Proc. Indian Acad. Sci. (Math. Sci.) Vol. 123, No. 2, May 2013, pp. 235–238.


# A note on 2-nilpotence of finite groups

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MS received 1 July 2012

**Abstract.** Ballester-Bolinches and Guo showed that a finite group $G$ is 2-nilpotent if $G$ satisfies: (1) a Sylow 2-subgroup $P$ of $G$ is quaternion-free and (2) $\Omega_1(P \cap G') \leq Z(P)$ and $N_G(P)$ is 2-nilpotent. In this paper, it is obtained that $G$ is a non-2-nilpotent group of order $16q$ for an odd prime $q$ satisfying (1) a Sylow 2-subgroup $P$ of $G$ is not quaternion-free and (2) $\Omega_1(P \cap G') \leq Z(P)$ and $N_G(P)$ is 2-nilpotent if and only if $q = 3$ and $G \cong GL_2(3)$.

**Keywords.** Finite group; 2-nilpotent group; quaternion-free.

## 1. Introduction

Let $G$ be a finite group and $P$ a Sylow 2-subgroup of $G$. Recall that $G$ is said to be 2-nilpotent if there exists a normal subgroup $K$ of $G$ such that $G = KP$ and $K \cap P = 1$. Let $H$ be a finite 2-group, denoting $\Omega_1(H) = \langle h \in H \mid o(h) = 2\rangle$. Ballester-Bolinches and Guo [1] gave a well-known result for the 2-nilpotence of finite groups, which is a classical generalization of Itô's lemma.

**Theorem (Theorem 2 of [1]).** *Let $P$ be a Sylow 2-subgroup of a finite group $G$. Suppose that $\Omega_1(P \cap G') \leq Z(P)$. If $P$ is quaternion-free and $N_G(P)$ is 2-nilpotent, then $G$ is 2-nilpotent.*

According to the above theorem, it seems interesting to investigate the non-2-nilpotent groups $G$ satisfying (1) a Sylow 2-subgroup $P$ of $G$ is not quaternion-free and (2) $\Omega_1(P \cap G') \leq Z(P)$ and $N_G(P)$ is 2-nilpotent. In this paper, such a group is called a special local 2-nilpotent group.

Note that the existence of special local 2-nilpotent groups can show that the hypothesis that $P$ is quaternion-free is necessary in Theorem 2 of [1].

Shi [5] and Shi *et al.* [3] showed that the general linear group $GL_2(3)$ and the special linear group $SL_2(q)$ where $q > 1$ and $q \equiv -1 (\mathrm{mod}\ 8)$ are special local 2-nilpotent groups respectively.

**Theorem [4].**

(1) *Let $G = SL_2(q)$ where $q > 3$, then $G$ is a special local 2-nilpotent group if and only if $q^2 \equiv 1 (\mathrm{mod}\ 16)$.*

(2) *Let $G = GL_2(q)$ where $q > 3$, then G is a special local 2-nilpotent group if and only if q is an odd number.*

In [4], we also showed that $PSL_2(q)$ and $PGL_2(q)$ are not special local 2-nilpotent groups for every $q > 3$.

As an extension of [5], we have the following result, the proof of which is given in §2.

**Theorem 1.1.** *Let G be a finite group of order $16q$, where q is an odd prime. Then G is a special local 2-nilpotent group if and only if $q = 3$ and $G \cong GL_2(3)$.*

## 2. Proof of Theorem 1.1

*Lemma* 2.1. $GL_2(3)$ *is the special local 2-nilpotent group of minimal order.*

*Proof.*

(1) Let $G$ be a special local 2-nilpotent group and $P$ a Sylow 2-subgroup of $G$.

*Claim.*

$$|P| \geq 16.$$

Suppose not. Assume $|P| < 16$. By the hypothesis, $P$ is isomorphic to the quaternion group $Q_8$. Since $G$ is non-2-nilpotent, there exists a subgroup $K$ of $G$ such that $K$ is a minimal non-2-nilpotent group. By Theorems 9.1.9 and 10.3.3 of [2], one has $K = S \rtimes T$, where $S \in \mathrm{Syl}_2(K)$ and $T \in \mathrm{Syl}_q(K)$, $q \neq 2$. Since $K$ is non-2-nilpotent and $P \cong Q_8$, we must have $|S| = |P|$ by Theorem 10.1.9 of [2]. Thus, we can assume $S = P$. Then $K = N_K(S) = N_K(P) \leq N_G(P)$ is 2-nilpotent, a contradiction. So $|P| \geq 16$.

(2) By part (1), we can easily get that $GL_2(3)$ is the special local 2-nilpotent group of minimal order. □

*Lemma* 2.2. $GL_2(3)$ *is the unique special local 2-nilpotent group of order* 48.

*Proof.* Let $G$ be a special local 2-nilpotent group of order 48. Let $Q$ be a Sylow 3-subgroup of $G$. Since $G$ is non-2-nilpotent and $N_G(P)$ is 2-nilpotent, we have $N_G(P) = P$ and $Q \ntrianglelefteq G$. It follows that $G/P_G \cong S_3$, the symmetric group of degree 3, where $P_G$ is the largest normal subgroup of $G$ that is contained in $P$.

Let $M = P_G \rtimes Q$. It is easy to see that $Q \ntrianglelefteq M$ since $M \trianglelefteq G$ and $Q \ntrianglelefteq G$. By the classifications of the groups of order 24, we can easily get that $M \cong SL_2(3)$ or $A_4 \times \mathbb{Z}_2$. However, by the classifications of the groups of order 16, it is easy to see that each group of order 16 in which a quotient of some subgroup is isomorphic to the quaternion group $Q_8$ does not contain ${\mathbb{Z}_2}^3$. Then $M \ncong A_4 \times \mathbb{Z}_2$ and so $M \cong SL_2(3)$.

Since $G/M \cong \mathbb{Z}_2$, we can easily get that $G \cong GL_2(3)$ or $SL_2(3) \times \mathbb{Z}_2$. However, since $P \ntrianglelefteq G$, we have $G \cong GL_2(3)$. It follows that $GL(2, 3)$ is the unique special local 2-nilpotent group of order 48. □

*Lemma* 2.3. *Suppose that G is a special local 2-nilpotent group of order $16q^n$, where q is an odd prime and $n \geq 1$. Then $q = 3$.*

*Proof.* Let $P$ be a Sylow 2-subgroup of $G$ and $Q$ a Sylow $q$-subgroup of $G$. Since $N_G(P)$ is 2-nilpotent, it is easy to see that $G$ cannot be a minimal non-2-nilpotent group. Moreover, since $G$ is non-2-nilpotent, one has $Q \ntrianglelefteq G$. By Sylow's theorem, we have $q = 3, 5$ or 7.

(1) Suppose $q = 5$. Let $K$ be a proper subgroup of $G$ such that $K$ is a minimal non-2-nilpotent group. Let $|K| = 2^m \cdot 5^l$, where $1 \leq l \leq n$. Since $N_G(P)$ is 2-nilpotent and $K$ is a minimal non-2-nilpotent group, we must have $m \leq 3$. Then, by Sylow's theorem the Sylow 5-subgroup of $K$ is normal in $K$, a contradiction.
(2) Suppose $q = 7$. Let $H$ be a proper subgroup of $G$ such that $H$ is a minimal non-2-nilpotent group. Then $H = H_2 \rtimes H_7$, where $H_2 \in \mathrm{Syl}_2(H)$, $H_7 \in \mathrm{Syl}_7(H)$ and $H_7$ is cyclic. Let $|H| = 2^t \cdot 7^s$, where $1 \leq s \leq n$. Arguing as above, we have $t \leq 3$. Moreover, since $H_7$ is not normal in $H$, we must have $t = 3$. We claim that $H_2$ *is a characteristically simple group*.

Otherwise, assume that $L$ is a non-trivial characteristic subgroup of $H_2$. It follows that $L \trianglelefteq H$ since $H_2 \trianglelefteq H$. Note that the subgroup $LH_7$ is nilpotent. Let $M$ be a subgroup of $L$ of order 2. Then $MH_7$ is a cyclic group of order $2 \cdot 7^s$. Thus, $H$ has $(2^3 - 1)$ elements of 2-power order, $\{8(7^s - 7^{s-1}) + 7^{s-1} - 1\} = (7^{s+1} - 1)$ elements of 7-power order, and at least $(7^s - 1)$ elements of order $2 \cdot 7^i$, where $1 \leq i \leq s$. However, it is obvious that $(2^3 - 1) + (7^{s+1} - 1) + (7^s - 1) + 1 > 2^3 \cdot 7^s = |H|$, a contradiction. So $H_2$ is a characteristically simple group, which implies that $H_2$ is isomorphic to ${\mathbb{Z}_2}^3$. However, by the classifications of the groups of order 16, each group of order 16 in which a quotient of some subgroup is isomorphic to $Q_8$ does not contain ${\mathbb{Z}_2}^3$, a contradiction.

Thus, by (1) and (2) we have $q = 3$. □

Lemmas 2.1, 2.2 and 2.3 combined together give Theorem 1.1. □

## 3. Remark

Motivated by Theorem 1.1, we propose the following problem:

*Question* 3.1. Let $G$ be a finite group of order $2^m q^n$ and $P$ a Sylow 2-subgroup of $G$ satisfying $\Omega_1(P \cap G') \leq Z(P)$ and $N_G(P)$ is 2-nilpotent. Suppose that $G$ is $GL_2(3)$-free, is it true that $G$ is 2-nilpotent?


## Acknowledgements

The first author was supported by the NSFC (Grant No. 11201401) and China Postdoctoral Science Foundation (Grant No. 201104027). The second author was supported by NSFC (Grant No. 11201403) and "Agencija za raziskovalno dejavnost Republike Slovenije", research program P1-0285.


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