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On rationality of moduli spaces of vector bundles on real Hirzebruch surfaces

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Abstract. Let X be a real form of a Hirzebruch surface. Let $M_H(r, c_1, c_2)$ be the moduli space of vector bundles on X. Under some numerical conditions on r, c_1 and c_2, we identify those $M_H(r, c_1, c_2)$ that are rational.

Keywords. Real variety; moduli space; rationality; Hirzebruch surface.

1. Introduction

Moduli spaces of semistable vector bundles on a smooth projective variety are studied from various points of view. One of the questions that is often addressed is the birational type of the moduli space, more precisely, the question of rationality. It is known that the moduli space of semistable vector bundles of rank r and fixed determinant of degree d over a smooth complex projective curve is rational if d is coprime to r [4].

Let Y be a smooth projective surface over $\mathbb{C}$; fix a very ample divisor H on Y. Let $M_H(r, c_1, c_2)$ be the moduli space of torsionfree sheaves on Y of rank r, with Chern classes c_i, semistable with respect to H. One may ask the following question: is $M_H(r, c_1, c_2)$ rational given that Y is rational? The answer to this question is not known in general. For $Y = \mathbb{P}^2$, Maruyama [7] and Ellingsrud and Strømme [2] proved that if $c_1^2 - 4 \cdot c_2 \neq 0 \pmod 8$, then the moduli space $M_H(2, c_1, c_2)$ is rational. Later on, Maeda in [6] proved that the rationality of $M_H(2, c_1, c_2)$, for $Y = \mathbb{P}^2$, holds if the moduli space is nonempty. In [1], the rationality of some moduli spaces of vector bundles on a Hirzebruch surface is proved. (A Hirzebruch surface is of the form $\mathbb{P}(\xi)$ where ξ is a direct sum of two line bundles on $\mathbb{P}^1_{\mathbb{C}}$.)

Our aim here is to address the question when the surface is defined over $\mathbb{R}$. More precisely, we extend the results of [1] when the surface is defined over $\mathbb{R}$. The moduli spaces are then defined over the field $\mathbb{R}$. There are exactly two non-isomorphic curves defined over $\mathbb{R}$ which become isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ after base change to $\mathbb{C}$; one of them is rational while the other is not.

Let $X_{\mathbb{R}} := \text{Proj}(\mathbb{R}[x_0, x_1, x_2]/x_0^2+x_1^2+x_2^2)$ be the anisotropic conic. Let ξ be a locally free sheaf of rank two on $X_{\mathbb{R}}$ which decomposes into a direct sum of line bundles. Define $Y_{\mathbb{R}} := \mathbb{P}(\xi)$. Let F denote a fiber of the natural map $Y_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$, where $Y_{\mathbb{C}}$ and $X_{\mathbb{C}}$ are the base changes, to $\mathbb{C}$, of $Y_{\mathbb{R}}$ and $X_{\mathbb{R}}$ respectively. Define $c_1 := C_0 + \text{d}F \in \text{Pic}(Y_{\mathbb{C}})$; the class C_0 is defined in §4. Let c_2, α and λ be integers satisfying

$$c_2 \equiv \alpha \mod r-1, \quad 0 < \alpha \leq r-1\,, \quad \lambda = \frac{c_2 - \alpha}{r-1}$$

and

$$\Delta(r, c_1, c_2) := \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right) \gg 0.$$

Define $m := d - c_2 - 1 - \lambda$.

We prove the following theorem (see Theorem 5.3).

Theorem 1.1. *The moduli space $M_H(r, c_1, c_2)$ is rational as a real variety if and only if one of the following holds*:

- *m is even and $r - \alpha - 1$ is even.*
- *m is odd and α is even.*

The proof of Theorem 1.1 uses the following lemma (see Lemma 3.1):

Lemma 1.2. *Let W be a complex vector space with a conjugate linear automorphism T such that $T \circ T = -\text{Id}_W$. For any $\alpha \in [1\,, \dim W]$, let*

$$\sigma : \text{Gr}(\alpha, W) \longrightarrow \text{Gr}(\alpha, W)$$

be the anti-holomorphic involution that sends any subspace to its image under T. Then the real variety $(\text{Gr}(\alpha, W)\,, \sigma)$ is rational if and only if α is even.

2. Preliminaries

We recall some basic properties of schemes defined over the field of real numbers. Let A_0 be a finitely generated $\mathbb{R}$–algebra. Let $A := A_0 \otimes_{\mathbb{R}} \mathbb{C}$. Then $\mathbb{C} \hookrightarrow A$ and A has a natural involution which acts on $\mathbb{C}$ by complex conjugation, namely, the involution given by sending $a \otimes \alpha$ to $a \otimes \bar{\alpha}$. Let us call this involution σ. We can recover the algebra A_0 as the subalgebra of A consisting of elements which are fixed by σ.

Conversely, assume that we are given a finitely generated $\mathbb{C}$-algebra A, with an involution σ which acts on $\mathbb{C}$ by complex conjugation. Let $A_0 \subset A$ be the subalgebra consisting of elements which are fixed by σ. Then A_0 is a finitely generated $\mathbb{R}$-algebra, and $A \cong A_0 \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathbb{C}$-algebras. Moreover, this isomorphism converts σ on the left to complex conjugation on the right. We note that it is an isomorphism because we can write every element as

$$a = \frac{a + \sigma(a)}{2} + \frac{a - \sigma(a)}{2}\,.$$

Let us carry this on to A-modules. Assume that we are given an A-module M with an involution δ, satisfying $\delta(am) = \sigma(a)\delta(m)$. Let M_0 be the subgroup consisting of elements which are fixed by δ. Then M_0 is an A_0-module, and there is a natural isomorphism $M = M_0 \otimes_{\mathbb{R}} \mathbb{C}$. Under this identification, δ corresponds to complex conjugation on $M_0 \otimes_{\mathbb{R}} \mathbb{C}$.

Denote by $\tilde{M}$ the sheaf on Spec A associated to M. Then giving an involution δ is equivalent to giving an $\mathcal{O}_{\mathrm{Spec}\,A}$ module homomorphism

$$\delta : \tilde{M} \longrightarrow \sigma_* \tilde{M} ,$$

satisfying the condition $\sigma_*(\delta) \circ \delta = \mathrm{Id}$. This is same as giving $\delta : \sigma^* \tilde{M} \longrightarrow \tilde{M}$ such that $\delta \circ \sigma^*(\delta) = \mathrm{Id}$.

Let $X_{\mathbb{R}}$ be a smooth, projective variety defined over the reals. We shall denote by $X_{\mathbb{C}}$ its base change to $\mathbb{C}$. A *quaternionic sheaf* on $X_{\mathbb{C}}$ is a pair (E , δ), where E is a coherent sheaf on $X_{\mathbb{C}}$, and

$$\delta : \sigma^* E \longrightarrow E$$

is an isomorphism such that $\delta \circ \sigma^*(\delta) = -\mathrm{Id}$.

3. Rationality of some real forms of Grassmannians

A *conjugate linear* homomorphism between complex vector spaces is a $\mathbb{C}$-antilinear homomorphism.

Lemma 3.1. *Let W be a complex vector space with a conjugate linear automorphism T such that $T^2 = -\mathrm{Id}_W$. For any $\alpha \in [1 , \dim W]$, the automorphism T gives rise to a real variety*

$$\mathrm{Gr}(\alpha, W) \xrightarrow{\sigma} \mathrm{Gr}(\alpha, W) .$$

This real variety is rational if and only if α is even.

Proof. First we choose a suitable basis for W. Let $w_1 \in W$ be a nonzero vector. If $T(w_1) = \mu w_1$, then $-w_1 = T^2(w_1) = \bar{\mu} T(w_1) = \bar{\mu}\mu w_1$, implying $\bar{\mu}\mu = -1$, which is a contradiction. Therefore, we conclude that w_1 and $T(w_1)$ are linearly independent.

Now we can go modulo the subspace generated by w_1 and $T(w_1)$, and repeat this process. In this way we construct a basis

$$\{w_1, T(w_1), w_2, T(w_2), \ldots , w_n, T(w_n)\} .$$

This shows that $\dim_{\mathbb{C}} W$ is even, say $2n$.

A point $x \in \mathrm{Gr}(\alpha, W)$ corresponds to a subspace V, and $\sigma(x)$ is the subspace $T(V)$. Applying the same analysis to a subspace of W which is left invariant under T, we conclude that its dimension is an even number. This shows that for σ to have any fixed points, a necessary condition is that α is even. Therefore, the real variety $(\mathrm{Gr}(\alpha, W) , \sigma)$ is not rational if α is an odd integer.

We now assume that α is even.

The matrix of T in the above basis is $\mathrm{Diag}(J, J, \ldots , J)$, where J is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Consider the open subset of $\mathrm{Gr}(\alpha, W)$ given in matrix form with respect to the above basis by the $2n \times \alpha$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x_{1,1}^{1,1} & x_{1,2}^{1,1} & \cdots & x_{1,1}^{1,\frac{\alpha}{2}} & x_{1,2}^{1,\frac{\alpha}{2}} \\ x_{2,1}^{1,1} & x_{2,2}^{1,1} & \cdots & x_{2,1}^{1,\frac{\alpha}{2}} & x_{2,2}^{1,\frac{\alpha}{2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,1}^{n-\frac{\alpha}{2},1} & x_{1,2}^{n-\frac{\alpha}{2},1} & \cdots & x_{1,1}^{n-\frac{\alpha}{2},\frac{\alpha}{2}} & x_{1,2}^{n-\frac{\alpha}{2},\frac{\alpha}{2}} \\ x_{2,1}^{n-\frac{\alpha}{2},1} & x_{2,2}^{n-\frac{\alpha}{2},1} & \cdots & x_{2,1}^{n-\frac{\alpha}{2},\frac{\alpha}{2}} & x_{2,2}^{n-\frac{\alpha}{2},\frac{\alpha}{2}} \end{pmatrix}.$$

This open subset is left invariant under σ. We want to understand how the coordinate functions transform. At the level of closed points, we have the following map:

$$\begin{pmatrix} a_{1,1}^{r,s} & a_{1,2}^{r,s} \\ a_{2,1}^{r,s} & a_{2,2}^{r,s} \end{pmatrix} \longmapsto \begin{pmatrix} \overline{a_{2,2}^{r,s}} & -\overline{a_{2,1}^{r,s}} \\ -\overline{a_{1,2}^{r,s}} & \overline{a_{1,1}^{r,s}} \end{pmatrix}.$$

This shows that the zero locus

- $\{x_{1,1}^{r,s} = 0\}$ maps to $\{x_{2,2}^{r,s} = 0\}$
- $\{x_{1,2}^{r,s} = 0\}$ maps to $\{x_{2,1}^{r,s} = 0\}$
- $\{x_{2,1}^{r,s} = 0\}$ maps to $\{x_{1,2}^{r,s} = 0\}$
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From the above, we conclude that at the level of rings, $x_{1,1}^{r,s} \longmapsto \beta x_{2,2}^{r,s}$. To find the value of β, we note that the zero locus $\{x_{1,1}^{r,s} = 1\}$ maps to $\{x_{2,2}^{r,s} = 1\}$, which shows that $\beta = 1$. Similarly, we conclude that

- $x_{1,1}^{r,s} \longmapsto x_{2,2}^{r,s}$
- $x_{1,2}^{r,s} \longmapsto -x_{2,1}^{r,s}$
- $x_{2,1}^{r,s} \longmapsto -x_{1,2}^{r,s}$
- $x_{2,2}^{r,s} \longmapsto x_{1,1}^{r,s}$.

This proves that

$$\mathrm{Gr}(\alpha, W) \xrightarrow{\sigma} \mathrm{Gr}(\alpha, W)$$

defines a rational real variety. □

Remark 3.2. Let $K_{\mathbb{R}} := \mathbb{R}(x_1, x_2, \ldots, x_n)$ and $K_{\mathbb{C}} := \mathbb{C}(x_1, x_2, \ldots, x_n)$. Then $K_{\mathbb{C}}$ is equipped with a complex conjugation given by conjugating the complex coefficients. This complex conjugation of $K_{\mathbb{C}}$ will be denoted by σ_K. If W in Lemma 3.1 were a $K_{\mathbb{C}}$ vector space, and T was σ_K-antilinear, with $T^2 = -\mathrm{Id}$, then the same argument would prove that

$$\mathrm{Gr}(\alpha, W) \xrightarrow{\sigma_K} \mathrm{Gr}(\alpha, W)$$

is a rational $K_{\mathbb{R}}$ variety if and only if α is even.

Lemma 3.3. *Let V be a complex vector space, and let $T : V \longrightarrow V$ be a conjugate linear map such that $T^2 = \mathrm{Id}$. Let*

$$\tilde{T} : \mathrm{Gr}(\alpha, V) \longrightarrow \mathrm{Gr}(\alpha, V)$$

be the map given by T. Then $(\mathrm{Gr}(\alpha, V), \tilde{T})$ is a real rational variety.

Proof. The involution T defines an $\mathbb{R}$-linear endomorphism of V. Since

$$v = \frac{v + T(V)}{2} + \frac{v - T(v)}{2},$$

we have $V = V_1 \oplus V_{-1}$, where V_i is the eigenspace of T corresponding to the eigenvalue i. Let $v_1, \ldots, v_n$ be a basis for V_1. Then $\sqrt{-1}v_1, \ldots, \sqrt{-1}v_n$ is a basis for V_{-1}, as T is conjugate linear. This shows that $v_1, \ldots, v_n$ is a $\mathbb{C}$-basis for V. In other words, $V \cong V_1 \otimes_{\mathbb{R}} \mathbb{C}$ and T acts by complex conjugation on the $\mathbb{C}$ factor. This evidently implies that $\mathrm{Gr}(\alpha, V) \cong \mathrm{Gr}(\alpha, V_1) \times_{\mathbb{R}} \mathbb{C}$, and $\tilde{T}$ acts by complex conjugation on the second factor. This proves the lemma. □

Remark 3.4. Let $K_{\mathbb{R}}$, $K_{\mathbb{C}}$ and σ_K be as in Remark 3.2. If in Lemma 3.3, W was a $K_{\mathbb{C}}$ vector space, and T was σ_K-antilinear, with $T^2 = \mathrm{Id}$, then the same argument would prove that

$$\mathrm{Gr}(\alpha, W) \xrightarrow{\sigma_K} \mathrm{Gr}(\alpha, W)$$

is a rational $K_{\mathbb{R}}$ variety.

4. Vector bundles on Hirzebruch surfaces

Let

$$X_{\mathbb{R}} := \mathrm{Proj}(\mathbb{R}[x_0, x_1, x_2]/x_0^2 + x_1^2 + x_2^2) \qquad \text{and} \qquad Y_{\mathbb{R}} := \mathbb{P}(\xi),$$

where ξ is a direct sum of line bundles on $X_{\mathbb{R}}$. The projective bundle $Y_{\mathbb{R}}$ remains unchanged if ξ is tensored with a line bundle. There is a unique line bundle ζ on $X_{\mathbb{R}}$ such that $\xi \otimes \zeta = \mathscr{O}_{X_{\mathbb{R}}} \oplus L$, where L is a line bundle on $X_{\mathbb{R}}$ with $\deg(L) \leq 0$. We will replace ξ by $\xi \otimes \zeta$. These projective bundles are called Hirzebruch surfaces. See Chapter 5 of [3] for a detailed exposition on Hirzebruch surfaces.

We recall the definition of the divisor C_0 on $\mathbb{P}(\xi)$. The inclusion of $\mathscr{O}_{X_{\mathbb{R}}}$ in $\mathscr{O}_{X_{\mathbb{R}}} \oplus L$ defines an irreducible effective divisor on $\mathbb{P}(\xi)$. This divisor will be denoted by C_0. Note that C_0 is defined over $\mathbb{R}$.

Note that $X_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = \mathbb{P}^1_{\mathbb{C}}$ and there is a natural map

$$\pi : Y_{\mathbb{R}} \longrightarrow X_{\mathbb{R}} .$$

Define $\xi_{\mathbb{C}} := \xi \otimes_{\mathbb{R}} \mathbb{C}$ and

$$Y_{\mathbb{C}} := Y_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} = \mathbb{P}(\xi_{\mathbb{C}}) .$$

The Picard group $\mathrm{Pic}(Y_{\mathbb{C}})$ is $\mathbb{Z} \oplus \mathbb{Z}$. The involution σ acts trivially on $\mathrm{Pic}(Y_{\mathbb{C}})$. Fix a very ample line bundle H on $Y_{\mathbb{C}}$. The *degree* of a torsionfree coherent sheaf W on $Y_{\mathbb{C}}$ is defined to be the degree of restriction of W to a smooth curve linearly equivalent to H.

We are interested in the moduli space of stable locally free sheaves on $Y_{\mathbb{C}}$. This moduli space comes with an involution which makes it into a variety defined over $\mathbb{R}$. This involution is given by $E \longmapsto \sigma^* E$.

Let F denote a fiber of the natural map $Y_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$, and define $c_1 := C_0 + \mathrm{d}F$. Let c_2, α, λ be integers satisfying

$$c_2 \equiv \alpha \mod r-1\,, \quad 0 < \alpha \leq r-1\,, \quad \lambda = \frac{c_2-\alpha}{r-1}$$

and

$$\Delta(r, c_1, c_2) := \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right) >> 0 .$$

Define

$$m := d - c_2 - 1 - \lambda .$$

Let $M_H(r, c_1, c_2)$ be the moduli space of stable locally free sheaves E on $Y_{\mathbb{C}}$ of rank r and $c_i(E) = c_i, i = 1, 2$. We will describe a birational model of space $M_H(r, c_1, c_2)$ as given in Remark 3.5 of [1].

Consider sheaves $E_{\alpha+1}$ satisfying the condition that there is a short exact sequence

$$0 \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}(C_0 + (d-c_2)F) \longrightarrow E_{\alpha+1} \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha} \longrightarrow 0 . \quad (4.1)$$

Let $V := H^1(Y_{\mathbb{C}}, \mathscr{O}_{Y_{\mathbb{C}}}(C_0 + mF))$. Isomorphism classes of non-split extensions of type (4.1) are parametrized by the projective space of lines in $V^{\oplus\alpha}$, which we will denote by P. Since

$$\mathrm{Hom}(\mathscr{O}_{Y_{\mathbb{C}}}(C_0 + (d-c_2)F)\,, \mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)) = 0 ,$$

from (4.1) we conclude that

$$\mathrm{Hom}(\mathscr{O}_{Y_{\mathbb{C}}}(C_0 + (d-c_2)F), E_{\alpha+1}) = \mathbb{C} .$$

Thus, if $E_{\alpha+1}$ sits in an extension of the above type, then this extension will be unique up to the natural action of $\mathrm{GL}_\alpha(\mathbb{C})$ on $\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha} = \mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F) \otimes_{\mathbb{C}} \mathbb{C}^\alpha$ (the group $\mathrm{GL}_\alpha(\mathbb{C})$ acts through the standard action on $\mathbb{C}^\alpha$).

Let $U \subset P$ be the open subset which consists of points $(v_1 : v_2 : \cdots : v_\alpha)$ such that $v_1, v_2, \ldots, v_\alpha$ are linearly independent. The group $\mathrm{GL}_\alpha(\mathbb{C})$ acts on U and

$$U/\mathrm{GL}_\alpha(\mathbb{C}) = \mathrm{Gr}(\alpha, V).$$

Points of $\mathrm{Gr}(\alpha, V)$ correspond to isomorphism classes of sheaves $E_{\alpha+1}$ which sit in extensions of type (4.1).

There is a universal sheaf $\mathscr{E}_{\alpha+1}$ over the space $P \times Y_{\mathbb{C}}$ which sits in a short exact sequence

$$\begin{aligned} 0 &\longrightarrow p_2^*\mathscr{O}_{Y_{\mathbb{C}}}(C_0 + (d-c_2)F) \longrightarrow \mathscr{E}_{\alpha+1} \\ &\longrightarrow p_2^*[\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha}] \otimes p_1^*\mathscr{O}_P(-1) \longrightarrow 0 \end{aligned} \tag{4.2}$$

(see [5]). Take any $A \in \mathrm{GL}_\alpha(\mathbb{C})$. We note that A gives rise to a map $A : P \longrightarrow P$. Pulling back the universal short exact sequence in (4.2) by $A \times \mathrm{Id}$ gives the following short exact sequence on $P \times Y_{\mathbb{C}}$:

$$\begin{aligned} 0 &\longrightarrow p_2^*\mathscr{O}_{Y_{\mathbb{C}}}(C_0 + (d-c_2)F) \longrightarrow (A \times Id)^*\mathscr{E}_{\alpha+1} \\ &\longrightarrow p_2^*[\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha}] \otimes p_1^*\mathscr{O}_P(-1) \longrightarrow 0\,. \end{aligned} \tag{4.3}$$

For any point $w \in P$, the restriction of the above exact sequence to $w \times Y_{\mathbb{C}}$ is the extension of type (4.1) corresponding to the point $A(w) \in P$.

Let $\mathscr{E}_1$ be the following pullback along the map A:

$$\begin{array}{ccc} \mathscr{E}_1 & \longrightarrow & p_2^*[\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha}] \otimes p_1^*\mathscr{O}_P(-1) \\ \big\downarrow & & \big\downarrow A \\ (A \times Id)^*\mathscr{E}_{\alpha+1} & \longrightarrow & p_2^*[\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha}] \otimes p_1^*\mathscr{O}_P(-1) \end{array}\,.$$

The fiber over the point $w \in P$ of $\mathscr{E}_1$ lies in the extension corresponding to the point $A^{-1}(A(w)) = w$. This shows that there is a natural isomorphism $\mathscr{E}_{\alpha+1} \longrightarrow \mathscr{E}_1$. It should be clarified that this natural isomorphism is not up to multiplication by some scalar. The composition

$$\mathscr{E}_{\alpha+1} \longrightarrow \mathscr{E}_1 \longrightarrow (A \times \mathrm{Id})^*\mathscr{E}_{\alpha+1}$$

gives an action of $\mathrm{GL}_\alpha(\mathbb{C})$ on $\mathscr{E}_{\alpha+1}$ which is compatible with the action on P.

Let

$$\mathscr{E} := (\mathscr{E}_{\alpha+1}|_U)/\mathrm{GL}_\alpha(\mathbb{C})$$

be the quotient sheaf on $\mathrm{Gr}(\alpha, V) \times Y_{\mathbb{C}}$. Consider the sheaf

$$\mathscr{F} := R^1 p_{1*}(\mathscr{E} \otimes p_2^*\mathscr{O}_{Y_{\mathbb{C}}}(-\lambda F))$$

on $\mathrm{Gr}(\alpha, V)$. For any point $x \in \mathrm{Gr}(\alpha, V)$, the fiber $\mathscr{F}_x$ is identified with

$$H^1(Y_{\mathbb{C}}, E_{\alpha+1} \otimes \mathscr{O}_{Y_{\mathbb{C}}}(-\lambda F))\,,$$

where $E_{\alpha+1}$ is the locally free sheaf corresponding to x. The vector space

$$H^1(Y_{\mathbb{C}}, E_{\alpha+1} \otimes \mathscr{O}_{Y_{\mathbb{C}}}(-\lambda F))^{\oplus r-\alpha-1}$$

parametrizes extensions of the type

$$0 \longrightarrow E_{\alpha+1} \stackrel{\beta}{\longrightarrow} E \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}(\lambda F)^{\oplus r-\alpha-1} \longrightarrow 0\,. \tag{4.4}$$

In [1] it is shown that for an extension as above,

$$\mathrm{Hom}(E_{\alpha+1}, E) = \mathbb{C}$$

and that if E sits in another extension

$$0 \longrightarrow E'_{\alpha+1} \stackrel{\beta'}{\longrightarrow} E \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}(\lambda F)^{\oplus r-\alpha-1} \longrightarrow 0\,,$$

then there is an isomorphism $\gamma : E_{\alpha+1} \to E'_{\alpha+1}$ such that $\beta = \beta' \circ \gamma$. Thus, points of $\mathrm{Gr}(r-\alpha-1, \mathscr{F})$ correspond to isomorphism classes of sheaves E which sit in an extension of type (4.4). This space is birational to $M_H(r, c_1, c_2)$.

Pulling back the short exact sequence (4.1) by σ, we get

$$0 \longrightarrow \sigma^*\mathscr{O}_{Y_{\mathbb{C}}}(C_0+(d-c_2)F) \longrightarrow \sigma^* E_{\alpha+1} \longrightarrow \sigma^*\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)^{\oplus\alpha} \longrightarrow 0.$$

There are natural isomorphisms

$$\sigma^*\mathscr{O}_{Y_{\mathbb{C}}}(C_0+(d-c_2)F) \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}(C_0+(d-c_2)F)$$

and

$$\sigma^*\mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F) \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}((1+\lambda)F)\,.$$

Using these we again get an extension of type (4.1). The functor σ^* takes U to itself and descends to an involution on $\mathrm{Gr}(\alpha, V)$; this involution of $\mathrm{Gr}(\alpha, V)$ will be denoted by $\widetilde{\sigma}$.

We describe an involution on the total space of the Grassmann bundle $\mathrm{Gr}(r-\alpha-1, \mathscr{F})$. A point of $\mathrm{Gr}(r-\alpha-1, \mathscr{F})$ in the fiber over $x \in \mathrm{Gr}(\alpha, V)$ corresponds to an extension of type (4.4). If we apply σ^* to it we get the short exact sequence

$$0 \longrightarrow \sigma^* E_{\alpha+1} \longrightarrow \sigma^* E \longrightarrow \sigma^*\mathscr{O}_{Y_{\mathbb{C}}}(\lambda F)^{\oplus r-\alpha-1} \longrightarrow 0\,.$$

There is a natural isomorphism $\sigma^*\mathscr{O}_{Y_{\mathbb{C}}}(\lambda F) \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}(\lambda F)$. Composing these we obtain an extension

$$0 \longrightarrow \sigma^* E_{\alpha+1} \longrightarrow \sigma^* E \longrightarrow \mathscr{O}_{Y_{\mathbb{C}}}(\lambda F)^{\oplus r-\alpha-1} \longrightarrow 0\,.$$

Let σ' be the involution of $\mathrm{Gr}(r-\alpha-1, \mathscr{F})$ obtained this way. This gives a birational model of the moduli space $M_H(r, c_1, c_2)$ along with its involution.

Note that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Gr}(r-\alpha-1, \mathscr{F}) & \stackrel{\sigma'}{\longrightarrow} & \mathrm{Gr}(r-\alpha-1, \mathscr{F}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(\alpha, V) & \stackrel{\widetilde{\sigma}}{\longrightarrow} & \mathrm{Gr}(\alpha, V) \end{array} \tag{4.5}$$

5. Rationality of moduli spaces

Let η denote the generic point of $\mathrm{Gr}(\alpha, V)$. We first describe the involution on

$$\mathscr{F}_\eta = H^1(\eta \times Y_{\mathbb{C}}, \mathscr{E}_\eta \otimes \mathscr{O}_{Y_{\mathbb{C}}}(-\lambda F))\,.$$

Recall that $\mathscr{E}$ was obtained as $(\mathscr{E}_{\alpha+1}|_U)/\mathrm{GL}_\alpha$, where $\mathscr{E}_{\alpha+1}$ is the universal extension on $P \times Y_{\mathbb{C}}$ as in (4.2). Let

$$W \ := \ V^{\oplus\alpha} \ = \ H^1(Y_{\mathbb{C}}, \mathscr{O}_{Y_{\mathbb{C}}}(C_0 + mF))^{\oplus\alpha}\,.$$

(i) If m is even, then there is a line bundle L on $Y_{\mathbb{R}}$ such that its pullback to $Y_{\mathbb{C}}$ is isomorphic to $\mathscr{O}_{Y_{\mathbb{C}}}(C_0 + mF)$. Denoting by

$$W_{\mathbb{R}} := H^1(Y_{\mathbb{R}}, L^{\oplus\alpha}),$$

we get

- $W = W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$
- $P = \mathbb{P}(W_{\mathbb{R}}^*) \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$.

We have the following natural isomorphisms

$$\begin{aligned}
&\operatorname{Ext}^1_{\mathbb{P}(W_{\mathbb{R}}^*)\times Y_{\mathbb{R}}}(p_1^*\mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)}(-1)^{\oplus\alpha}, p_2^*L)\\
&\quad\cong H^1(\mathbb{P}(W_{\mathbb{R}}^*) \times Y_{\mathbb{R}}, [p_1^*\mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)}(1) \otimes p_2^*L]^{\oplus\alpha})\\
&\quad\cong \operatorname{End}(W_{\mathbb{R}}).
\end{aligned}$$

There is a universal extension on $\mathbb{P}(W_{\mathbb{R}}^*) \times Y_{\mathbb{R}}$ corresponding to the identity in $\operatorname{End}(W_{\mathbb{R}})$,

$$0 \longrightarrow p_2^*L \longrightarrow \mathscr{G} \longrightarrow p_1^*\mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)}(-1)^{\oplus\alpha} \longrightarrow 0\,.$$

Remark 5.1. The pullback of $\mathscr{G}$ to $P \times Y_{\mathbb{C}}$ is $\mathscr{E}_{\alpha+1} \otimes p_2^*\mathscr{O}_{Y_{\mathbb{C}}}(-(1+\lambda)F)$. This proves that when m is even, $\mathscr{E}_{\alpha+1}$ is real (respectively, quaternionic) if λ is odd (respectively, even). Moreover, this shows that for m even, $\mathscr{F}$ is always quaternionic.

(ii) If m is odd and α is even, then there is a locally free sheaf E of rank 2 on $Y_{\mathbb{R}}$ such that its pullback to $Y_{\mathbb{C}}$ is isomorphic to $\mathscr{O}_{Y_{\mathbb{C}}}(-C_0 - mF)^{\oplus 2}$. Denoting by

$$W_{\mathbb{R}} := H^1(Y_{\mathbb{R}}, E^{*\oplus\alpha/2}),$$

we get

- $W = W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$
- $P = \mathbb{P}(W_{\mathbb{R}}^*) \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$.

We have the following natural isomorphisms

$$\begin{aligned}
&\operatorname{Ext}^1_{\mathbb{P}(W_{\mathbb{R}}^*)\times Y_{\mathbb{R}}}([p_1^*\mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)}(-1) \otimes p_2^*E]^{\oplus\alpha/2}, \mathscr{O})\\
&\quad\cong H^1(\mathbb{P}(W_{\mathbb{R}}^*) \times Y_{\mathbb{R}}, [p_1^*\mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)}(1) \otimes p_2^*E^*]^{\oplus\alpha/2})\\
&\quad\cong \operatorname{End}(W_{\mathbb{R}}).
\end{aligned}$$

There is a universal extension on $\mathbb{P}(W_{\mathbb{R}}^*) \times Y_{\mathbb{R}}$ corresponding to the identity in $\operatorname{End}(W_{\mathbb{R}})$,

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)\times Y_{\mathbb{R}}} \longrightarrow \mathscr{G} \longrightarrow [p_1^*\mathscr{O}_{\mathbb{P}(W_{\mathbb{R}}^*)}(-1) \otimes p_2^*E]^{\oplus\alpha/2} \longrightarrow 0\,.$$

Remark 5.2. The pullback of $\mathscr{G}$ to $P \times Y_{\mathbb{C}}$ is isomorphic to $\mathscr{E}_{\alpha+1} \otimes p_2^*\mathscr{O}_{Y_{\mathbb{C}}}(-C_0 - (d - c_2)F)$. This implies that when m is odd and α is even, then $\mathscr{E}_{\alpha+1}$ is real (respectively, quaternionic) if $d - c_2$ is even (respectively, odd). Moreover, since $\lambda = d - c_2 - m - 1$, this shows that in this case $\mathscr{F}$ is real.

Let $X_{\mathbb{R}} := \text{Proj}(\mathbb{R}[x_0, x_1, x_2]/x_0^2 + x_1^2 + x_2^2)$. Let ξ be a locally free sheaf of rank two on $X_{\mathbb{R}}$ which is a direct sum of line bundles on $X_{\mathbb{R}}$. Define

$$Y_{\mathbb{R}} := \mathbb{P}(\xi)\,.$$

Let $Y_{\mathbb{C}}$ (respectively, $X_{\mathbb{C}}$) denote the base change of $Y_{\mathbb{R}}$ (respectively, $X_{\mathbb{R}}$) to the field $\mathbb{C}$. Let F denote a fiber of the natural map $Y_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$. Let $c_1 \in \text{Pic}(Y_{\mathbb{C}})$ be

$$c_1 := C_0 + \mathrm{d}F\,.$$

Let c_2, α and λ be integers satisfying

$$c_2 \equiv \alpha \mod r-1\,, \quad 0 < \alpha \leq r-1\,, \quad \lambda = \frac{c_2 - \alpha}{r-1}$$

and

$$\Delta(r, c_1, c_2) := \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right) >> 0\,.$$

Define

$$m := d - c_2 - 1 - \lambda\,.$$

Theorem 5.3. *The moduli space $M_H(r, c_1, c_2)$ is rational as a real variety if and only if one of the following holds*:

- *m is even and $r - \alpha - 1$ is even.*
- *m is odd and α is even.*

Proof. We have already seen that C_0 is defined over $\mathbb{R}$.

First assume that m is even.

The line bundle $\mathscr{O}_{Y_{\mathbb{C}}}(mF)$ is defined over $\mathbb{R}$ and so the vector space

$$V = H^1(Y_{\mathbb{C}}, \mathscr{O}_{Y_{\mathbb{C}}}(C_0 + mF))$$

has a conjugate linear endomorphism T such that $T^2 = \text{Id}$. Now, Lemma 3.3 tells us that $(\text{Gr}(\alpha, V)\,, T)$ is a real rational variety.

We need to check the rationality of $\text{Gr}(r - \alpha - 1, \mathscr{F}_\eta)$.

For that, in view of Remarks 3.2 and 5.1, $\text{Gr}(r - \alpha - 1, \mathscr{F}_\eta)$ is rational if and only if $r - \alpha - 1$ is even. If $r - \alpha - 1$ is odd, then the fibers of $\text{Gr}(r - \alpha - 1, \mathscr{F}_\eta)$ over real points of $(\text{Gr}(\alpha, V), T)$ does not have any real points. Hence in this case $M_H(r, c_1, c_2)$ is not rational.

Now assume that m is odd.

The vector space $V = H^1(Y_{\mathbb{C}}, \mathscr{O}_{Y_{\mathbb{C}}}(C_0 + mF))$ has a conjugate linear endomorphism T such that $T^2 = -\text{Id}$. From Lemma 3.1 it follows that $(\text{Gr}(\alpha, V), T)$ is rational if and only if α is even. Now Remarks 3.4 and 5.2 show that $\text{Gr}(r - \alpha - 1, \mathscr{F}_\eta)$ is rational. If

α is odd, then $\mathrm{Gr}(r - \alpha - 1, \mathscr{F}_\eta)$ has no real points because $(\mathrm{Gr}(\alpha, V), T)$ has no real points. □

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