2 in $\overline{\Sigma_{\mathbb{H}}^1}$ is defined as

$$[z_1,z_2,w_1,w_2]=\langle \mathbf{w_1},\ \mathbf{z_1}\rangle\langle \mathbf{w_1},\ \mathbf{z_2}\rangle^{-1}\langle \mathbf{w_2},\ \mathbf{z_2}\rangle\langle \mathbf{w_2},\ \mathbf{z_1}\rangle^{-1}, \tag{8}$$

where $\mathbf{z_i}\in P^{-1}(z_i),\mathbf{w_i}\in P^{-1}(w_i),\ i=1,2.$

As in [3], the quaternionic cross-ratio $[z_1,z_2,w_1,w_2]$ depends on the choice $\mathbf{z}_1\in P^{-1}(z_1)$. However its absolute value

$$\big|[z_1,z_2,w_1,w_2]\big|=\frac{|\langle \mathbf{w_1},\ \mathbf{z_1}\rangle\langle \mathbf{w_2},\ \mathbf{z_2}\rangle|}{|\langle \mathbf{w_1},\ \mathbf{z_2}\rangle\langle \mathbf{w_2},\ \mathbf{z_1}\rangle|} \tag{9}$$

is independent of the preimage of z_i and w_i in $\mathbb{H}^{1,1}$.

The following lemma is crucial to us.

Lemma 2.1 (*cf. Theorem* 1.1 *of* [3]). *Let g be a loxodromic element of $\hat{U}(1,1,\mathbb{H})$ with $M_g<1$ and with fixed points $u,\ v\in\partial\Sigma_{\mathbb{H}}^1$. Let h be any other element of $\hat{U}(1,1,\mathbb{H})$. If*

$$\big|[h(u),u,v,h(v)]\big|^{1/2}\big|[h(u),v,u,h(v)]\big|^{1/2}<\frac{1-M_g}{M_g^2}, \tag{10}$$

then the group $\langle g,h\rangle$ is either elementary or not discrete.

3. Proof of Theorem 1.1

As in [2, 4, 5], we can regard Sp(1, 1) as the isometries of hyperbolic 4-space $\mathbf{H}_{\mathbb{R}}^4$, whose model is the unit ball in the quaternions $\mathbb{H}$. $SL(2,\mathbb{C})$, the isometries of hyperbolic 3-space $\mathbf{H}_{\mathbb{R}}^3$, can be embedded as a subgroup of $\hat{U}(1,1;\mathbb{H})$ as follows:

$$f\in SL(2,\mathbb{C})\hookrightarrow TfT^{-1}\in U(1,1;\mathbb{H})\rightleftharpoons CTfT^{-1}C^{-1}\in\hat{U}(1,1;\mathbb{H}), \tag{11}$$

where $T=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathbf{j}\\ -\mathbf{j} & 1\end{pmatrix}$ and C be the Cayley transform given by (5).

The following lemma can be verified directly.

Lemma 3.1. *Let g and h be as in* (1) *and* $\hat{g} = CTgT^{-1}C^{-1}$, $\hat{h} = CThT^{-1}C^{-1}$. *Then*

$$\hat{g} = \frac{1}{4}\begin{pmatrix} (1+\mathbf{j})(re^{\mathbf{i}\theta}+r^{-1}e^{-\mathbf{i}\theta})(1-\mathbf{j}) & (1+\mathbf{j})(re^{\mathbf{i}\theta}-r^{-1}e^{-\mathbf{i}\theta})(1+\mathbf{j}) \\ (1-\mathbf{j})(re^{\mathbf{i}\theta}-r^{-1}e^{-\mathbf{i}\theta})(1-\mathbf{j}) & (1-\mathbf{j})(re^{\mathbf{i}\theta}+r^{-1}e^{-\mathbf{i}\theta})(1+\mathbf{j}) \end{pmatrix} \tag{12}$$

and

$$\hat{h} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \tag{13}$$

where

$$\hat{a} = \frac{1}{4}(1+\mathbf{j})(a-c-b+d)(1-\mathbf{j}),\ \hat{b} = \frac{1}{4}(1+\mathbf{j})(a-c+b-d)(1+\mathbf{j}),$$

$$\hat{c} = \frac{1}{4}(1-\mathbf{j})(a+c-b-d)(1-\mathbf{j}),\ \hat{d} = \frac{1}{4}(1-\mathbf{j})(a+c+b+d)(1+\mathbf{j}).$$

Proof of Theorem 1.1. As in Lemma 3.1, by the embedding given by (11), we get the corresponding elements $\hat{g}$ and $\hat{h}$ in $\hat{U}(1,1;\mathbb{H})$ of g and h. That is $\hat{g}$ and $\hat{h}$ are given by (12) and (13), respectively.

Let $\mathbf{u} = \frac{1}{2}\begin{pmatrix} 1+\mathbf{j} \\ 1-\mathbf{j} \end{pmatrix}$ and $\mathbf{v} = \frac{1}{2}\begin{pmatrix} -1-\mathbf{j} \\ 1-\mathbf{j} \end{pmatrix}$. Then $\mathbf{u}$ and $\mathbf{v}$ are the preimages of the fixed points u, v of $\hat{g}$, respectively. By direct computation, we have

$$\begin{aligned} &\left|[\hat{h}(u), v, u, \hat{h}(v)]\right| = |ad|, \quad \left|[\hat{h}(u), u, v, \hat{h}(v)]\right| = |bc|, \\ &\left|[u, v, \hat{h}(u), \hat{h}(v)]\right| = \frac{|bc|}{|ad|}. \end{aligned} \tag{14}$$

As in [3], the two right eigenvalues of $\hat{g}$ can chosen to be $re^{\mathbf{i}\theta}$ and $r^{-1}e^{-\mathbf{i}\theta}$, which implies that

$$M_{\hat{g}} = |re^{\mathbf{i}\theta} - 1| + |r^{-1}e^{-\mathbf{i}\theta} - 1| = \sqrt{(r + r^{-1} + 2)(r + r^{-1} - 2\cos\theta)}.$$

Applying Lemma 2.1 to the two generator subgroup $\langle \hat{g}, \hat{h} \rangle$, we conclude the proof of Theorem 1.1. □

4. Comparisons

In this section, we will give an example to show that our result gives an improvement over the Jørgensen's inequality. Without loss of generality, we assume $r > 1$ in the following case.

Example 4.1.

$$g = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} x\mathbf{i} & y\mathbf{i} \\ y\mathbf{i} & x\mathbf{i} \end{pmatrix}, \text{ where } x, y > 0, y^2 - x^2 = 1. \tag{15}$$

In this case, g and h are loxodromic and $\langle g, h \rangle$ is non-elementary.

In this case $M_g = r - r^{-1}$. We can assert that Theorem 1.1 is better than Jørgensen's inequality if we can find r and $x > 0$ and some $m = r - \frac{1}{r} < 1$ satisfy the following inequalities:

$$x\sqrt{1+x^2} < \frac{1-m}{m^2}, \quad 2 + x^2 \geq \frac{1}{m^2}. \tag{16}$$

Since the above inequalities are equivalent to

$$x^2 < \frac{\sqrt{m^4 + 4(1-m)^2} - m^2}{2m^2}, \quad x^2 \geq \frac{1-2m^2}{m^2},$$

and for $\frac{\sqrt{5}-1}{2} < m < 1$, i.e. $1.3557 \approx \frac{\sqrt{5}-1+\sqrt{22-2\sqrt{5}}}{4} < r < \frac{\sqrt{5}+1}{2} \approx 1.6180$,

$$\frac{1-2m^2}{m^2} < \frac{\sqrt{m^4 + 4(1-m)^2} - m^2}{2m^2}, \tag{17}$$

we know that for r, x in (15) satisfying

$$\frac{\sqrt{5}-1+\sqrt{22-2\sqrt{5}}}{4} < r < \frac{\sqrt{5}+1}{2},$$
$$\frac{1-2m^2}{m^2} \leq x^2 < \frac{\sqrt{m^4 + 4(1-m)^2} - m^2}{2m^2},$$

Theorem 1.1 implies that the subgroup $\langle g, h\rangle$ is not discrete, while Jørgensen's inequality (Proposition 1.1) fails to assert the non-discreteness of $\langle g, h\rangle$.

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