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Conjugacy class sizes and solvability of finite groups

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Abstract. Let G be a finite group and G^* be the set of primary, biprimary and triprimary elements of G. We prove that if the conjugacy class sizes of G^* are $\{1, m, n, mn\}$ with positive coprime integers m and n, then G is solvable. This extends a recent result of Kong (*Manatsh. Math.* **168(2)** (2012) 267–271).

Keywords. Conjugacy class sizes; primary; biprimary and triprimary elements; solvable groups; finite groups.

1. Introduction

Throughout this paper all groups considered are finite and G always denotes a group. A primary element is an element of prime power order and a biprimary (triprimary) element is an element whose order is divisible by precisely two (three) distinct primes. We will denote by x^G the conjugacy class of x, its size $|x^G|$, is called the index of x in G. The rest of the notations and terminologies are standard and readers may refer to [6].

A well-known problem in group theory is to study the influence of conjugacy class sizes on the structure of a group. For instance, as regards to groups with conjugacy class sizes $\{1, m\}$, Itô [3] proved that G is nilpotent, $m = p^a$ for some prime p, and $G = P \times A$, where P is a Sylow p-subgroup of G. Moreover, $A \leq Z(G)$. Beltrán and Felipe [1, 2] proved that if the conjugacy class sizes of G are $\{1, m, n, mn\}$, where $m, n > 1$ are coprime integers, then G is nilpotent. Meanwhile, $m = p^a$ and $n = q^b$ for distinct primes p and q.

On the other hand, using partial conjugacy class sizes to investigate the structure of a group is also studied extensively. For instance, in a very recent paper [5], Kong proved that G is solvable if the conjuagcy class sizes of primary, biprimary and triprimary elements are $\{1, p^a, n, p^a n\}$, where p is a prime, a is an integer and n is a positive integer coprime to p.

He proposed that at that moment he could not prove that G is solvable when the conjugacy class sizes of primary, biprimary and triprimary elements of G are exactly $\{1, m, n, mn\}$ with $(m, n) = 1$. In this paper, this problem is solved and our main theorem is the following:

Main theorem. *Let G be a group and G^* be the set of primary, biprimary and triprimary elements of G. If the conjugacy class sizes of G^* are $\{1, m, n, mn\}$, then G is a solvable, where m, n are two positive coprime integers.*

2. Preliminaries

In this section we list some basic and known results which will be used in the sequel.

Lemma 2.1. *Let G be a group. If $x, y \in G$ be such that $[x, y] = 1$ and $(o(x), o(y)) = 1$. Then $C_G(xy) = C_G(x) \cap C_G(y)$.*

Proof. Obviously, $C_G(x) \cap C_G(y) \subseteq C_G(xy)$. Now we prove another inclusion. Assume that $o(x) = a$, $o(y) = b$. Since $(a, b) = 1$, there are two integers s, t such that $as+bt = 1$. For every $v \in C_G(xy)$, we have that $(xy)^v = x^v y^v$. Note that $[x, y] = 1$, we have that $(xy)^{as} = x^{as}y^{as} = y^{as}$. Hence $((xy)^{as})^v = (x^{as})^v(y^{as})^v = (y^{as})^v = (xy)^{as} = y^{as}$. On the other hand, $y = y^{as+bt} = y^{as}$. Hence $y^v = (y^{as})^v = y^{as} = y$. By the same reason, we have that $x^v = x$. This shows that $v \in C_G(x) \cap C_G(y)$. Therefore, $C_G(xy) = C_G(x) \cap C_G(y)$. □

Lemma 2.2 (*Theorem* 5 *of* [7]). *If for some prime p every primary p'-element of a group G has index prime to p, then the Sylow p-subgroup of G is a direct factor of G.*

Lemma 2.3 (*Lemma* 2.7 *of* [4]). *If 1 and $m > 1$ are the only sizes of conjugacy classes of elements of primary and biprimary orders of a group G, then $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ and A is abelian. In particular, m is a power of p.*

Lemma 2.4. *Suppose that the three smallest non-trivial indices of elements of primary, biprimary and triprimary elements are $a < b < c$, with $(a, b) = 1$ and $a^2 < c$. Then the set $\{g \in G \mid |g^G| = 1 \text{ or } a\}$ is a normal subgroup of G.*

Proof. The proof is similar to Lemma 2.4 of [8]. □

Lemma 2.5 (*Theorem A of* [5]). *Let G be a finite group and G^* the set of elements of primary, biprimary and triprimary order of G. If the sizes of elements of G^* are $\{1, p^a, n, p^a n\}$ with p, n being coprime positive integers, then G is solvable.*

Lemma 2.6 (*Theorem* 8.2.8 *of* [6]). *Let $P \times Q$ be the direct product of a p-group P and a p'-group Q. Suppose that G is a p-group such that $C_G(P) \leq C_G(Q)$, then Q acts trivially on G.*

Lemma 2.7. *Let G be a π-solvable group with $\pi \subseteq \pi(G)$. Then $|x^G|$ is a π-number for every π'-element x of primary order if and only if G has an abelian Hall π'-subgroup.*

Proof. We only prove the necessity. Without loss of generality, we may assume that $O_{\pi'}(G) \neq 1$ since otherwise, the conclusion holds by an inductive argument on the quotient group $G/O_\pi(G)$. Let $g \in G$ be an arbitrary π'-element of primary order. Then there exists a Hall π'-subgroup K_1 of G such that $K_1 \leq C_G(g)$. Therefore,

$$g \in C_G(K_1) \leq C_G(O_{\pi'}(G)) \leq O_{\pi'}(G),$$

which follows that $O_{\pi'}(G)$ is an abelian Hall π'-subgroup of G. □

3. Proof of the main theorem

Proof. For every prime $r \in \pi(G)-(\pi(m)\bigcup\pi(n))$, it follows from Lemma 2.3 of [4] that there exists a Sylow r-subgroup R of G such that $R \leq Z(G)$. Consequently, $G = H \times K$, where K is a Hall $\pi(m)\bigcup\pi(n)$-subgroup of G. Moreover, the hypothesis is inherited by K. As a result, without loss of generality, we may assume that G is a $\pi(m)\bigcup\pi(n)$-group. For convenience, we denote $\pi := \pi(m)$ and $\pi' := \pi(n)$. The proof will be completed in the following steps:

Step 1. We may assume that G has no π-elements of index m and no π'-elements of index n.

Suppose that x is a π-element of index m. By considering the primary decomposition of x we may assume that x is a p-element for some $p \in \pi$. Now if y_1 is a primary p'-element of $C_G(x)$, we see from Lemma 2.1 that $C_G(xy_1) = C_G(x) \cap C_G(y_1) \leq C_G(x)$, which forces $|C_G(x) : C_G(x) \cap C_G(y_1)| = 1$ or n. Hence $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$ by Lemma 2.2. If $C_G(x)_{p'}$ is not abelian, then class sizes of primary and biprimary elements of such a p'-subgroup are exactly the two numbers 1 and n in $C_G(x)_{p'}$. Note that n must occur. It follows from Lemma 2.3 that $n = q^b$ and therefore, G is solvable by Lemma 2.5, and we are done.

Now we prove that $C_G(x)_{p'}$ is not abelian. Otherwise, we have that $C_G(x)_{p'}$ has a π-complement H, which is also a π-complement of G. Moreover, $H \nleq Z(G)$. Let $v \in H - Z(G)$ be a primary element. Then $|v^G| = m$ and $C_G(x) = C_G(v)$. For any $w \in C_G(x)_p$, we have that $|w^G| = 1$ or m. Since $|C_G(x) : C_G(x) \cap C_G(w)| = |C_G(v) : C_G(v) \cap C_G(w)| = 1$ or n and $C_G(x)_{p'} \leq C_G(w)$, we have that $C_G(x) = C_G(w)$. Hence $C_G(x)$ is abelian. Let $y \in G$ be a primary element of index n. By conjugation, there is some $g \in G$ such that $x^{g^{-1}} \in C_G(y)$, that is, $y^g \in C_G(x)$. Hence $C_G(x) \leq C_G(y^g)$. It follows that $|y^G| \mid |x^G|$, a contradiction.

The second assertion holds because the hypotheses are symmetric in m and n.

Step 2. If x is a primary or biprimary π-element of index mn, then $C_G(x) = C_G(x)_\pi \times C_G(x)_{\pi'}$ with $C_G(x)_{\pi'} \nleq Z(G)$ abelian. Similarly, if y is a primary or biprimary π'-element of index mn, then $C_G(y) = C_G(y)_\pi \times C_G(y)_{\pi'}$ with $C_G(y)_\pi \nleq Z(G)$ is abelian.

Note that mn is the maximal index. For any π'-element $y \in C_G(x)$ of primary order, by Lemma 2.1, it follows that $C_G(xy) = C_G(x) \cap C_G(y) \leq C_G(x)$ and $C_G(xy) = C_G(x) \leq C_G(y)$. Moreover, $y \in Z(C_G(x))$ and hence $C_G(x) = C_G(x)_\pi \times C_G(x)_{\pi'}$, where $C_G(x)_{\pi'}$ is an abelian Hall π'-subgroup of $C_G(x)$. We claim that $C_G(x)_{\pi'} \nleq Z(G)$. Otherwise, $C_G(x)_{\pi'} = Z(G)_{\pi'}$, which implies that $|G : Z(G)|_{\pi'} = n$. Let z be an arbitrary non-central primary or biprimary π-element, then z is of index n or mn by the hypothesis. Since both cases imply that $Z(G)_{\pi'}$ is a π'-subgroup of $C_G(z)$, we see that for any non-central π'-element w of primary order of G, $C_G(w)_\pi = Z(G)_\pi$, yielding that $|G : Z(G)|_\pi = m$. Hence $|G : Z(G)| = |G : Z(G)|_\pi |G : Z(G)|_{\pi'} = mn$, a contradiction to the fact that there is an element of primary order of index mn, and the claim holds. Analogously, we have that $C_G(y) = C_G(y)_\pi \times C_G(y)_{\pi'}$, where $C_G(y)_\pi \nleq Z(G)$ is abelian.

Without loss of generality, we assume that $n < m$ from now on.

Step 3. Write $L_\pi := \{x | x$ is a π-element such that $|x^G| = 1$ or $n\}$. Then L_π is a non-trivial abelian normal π-subgroup of G.

By Lemma 2.4, the set $W := \{x \mid |x^G| = 1 \text{ or } n\}$ is a normal subgroup of G. Let x be an element in G^* of index n. Write $x = x_\pi x_{\pi'}$, where x_π and $x_{\pi'}$ are the π-part and the π'-part of x. Applying Step 1, $x_{\pi'}$ must be central, whence $x \in L_\pi \times Z(G)_{\pi'}$. Therefore $W = L_\pi \times Z(G)_{\pi'}$ and consequently L_π is a normal π-subgroup of G.

Notice that L_π is abelian, as if primary element $y \in L_\pi$, then $|L_\pi : C_{L_\pi}(y)|$ divides $(|L_\pi|, n) = 1$. Finally, by considering the decomposition of any element of index n and taking into account Step 1, we see immediately that L_π is non-trivial.

For a prime $q \in \pi$, we define $L_q := \{x \mid x \text{ is a } q\text{-element such that } |x^G| = 1 \text{ or } n\}$. Notice that L_π is the direct product of the subgroups L_q for all primes $q \in \pi$, and consequently, L_q is an abelian normal subgroup of G.

Step 4. Let $q \in \pi$. If L_q is not central in G, then L_q is a Sylow q-subgroup of G.

Assume false, we may choose a q-element w of index mn. By Step 2, we write $C_G(w) = C_G(w)_\pi \times C_G(w)_{\pi'}$, with $C_G(w)_{\pi'} \nleq Z(G)$ abelian. If $u \in C_G(w)_{\pi'}$, then $C_G(w) \leq C_G(u)$ and in particular, $C_{L_q}(w) \leq C_{L_q}(u)$, so by applying Lemma 2.6, we get $u \in N := C_G(L_q)$. Thus $C_G(w)_{\pi'} \leq N$.

We fix some non-central element $y \in L_q$, so that $N \leq C_G(y)$. As w has index mn and y has index n, we see that $|C_G(y) : N|$ and $|N : C_G(w)_{\pi'}|$ are both π-numbers, and so $C_G(w)_{\pi'}$ is a Hall π'-subgroup of N and $C_G(y)$.

We show now that any q-element of G lies in N. Trivially, $L_q \leq N$, so we consider a q-element $z \notin L_q$. Thus z must have index mn. By Step 2, we may write $C_G(z) = C_G(z)_\pi \times C_G(z)_{\pi'}$, where $C_G(z)_{\pi'}$ is a non-central abelian Hall π'-subgroup of $C_G(z)$. Arguing about z as we did above with w, we obtain that $C_G(z)_{\pi'} \leq N$. Take a non-central primary element $u \in C_G(z)_{\pi'}$, then $C_G(z) \leq C_G(u)$. In addition, by Step 1, u has index m or mn, so that $|C_G(u) : C_G(z)|$ is equal to 1 or n. Since $C_G(z)_{\pi'} \leq N$ we have $L_q \leq C_G(u)$ so that $L_q \leq C_G(z)$ and consequently $z \in N$, as we wanted to prove.

Now let $t \in C_G(y) \cap G^*$. Consider the usual factorization $t = t_q t_{q'}$. We have $C_G(t_{q'}) \cap C_G(y) = C_G(t_{q'}y) \leq C_G(y)$ by Lemma 2.1 and $|C_G(y) : C_G(t_{q'}y)|$ must be equal to 1 or m because y has index n. If for every $r \in \pi'$ and $C_G(y)_r \in \mathrm{Syl}_r(C_G(y))$, we have that $C_G(y)_r \leq Z(G)$. Then $C_G(y)_{\pi'} \leq Z(G)$. From the first paragraph of this step, we have that $C_G(y)_{\pi'} < C_G(w)_{\pi'}$ since $C_G(w)_{\pi'} \nleq Z(G)$. But $|C_G(y)_{\pi'}| = |C_G(w)_{\pi'}|$, a contradiction. For some $r \in \pi'$, we can choose a non-central Sylow r-subgroup R of $C_G(y)$, which is also a Sylow r-subgroup of N. Then there exists some $g \in C_G(y)$ such that $R^g \leq C_G(t_{q'}y)$, so that $t_{q'} \in C_G(R^g)$. Now we distinguish two cases for t_q. Suppose $t_q \in L_q$, so that $R^g \leq N \leq C_G(t_q)$ from the second paragraph of this step, we conclude that $t \in C_G(R^g)$. In the other case, when t_q is not in L_q, then it has index mn, and we can again write $C_G(t_q) = C_G(t_q)_\pi \times C_G(t_q)_{\pi'}$, where $C_G(t_q)_{\pi'}$ is an abelian non-central Hall π'-subgroup of $C_G(t_q)$. Furthermore, notice that the property of w given at the beginning of this step holds for t_q, that is, $C_G(t_q)_{\pi'}$ is also a π-complement of N and $C_G(y)$. As we know that N has non-central π-complements and non-central Sylow r-subgroups, we may consider the Sylow r-subgroup R_1 of $C_G(t_q)_{\pi'}$, which is not central. Then $t_{q'} \in C_G(R_1)$ and trivially $t_q \in C_G(R_1)$ whence $t \in C_G(R_1)$. Since $R_1 = R^g$ for some $g \in C_G(y)$, we have that $t \in C_G(R^g)$.

Let $v \in C_G(y)$ be an arbitrary element. Considering its primary decomposition of $v = v_1 \cdots v_r w_1 \cdots w_s$, where v_i are π-elements and w_j are π'-elements, respectively.

If there is some component of v with index mn, say v_1, then $C_G(v_1 v_i) = C_G(v_1) \leq C_G(v_i)$ for all $i \geq 2$ and $C_G(v_1 w_j) = C_G(v_1) \leq C_G(w_j)$ for all j. Hence $C_G(v) =$

$C_G(v_1) \cap \cdots \cap C_G(w_s) = C_G(v_1)$. From the above, we have that $v_1 \in C_G(R^g)$ for some $g \in C_G(y)$ and thus $R^g \leq C_G(v_1) = C_G(v)$. Therefore, $v \in C_G(R^g)$.

On the other hand, we assume that there is no component of v with index mn. Then by Step 1, we have that $|v_i^G| = 1$ or n and $|w_j^G| = 1$ or m. Assume that $v_1, w_1 \notin Z(G)$. Then $C_G(v_1 v_i) = C_G(v_1) \cap C_G(v_i)$ for $i \geq 2$. If $|v_i^G| = 1$, then $C_G(v_1 v_i) = C_G(v_1) \leq C_G(v_i)$. If $|v_i^G| = n$, then $C_G(v_1 v_i) = C_G(v_1) \cap C_G(v_i)$ and thus $|(v_1 v_i)^G| \leq |v_1^G||v_i^G| = n^2 < mn$. This gives $|(v_1 v_i)^G| = |v_1^G| = n$ and $C_G(v_1 v_i) = C_G(v_1) = C_G(v_i)$. Moreover, $C_G(v_1 \cdots v_r) = C_G(v_1)$. On the other hand, we see from $C_G(w_1 w_j) = C_G(w_1) \cap C_G(w_j)$ that $|(w_1 w_j)^G| \leq |w_1^G||w_j^G| = m^2$. Hence $|(w_1 w_j)^G| = mn$ or m. If there is a non-central w_j, say w_2, such that $|(w_1 w_2)^G| = mn$, we can get that $C_G(w_1 \cdots w_s) = C_G(w_1 w_2)$. If there is no non-central w_j such that $|(w_1 w_j)^G| = mn$, then $C_G(w_1 w_j) = C_G(w_1) \leq C_G(w_j)$ by Lemma 2.1 and thus $C_G(w_1 \cdots w_s) = C_G(w_1) \cap \cdots \cap C_G(w_s) = C_G(w_1)$.

Since we have proved that $v_1 w_1 \in C_G(R^g)$ for some $g \in G$, it follows that $R^g \leq C_G(v_1 w_1) = C_G(v)$. Hence $v \in C_G(R^g)$. Further, $C_G(y) = \bigcup_{g \in C_G(y)} C_{C_G(y)}(R)^g$, which implies that $C_G(y) = C_{C_G(y)}(R)$. Hence R must be central in $C_G(y)$. But we know that R is not central in G, and so if we take some non-central $u_1 \in R$, we have $C_G(y) \leq C_G(R) \leq C_G(u_1)$. This provides a π'-element u of index n, contradicting Step 1.

Step 5. G is solvable. Let $y \in L_q - Z(G)$. It is easy to check that any primary or biprimary q'-element of $C_G(y)$ has index 1 or m in $C_G(y)$. We will assume first that there exists a non-central Sylow r-subgroup R of $C_G(y)$ for some prime $r \in \pi'$. If w is a primary q'-element of $C_G(y)$, then there exists some $g \in C_G(y)$ such that $R^g \leq C_G(w)$, that is, $w \in C_{C_G(y)}(R)^g$.

Let $v \in C_G(y)$ be a q'-element. Similar as in Step 4, we can obtain that $v \in C_G(R)^g$ for some $g \in C_G(y)$.

Thus, if we consider the $\{q, q'\}$-decomposition of any element of $C_G(y)$, taking into account that L_q is a Sylow q-subgroup of G, we have

$$C_G(y) = \bigcup_{g \in C_G(y)} C_{C_G(y)}(R)^g L_q = \bigcup_{g \in C_G(y)} (C_{C_G(y)}(R) L_q)^g,$$

which implies that $C_G(y) = C_{C_G(y)}(R) L_q$, and accordingly, $|C_G(y) : C_{C_G(y)}(R)|$ is a q-number. Now, we take some non-central $u \in R$, which has index m or mn. Observe that $C_{C_G(y)}(R) \leq C_G(u) \cap C_G(y) = C_G(uy) \leq C_G(y)$, so that uy has index n or mn. The first case leads to $C_G(y) \leq C_G(u)$, which is a contradiction, and so uy has index mn and it follows that m is a q-power. By Lemma 2.5, we obtain that G is solvable and the theorem is proved.

Therefore, we will assume that for each prime $r \in \pi'$, Sylow r-subgroups of $C_G(y)$ are central in G and we will obtain a contradiction. In this case, we have $C_G(y) = C_G(y)_\pi \times Z(G)_{\pi'}$, and also $|G : Z(G)|_{\pi'} = n$. If there exists a primary or biprimary π-element v of index mn, then $C_G(v) = C_G(v)_\pi \times C_G(v)_{\pi'}$ with $C_G(v)_{\pi'} \not\leq Z(G)$ abelian. Since $C_G(v)_{\pi'} \geq Z(G)_{\pi'}$ and $|C_G(v)_{\pi'}| = |C_G(y)_{\pi'}|$, we have that $C_G(v)_{\pi'} = Z(G)_{\pi'}$, a contradiction. Notice that L_π is a normal abelian subgroup and all the primary and biprimary π-elements are contained in L_π, we assume that there are no π-elements of index

mn, and consequently that L_π is a Hall π-subgroup of G. By Schur–Zassenhaus, we have that G has a π-complement H. Write $G = L_\pi H$. Since $1 \trianglelefteq L_\pi \trianglelefteq G$ and L_π is a characteristic subgroup of G and G/L_π is a π'-group, we have have that G is π-separable by Theorem 6.4.2 of [6]. Since $|G : Z(G)|_{\pi'} = n$, we have that every π'-element in $G^* \cap H$ with index 1 or m. By Lemma 2.7, we have that H is abelian. This implies that there is a biprimary or triprimary element w with index mn. Write $w = w_1 w_2 w_3$. It is easy to see that w is neither a π-element nor a π'-element. We may assume that $w_1 w_2$ is a π-element. Then $C_G(w_1 w_2) = C_G(w_1) \cap C_G(w_2) = C_G(w_1)$, and thus we may assume that w is a biprimary element and write $w = w_1 w_2$ with w_1 being a π-element and w_2 being a π'-element. On the other hand, $|G : C_G(w)| = mn$ and $C_G(w)_{\pi'} = Z(G)_{\pi'}$, we get that $C_G(w) = C_G(w)_\pi \times Z(G)_{\pi'}$. Since w_2 is a π'-element of $C_G(w)$, we have that $w_2 \in Z(G)_{\pi'}$. This gives $|w^G| = |w_1^G| = mn$, which contradicts that L_π is abelian. □

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References

[1] Beltrán A and Felipe M J, Variations on a theorem by Alan Camina on conjugacy class sizes, *J. Algebra* **296** (2006) 253–266

[2] Beltrán A and Felipe M J, Some class size conditions implying solvability of finite groups, *J. Group Theory* **48(10)** (2006) 1–11

[3] Itô N, On finite groups with given conjugate types I, *Nagoya Math. J.* **6** (1953) 17–28

[4] Kong Q J and Guo X Y, On an extension of a theorem on conjugacy class sizes, *Israel J. Math.* **179** (2010) 279–284

[5] Kong Q J, Conjugacy class sizes and solvability of finite groups, *Manatsh. Math.* **168(2)** (2012) 267–271

[6] Kurzweil H and Stellmacher B, The theory of finite groups: An introduction (2004) (Berlin-Heidelberg-New York: Springer-Verlag)

[7] Liu X, Wang Y and Wei H, Notes on the length of conjugacy classes of finite groups, *J. Pure Appl. Algebra* **196** (2005) 111–117

[8] Shao C G and Jiang Q H, On conjugacy class sizes of primary and biprimary elements of a finite group, *Science China Math.* (to appear)