Proc. Indian Acad. Sci. (Math. Sci.) Vol. 123, No. 2, May 2013, pp. 193–201.

On partial sums of a spectral analogue of the Möbius function

KALYAN CHAKRABORTY[1] and MAKOTO MINAMIDE[2]

[1]Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211 019, India
[2]Faculty of Science, Kyoto Sangyo University, Kamigamo, Kita-ku, Kyoto 603-8555, Japan
E-mail: kalyan@hri.res.in; minamide@cc.kyoto-su.ac.jp

MS received 11 February 2012; revised 4 September 2012

Abstract. Sankaranarayanan and Sengupta introduced the function $\mu^*(n)$ corresponding to the Möbius function. This is defined by the coefficients of the Dirichlet series $1/L_f(s)$, where $L_f(s)$ denotes the L-function attached to an even Maaß cusp form f. We will examine partial sums of $\mu^*(n)$. The main result is $\sum_{n\le x}\mu^*(n) = O(x\exp(-A\sqrt{\log x}))$, where A is a positive constant. It seems to be the corresponding prime number theorem.

Keywords. Riemann zeta function; Maaß forms; L-functions.

1. Introduction

It is well known that the prime number theorem is equivalent to the following estimation for the partial sums of the Möbius function $\mu(n)$,

$$M(x) := \sum_{n\le x}\mu(n) = o(x), \quad \text{as } x\to\infty.$$

It was conjectured by Mertens that $|M(n)| < \sqrt{n}$ $(n\ge 2)$, which was later disproved by Odlyzko and te Riele [11]. However it is known that a weaker estimate

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right), \quad \text{for any } \epsilon > 0$$

is equivalent to the Riemann hypothesis, which is due to Littlewood [7]. In this paper we shall consider a spectral analogue of the partial summations for a certain analogue of $\mu(n)$, which was defined by Sankaranarayanan and Sengupta [12].

Let $f(z)$, $z = x+iy$, $x\in\mathbb{R}$, $y>0$ be an even Maaß cusp form with respect to $\Gamma = PSL(2,\mathbb{Z})$, which is also a Hecke eigenfunction. Such an f has Fourier expansion of the form

$$f(z) = \sum_{n=1}^{\infty} a(n)\sqrt{y}K_{ir}(2\pi ny)\cos(2\pi nx), \tag{1}$$

where $K_{ir}(\cdot)$ is the Bessel function. The L-function attached to such an f is defined by

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(1 - \frac{a(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}, \quad \mathrm{Re}\,(s) > 1, \tag{2}$$

where $a(n)$ are the Fourier coefficients in (1). It is well-known that $L_f(s)$ is analytically continued to the whole $\mathbb{C}$ plane, and it has no poles. To study the zero-density estimate of $L_f(s)$, Sankaranarayanan and Sengupta introduced the function $\mu^*(n)$ (p. 275 of [12]), where

$$\frac{1}{L_f(s)} = \sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s}, \quad \mathrm{Re}\,(s) > 1.$$

We note down the following properties of $\mu^*(n)$, which can easily be derived by using the Euler product (2):

(A) μ^* is a multiplicative function.
(B) For any prime p,

$$\mu^*\left(p^i\right) = \begin{cases} 1, & i = 0, \\ -a(p), & i = 1, \\ 1, & i = 2, \\ 0, & i \geq 3. \end{cases}$$

As $\mathrm{Re}\,(s) > 1$ one has $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$. The function $\mu^*(n)$ can be thought of as a spectral analogue of $\mu(n)$. Here we shall discuss the partial sums of $\mu^*(n)$:

$$M^*(x) := \sum_{n \leq x} \mu^*(n).$$

Using the above properties and $\sum_{n \leq x} |a(n)|^2 \ll x$, we easily get the following.

PROPOSITION 1.1

(1) $\sum_{n \leq x} |\mu^*(n)|^2 = O(x)$ *as* $x \to \infty$.
(2) $|M^*(x)| \leq \sum_{n \leq x} |\mu^*(n)| = O(x)$ *as* $x \to \infty$.
(3) $\left|\frac{1}{L_f(s)}\right| \leq \sum_{n=1}^{\infty} |\mu^*(n)| n^{-\sigma} = O\left(\frac{1}{\sigma - 1}\right)$ *as* $\sigma \to 1^+$.

The main theorem of this paper is an improvement of the second estimate of this proposition.

Theorem 1.2. *There is a positive constant* $A > 0$ *satisfying*

$$M^*(x) = O(x \exp(-A\sqrt{\log x})),$$

as $x \to \infty$.

This is a corresponding theorem of the prime number theorem in our set up. Moreno in [8] was concerned with analogous prime number theorems in the set up of modular forms and he discussed the partial sums of the coefficient of the Dirichlet series $-L'/L(s)$, where L denotes the L-function attached to a cusp form. We use the Perron formula due to Liu and Ye [6] to obtain Theorem 1.2. We also prove the following equivalence of the generalized Riemann hypothesis (GRH) for $L_f(s)$.

Theorem 1.3.

(1) *GRH* $\Longleftrightarrow \sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s}$ *converges for* $\mathrm{Re}\,(s) > 1/2$.
(2) *GRH* $\Longleftrightarrow \sum_{n\le x} \mu^*(n) = O(x^{\frac{1}{2}+\epsilon})$ *for any* $\epsilon > 0$.

This is a kind of a non-holomorphic version of Goldstein's work (p. 212, Theorem 3.5 of [3]). An analogue of Möbius function is defined in the above mentioned paper for a cusp form of weight k associated Γ. The results (p. 211, Propositions 3.3 and 3.4 of [3]) are similar to the above properties (1) and (2), and we use similar method as in [3] to prove the above theorem.

It is well known that the estimate $M(x) = o(x)$ is equivalent to the formula $\psi(x) := \sum_{n\le x} \Lambda(n) \sim x$, where $\Lambda(n)$ is the von Mongoldt function. In our case, we will obtain a similar result. We shall write

$$-\frac{L_f'}{L_f}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)c(n)}{n^s}, \quad \mathrm{Re}\, s > 1,$$

and define $\psi^*(n)$ by

$$\psi^*(x) := \sum_{n\le x} \Lambda(n)c(n).$$

We have the following theorem.

Theorem 1.4. *The estimate* $M^*(x) = o(x)$ *is equivalent to* $\psi^*(x) = o(x)$.

2. Basic properties of $L_f(s)$

First of all, we shall prove Theorem 1.3. For this we review some basic properties of $L_f(s)$.

Lemma 2.1. *For* $\mathrm{Re}\,(s) = \sigma > 1$, *we have*

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = O\left(\frac{1}{\sigma - 1}\right).$$

This is easily proved by the partial summation technique. By using the estimate $a(p) \ll p^{1/2}$ we obtain the following lemma.

Lemma 2.2. *For* $\mathrm{Re}\,(s) > 1$, *one has*

$$\log L_f(s) = O\left(\frac{1}{\sigma - 1}\right).$$

The functional equation of $L_f(s)$ and the Phragmén–Lindelöf theorem give the following estimate.

Lemma 2.3 (p. 116, Proposition 2.1 of [2]). *For any* $\epsilon > 0$, *and* $1/2 \le \sigma \le 1$ *we have*

$$|L_f(\sigma + it)| \ll (|t| + 2)^{1-\sigma+\epsilon}.$$

We also need the following zero-free region of $L_f(s)$.

Lemma 2.4 (p. 135, Theorem 5.39 of [5]). *There exist positive constants* c_0 *and* $|t_0|$ *which satisfy*

$$L_f(s) \neq 0 \quad \textit{for} \operatorname{Re}(s) > 1 - \frac{c_0}{\log |t|} \quad \textit{and} \quad |t| > t_0.$$

3. Partial sums of $\mu^*(n)$ with GRH

In this section we assume GRH, i.e. $L_f(s)$ has no zeros for $\sigma > 1/2$. Hence we remark that $\log L_f(s)$ is holomorphic for $\sigma > 1/2$. Following Titchmarsh (Ch. 14 of [13]), we can prove Theorem 1.2. To do that, firstly we get an estimate of $\log L_f(\sigma + it)$ for $\sigma > 1/2$.

Theorem 3.1. *Assume the truth of GRH. Then for any* $\epsilon > 0$ *we have*

$$\log L_f(\sigma + it) = O((\log t)^{2-2\sigma+\epsilon})$$

for $\sigma > 1/2$ *and* $|t| > t_0 > 0$. *From this estimate we see that*

$$\begin{aligned} L_f(\sigma + it) &= O\left(|t|^{\epsilon}\right), \quad |t| > t_0 > 0, \\ \frac{1}{L_f(\sigma + it)} &= O\left(|t|^{\epsilon}\right), \quad |t| > t_0 > 0. \end{aligned}$$

Proof. Using Lemmas 2.1, 2.2, 2 and proceeding as in pp. 336–337 of [13] one can complete the proof.

The following estimate $\mu^*(n)$ will be used later.

Lemma 3.2. *For any* $n \in \mathbb{N}$, *we have*

$$\mu^*(n) = O(n^{1/2}).$$

This could be easily shown by using the multiplicative property of $\mu^*(n)$ and $a(p) \ll p^{1/2}$.

Proof (*Proof of Theorem* 1.3).

(1) We shall apply the formula (p. 60, Lemma 3.12 of [13]) to $1/L_f(s) = \sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s}$ and use Lemma 3.2 and obtain

$$\sum_{n<x} \frac{\mu^*(n)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^{s+w}} \frac{x^w}{w} \mathrm{d}w + O\left(\frac{x^2}{T}\right).$$

Now using the residue theorem for the rectangle with the vertexes $2 - iT, 2 + iT, \frac{1}{2} - \sigma + \delta + iT, \frac{1}{2} - \sigma + \delta - iT$ $(0 < \delta < \sigma - \frac{1}{2})$, GRH, and Theorem 3.1, for any $\epsilon > 0$ we have

$$\frac{1}{L_f(s)} = \sum_{n<x} \frac{\mu^*(n)}{n^s} + O\left(\frac{T^{\epsilon}}{T} x^2\right) + O(x^{\frac{1}{2}-\sigma+\delta} T^{\epsilon}).$$

Now, if $T = x^3$ and $x \to \infty$, then, the series $\sum_{n=1}^{\infty} \mu^*(n)n^{-s}$ converges for $\sigma > 1/2$. Conversely, assume the above. Since it is uniformly convergent for $\sigma \geq \sigma_0 > 1/2$, also analytic for $\sigma > 1$, one has $1/L_f(s)$ is analytic for $\sigma > 1/2$, that is, GRH holds.
(2) By the previous Perron's formula, again, for $x = N + 1/2$ and $\delta > 0$ we can get

$$\begin{aligned}\sum_{n<x} \mu^*(n) &= \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{L_f(w)}\frac{x^w}{w}\mathrm{d}w + O\left(\frac{x^2}{T}\right)\\ &= \frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+\delta+iT} + \int_{\frac{1}{2}+\delta+iT}^{\frac{1}{2}+\delta-iT} + \int_{\frac{1}{2}+\delta-iT}^{2-iT}\right)\frac{1}{L_f(w)}\frac{x^w}{w}\mathrm{d}w\\ &\quad + O\left(\frac{x^2}{T}\right).\end{aligned}$$

Estimating these integrals with GRH and Theorem 3.1 we have

$$\sum_{n<x} \mu^*(n) = O\left(\frac{T^\epsilon}{T}x^2\right) + O\left(T^\epsilon x^{\frac{1}{2}+\delta}\right).$$

Choose $T = x^2$, and by Lemma 3.2 the assertion is proved. Conversely, if the above estimate holds, by the partial summation technique it is shown that $\sum_{n=1}^{\infty} \mu^*(n)n^{-s}$ converges for $\sigma > 1/2$. Therefore $1/L_f(s)$ is analytic for $\sigma > 1/2$ and thus holds truth of the GRH. □

4. Prime number theorem

In order to prove our prime number theorem, we need an estimation of $1/L_f(s)$ near $s = 1$. To lead to the desired estimate we follow the argument of Montgomery and Vaughan (pp. 170–171 of [9]) on the estimate of $1/\zeta(s)$.

Lemma 4.1 (cf. p. 171, Lemma 6.4 of [9]). *For* $s = \sigma + it$ *which satisfies* $5/6 \leq \sigma \leq 2$ *and* $|t| > 2$ *we have*

$$\frac{L_f'}{L_f}(s) = \sum_\rho \frac{1}{s-\rho} + O\left(\log(|t|+2)\right), \tag{3}$$

where the summation is extended over all zeros ρ *of* $L_f(s)$ *satisfying* $|\rho - (3/2+it)| \leq 5/6$.

Applying (3) and the result on the zero-free region of $L_f(s)$ we show the following.

Lemma 4.2. *There exist positive constants* c_1, c_2, c_3 *and* t_0 *satisfying*

$$\left|\frac{L_f'}{L_f}\left(1 + \frac{1}{\log|t|} + it\right)\right| \leq \log|t| + O(1) \quad \textit{for } |t| > t_0, \tag{4}$$

$$\left|\frac{L_f'}{L_f}(s)\right| = O\left(\log|t|\right) \quad \textit{for } \mathrm{Re}\,(s) > 1 - \frac{c_1}{\log|t|},\ |t| > t_0 \tag{5}$$

$$\left|\log L_f(s)\right| \leq \log\log|t| + O(1) \quad \textit{for } \mathrm{Re}\,(s) > 1 - \frac{c_2}{\log|t|},\ \ |t| > t_0 \tag{6}$$

$$\left|\frac{1}{L_f(s)}\right| = O\left(\log|t|\right) \quad \textit{for } \mathrm{Re}\,(s) > 1 - \frac{c_3}{\log|t|},\ |t| > t_0. \tag{7}$$

Proof. By the logarithmic derivative of (2) for $\mathrm{Re}(s) > 1$ and using the estimates $\alpha(p),\ \beta(p) \ll p^{1/5}$ [10], we have

$$-\frac{L'_f}{L_f}(s) = \sum_p \frac{a(p)\log p}{p^s} + O(1).$$

Now the Cauchy–Schwarz inequality implies that for $\sigma > 1$,

$$\left|\frac{L'_f}{L_f}(s)\right| \le \left(\sum_p \frac{|a(p)|^2 \log p}{p^\sigma}\right)^{\frac{1}{2}} \left(\sum_p \frac{\log p}{p^\sigma}\right)^{\frac{1}{2}} + O(1). \tag{8}$$

Next we shall estimate the above two Dirichlet series. Note that the series $\sum_p |a(p)|^2(\log p)p^{-\sigma}$ appears in the logarithmic derivative of

$$\begin{aligned} L_{f\otimes\bar{f}}(s) &= \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s} \\ &= \frac{1}{\zeta(2s)} \prod_p \left(1 - \frac{\alpha(p)\overline{\alpha(p)}}{p^s}\right)^{-1} \left(1 - \frac{\alpha(p)\overline{\beta(p)}}{p^s}\right)^{-1} \\ &\quad \times \left(1 - \frac{\beta(p)\overline{\alpha(p)}}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\overline{\alpha(p)}}{p^s}\right)^{-1} \end{aligned}$$

for $\sigma > 1$. In fact, using the estimates $\alpha(p),\ \beta(p) \ll p^{1/5}$, we get

$$-\frac{L'_{f\otimes\bar{f}}}{L_{f\otimes\bar{f}}}(s) = \sum_p \frac{|a(p)|^2\log p}{p^s} + O(1).$$

Hence,

$$\left|\sum_p \frac{|a(p)|^2\log p}{p^s}\right| \le \left|\frac{L'_{f\otimes\bar{f}}}{L_{f\otimes\bar{f}}}(s)\right| + O(1).$$

Since $L_{f\otimes\bar{f}}(s)$ has a simple pole at $s = 1$, if we put $s = 1 + \frac{1}{\log|t|}$, then we get

$$\sum_p \frac{|a(p)|^2\log p}{p^{1+\frac{1}{\log|t|}}} \le \log|t| + O(1) \quad \text{for } |t| > 2. \tag{9}$$

On the other hand, for $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(s) = \sum_p \frac{\log p}{p^s} + O(1)$$

and $\zeta(s)$ has the simple pole at $s = 1$. These facts lead to

$$\sum_p \frac{\log p}{p^{1+\frac{1}{\log|t|}}} \le \left|\frac{\zeta'}{\zeta}\left(1 + \frac{1}{\log|t|}\right)\right| + O(1) \le \log|t| + O(1). \tag{10}$$

Combining (8), (9) and (10), we have (4). The assertion (5) is deduced from the usual argument together with (4), Lemmas 4 and 2.

We note that

$$\log L_f\left(1+\frac{1}{\log|t|}\right) = -\sum_p \frac{1}{p^{1+\frac{1}{\log|t|}}} + O(1),$$

i.e.

$$\left|\log L_f\left(1+\frac{1}{\log|t|}\right)\right| \le \left(\sum_p \frac{1}{p^{1+\frac{1}{\log|t|}}}\right)^{1/2} \left(\sum_p \frac{|a(p)|^2}{p^{1+\frac{1}{\log|t|}}}\right)^{1/2} + O(1). \tag{11}$$

On the other hand, $\log L_{f\otimes\bar{f}}(s)$ is expressed for $\mathrm{Re}\,(s) > 1$ as

$$\log L_{f\otimes\bar{f}}(s) = \sum_p \frac{|a(p)|^2}{p^s} + O(1) \tag{12}$$

by the Euler product and the estimates $\alpha(p), \beta(p) \ll p^{1/5}$. If $s = 1 + \frac{1}{\log|t|}$, then

$$\sum_p \frac{|a(p)|^2}{p^{1+\frac{1}{\log|t|}}} = \log L_{f\otimes\bar{f}}\left(1+\frac{1}{\log|t|}\right) + O(1).$$

Using the fact that $L_{f\otimes\bar{f}}(s)$ and $\zeta(s)$ have simple poles at $s = 1$, we get

$$\sum_p \frac{|a(p)|^2}{p^{1+\frac{1}{\log|t|}}} \le \log\log|t| + O(1), \quad \sum_p \frac{1}{p^{1+\frac{1}{\log|t|}}} = \log\log|t| + O(1), \tag{13}$$

respectively. Now combining (11), (12) and (13) we get

$$\left|\log L_f\left(1+\frac{1}{\log|t|}\right)\right| \le \log\log|t| + O(1). \tag{14}$$

Let $s = \sigma + it$ be such that $1 - \frac{c}{\log|t|} \le \sigma \le 1 + \frac{1}{\log|t|} = s_0$ and $|t| > t_0$ is sufficiently large. By (4) and (14) we have the assertions (6) and (7).

To prove our main theorem (Theorem 1.2), we shall use the following version of the Perron formula due to Liu and Ye [6].

Theorem 4.3 (p. 483, Theorem 2.1 of [6]). *Let $D(s)$ be a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ which converges absolutely for $\mathrm{Re}\,(s) > \sigma_a$, and $B(\sigma) := \sum_{n=1}^{\infty} |a_n| n^{-\sigma}$. Then, for $b > \sigma_a$, $x \ge 2$, $T \ge 2$ and $H \ge 2$ we have*

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} D(s)\frac{x^s}{s}\mathrm{d}s + O\left(\sum_{x-x/H<n\le x+x/H} |a_n|\right) + O\left(\frac{x^b H B(b)}{T}\right).$$

Proof (*Proof of Theorem* 1.2). We shall apply this theorem to the case $D(s) = 1/L_f(s)$, $b = 1 + 1/\log x$ and $H = \sqrt{T}$. Then

$$\sum_{n \le x} \mu^*(n) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{1}{L_f(w)} \frac{x^w}{w} \mathrm{d}w + O\left(\frac{x}{T^{1/4}}\right) + O\left(\frac{x \log x}{T^{1/2}}\right). \tag{15}$$

Here we remark that in our situation there are no Siegel's zeros [4]. The above integral is estimated as $O(x(\log T)^2 \exp(-c \log x / \log T))$ by Lemmas 2 and 4.2. Hence, when $T = \exp((\log x)^{1/2})$, we have the assertion of Theorem 1.2. □

The assertion of Theorem 1.2 gives the following corollary.

COROLLARY 4.4

For $\mathrm{Re}\,(s) = 1$, *the series* $\sum_{n=1}^{\infty} \mu^*(n) n^{-s}$ *converges. Especially, using the integral expression of* $L_f(s)$ *we have*

$$\sum_{n=1}^{\infty} \frac{\mu^*(n)}{n} = \frac{1}{L_f(1)} = \frac{\Gamma\left(\frac{1+ir}{2}\right)\Gamma\left(\frac{1-ir}{2}\right)}{4\pi} \frac{1}{\int_0^\infty \frac{f(iy)}{y^{1/2}} \mathrm{d}y}.$$

Finally, we shall prove Theorem 1.4. We recall the following well known relations:

$$-\mu^*(n) \log n = \sum_{d|n} \mu^*(d) \Lambda\left(\frac{n}{d}\right) c\left(\frac{n}{d}\right),$$

$$\Lambda(n) c(n) = \sum_{d|n} \mu^*\left(\frac{n}{d}\right) a(d) \log d. \tag{16}$$

The above relations are obtained by Dirichlet convolutions $(1/L_f(s))' = (-L'_f/L_f(s))(1/L_f(s))$ and $-L'_f/L_f(s) = (1/L_f(s))(-L'_f(s))$.

Proof (*Proof of Theorem* 1.4). We assume $\psi^*(x) = o(x)$ and put $H^*(x) = \sum_{n \le x} \mu^*(n) \log n$. Partial summation for $H^*(x)$ implies

$$\lim_{x \to \infty} \left(\frac{M^*(x)}{x} - \frac{H^*(x)}{x \log x} \right) = 0. \tag{17}$$

On the other hand, (16) gives

$$-H^*(x) = \sum_{n \le x} \mu^*(n) \psi^*\left(\frac{x}{n}\right) = o(x \log x).$$

Hence, by (17) we obtain $M^*(x) = o(x)$.

Next we assume $M^*(x) = o(x)$. By (16) we have

$$\psi^*(x) = \sum_{k \le K} (a(k) \log k)\, M^*\left(\frac{x}{k}\right) + \sum_{d \le \frac{x}{K}} \mu^*(d)\, (a(m) \log m) - \sum_{l \le x} (a(l) \log l)\, M^*\left(\frac{x}{K}\right),$$

where K satisfies $K \le x$. Using the sharp estimate $\sum_{n \le x} a(n) \ll x^{1/2+\epsilon}$ (p. 118, Theorem 3.1 of [2]) our assertion is proved (the details of the proof is similar to the argument in p. 32 of [5] or in pp. 91–97 of [1]). □

Acknowledgements

The authors are grateful to Professor Jianya Liu of Shandong University for providing his notes on results of zero-free regions for L-functions attached to Maaß forms. The authors express their gratitude to Prof. Yoshio Tanigawa and Dr. Yashiro of Nagoya University for their seminars which provided hints to the theme of this paper. The essential part of this paper was obtained during the second author's visit to Harish-Chandra Research Institute (HRI). The second author would like to thank the hospitality of the people who work for HRI. Finally the second author expresses his sincere thanks to Mr. Rian Chakraborty for his kindness and hospitality. The second author is supported by JSPS No. 23740009.

References

[1] Apostol T M, Introduction to Analytic Number Theory (1976) (Springer)

[2] Epstein C, Hafner J L and Sarnak P, Zeros of L-functions attached to Maaß forms, *Math. Z.* **190** (1985) 113–128

[3] Goldstein L J, A necessary and sufficient condition for the Riemann hypothesis for zeta functions attached to eigenfunctions of the Hecke operators, *Acta Arith.* **15** (1968/1969) 205–215

[4] Hoffstein J and Ramakrishnan D, Siegel zeros and cusp forms, *IMRN* **6** (1995) 279–308

[5] Iwaniec H and Kowalski E, Analytic Number Theory, AMS Coll. Publ. 53 (2004)

[6] Liu J and Ye Y, Perron's formula and the prime number theorem for automorphic L-functions, *Pure Appl. Math. Quart.* **3** (2007) 481–497

[7] Littlewood J E, Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $\mathbf{R}(s) > \frac{1}{2}$, *Comp. Rend. Acad. Sci. (Paris)* **154** (1912) 263–266

[8] Moreno C J, Prime number theorems for the coefficients of modular forms, *Bull. Am. Math. Soc.* **78** (1972) 796–798

[9] Montgomery H L and Vaughan R C, Multiplicative Number Theory I. Classical Theory (2007) (Cambridge Univ. Press)

[10] Murty M R, On the estimation of eigenvalues of Hecke operators, *Rocky Mountain J. Math.* **15** (1985) 521–533

[11] Odlyzko A M and te Riele H J, Disproof of the Mertens conjecture, *J. Reine Angew. Math.* **357** (1985) 138–160

[12] Sankaranarayanan A and Sengupta J, Zero-density estimate of L-functions attached to Maaß forms, *Acta Arith.* **127** (2007) 273–284

[13] Titchmarsh E C, The Theory of the Riemann Zeta-Function, 2nd ed. revised by D R Heath-Brown (1986) (Oxford Univ. Press)