Proc. Indian Acad. Sci. (Math. Sci.) Vol. 123, No. 2, May 2013, pp. 177–192.

A new generalization of Hardy–Berndt sums

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MS received 17 February 2012

Abstract. In this paper, we construct a new generalization of Hardy–Berndt sums which are explicit extensions of Hardy–Berndt sums. We express these sums in terms of Dedekind sums $s_r(h, k : x, y|\lambda)$ with $x = y = 0$ and obtain corresponding reciprocity formulas.

Keywords. Dedekind sums; Hardy–Berndt sums; Bernoulli numbers and polynomials.

1. Introduction

The three of the six different arithmetic sums, called Hardy or Hardy–Berndt sums [3,11], are defined by

$$S(h,k) = \sum_{j=1}^{k-1} (-1)^{j+1+[hj/k]},$$

$$s_3(h,k) = \sum_{j=1}^{k-1} (-1)^j \left(\left(\frac{hj}{k}\right)\right),$$

$$s_4(h,k) = \sum_{j=1}^{k-1} (-1)^{[hjk]},$$

where h and k are integers with $k > 0$, $[x]$ denotes the greatest integer not exceeding x and $((x)) = 0$ or $x - [x] - \frac{1}{2}$, according as x is or is not an integer, respectively. In this paper we only study on the sums given above. Details of other sums can be found in [3,4,11,18]. Berndt and Goldberg [4] found analytic properties of these sums and established infinite trigonometric series representations for them. The most important properties of Hardy–Berndt sums are reciprocity theorems due to Berndt [3] and Goldberg [11]. These sums were studied in [2–5,11,13–15,17,18,20]. In [18], Sitaramachandrarao explicitly deduced each of the Hardy–Berndt sums in terms of the Dedekind sums $s(h,k)$ defined by

$$s(h,k) = \sum_{j=1}^{k-1} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right)$$

(for basic properties, see [16]). Dedekind sums were generalized by various mathematicians [1,7,9,16,19] and the corresponding reciprocity laws were obtained. One of these generalizations, due to Apostol [1], is

$$s_r(h,k)=\sum_{j=1}^{k-1}\frac{j}{k}\bar{B}_r\left(\frac{hj}{k}\right).$$

Here $\bar{B}_n(x)$ is the n-th Bernoulli function defined by

$$\begin{aligned}\bar{B}_n(x) &= B_n(\{x\}), \qquad \text{if } n>1\\ \bar{B}_1(x) &= \begin{cases} B_1(\{x\}), & x\notin\mathbb{Z},\\ 0, & x\in\mathbb{Z},\end{cases}\end{aligned} \tag{1.1}$$

where $\{x\}$ denotes fractional part of a real number x, and $B_n(x)$ is the n-th Bernoulli polynomial given by means of the generating function

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\qquad |t|<2\pi$$

where $B_n(0)=B_n$ is the n-th Bernoulli number.

Cenkci *et al* [9] generalized Carlitz's, Takacs' and of course Apostol's Dedekind sums by defining $s_r(h,k:x,y|\lambda)$ by

$$s_r(h,k:x,y|\lambda)=\sum_{j=0}^{|k|-1}\bar{\beta}_1\left(\lambda,\frac{j+y}{k}\right)\bar{\beta}_r\left(\lambda,h\frac{j+y}{k}+x\right),$$

which is called degenerate Dedekind sums, and obtained reciprocity laws for these sums as

$$\begin{aligned}&(r+1)\{hk^r s_r(h,k:x,y|h\lambda)+kh^r s_r(k,h:y,x|k\lambda)\}\\ &\quad=\sum_{j=0}^{r+1}\binom{r+1}{j}h^j\bar{\beta}_j(k\lambda,y)k^{r+1-j}\bar{\beta}_{r+1-j}(h\lambda,x)\\ &\qquad+\frac{\lambda}{2}hkd^r(r+1)(h+k+2r-2)\bar{\beta}_r\left(\frac{hk}{d}\lambda,\frac{hy+kx}{d}\right)\\ &\qquad+rd^{r+1}\bar{\beta}_{r+1}\left(\frac{hk}{d}\lambda,\frac{hy+kx}{d}\right),\end{aligned} \tag{1.2}$$

where d stands for the greatest common divisor of h and k. Here $\bar{\beta}_n(\lambda,x)$ is the n-th generalized Bernoulli function defined by

$$\bar{\beta}_n(\lambda,x)=\beta_n(\lambda,\{x\}) \tag{1.3}$$

and, $\beta_n(\lambda,x)$ is the n-th generalized (or degenerate) Bernoulli polynomial defined by means of generating function [8]

$$\frac{t}{(1+\lambda t)^{\mu}-1}(1+\lambda t)^{x\mu}=\sum_{n=0}^{\infty}\beta_n(\lambda,x)\frac{t^n}{n!}, \tag{1.4}$$

where $\lambda \neq 0$ and $\mu = \frac{1}{\lambda}$. For $x = 0$, we have $\beta_n(\lambda, 0) = \beta_n(\lambda)$ called the generalized Bernoulli numbers and for $\lambda \to 0$ we have $\beta_n(0, x) = B_n(x)$ ordinary Bernoulli polynomials and $\beta_n(0) = B_n$ ordinary Bernoulli numbers. Similar to Bernoulli functions, $\bar{\beta}_n(\lambda, x)$ satisfies Raabe's theorem

$$\sum_{j=0}^{m-1} \bar{\beta}_n\left(\lambda, x + \frac{j}{m}\right) = m^{1-n} \bar{\beta}_n(m\lambda, mx) \tag{1.5}$$

for any x.

From now on, $s_r(h, k|\lambda)$ will be denoted by

$$s_r(h, k|\lambda) = s_r(h, k : 0, 0|\lambda).$$

In [6], Can *et al* generalized the Hardy–Berndt sums by

$$\begin{aligned} S_r(h, k) &= 4\sum_{j=1}^{k-1} \bar{B}_r\left(\frac{(h+k)j}{2k}\right), \\ s_{3,r}(h, k) &= \sum_{j=1}^{k-1} (-1)^j \bar{B}_r\left(\frac{hj}{k}\right), \\ s_{4,r}(h, k) &= -4\sum_{j=1}^{k-1} \bar{B}_r\left(\frac{hj}{2k}\right), \end{aligned}$$

proved the corresponding reciprocity theorems, established infinite and finite series representations and derived some properties of these sums. Furthermore, they explicitly deduced generalized Hardy–Berndt sums in terms of $s_r(h, k)$.

In this paper we define $S_r(h, k|\lambda)$, $s_{3,r}(h, k|\lambda)$ and $s_{4,r}(h, k|\lambda)$ Hardy–Berndt sums with generalized Bernoulli functions by

$$\begin{aligned} S_r(h, k|\lambda) &= 4\sum_{j=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{(h+k)j}{2k}\right), \\ s_{3,r}(h, k|\lambda) &= \sum_{j=0}^{k-1} (-1)^j \bar{\beta}_r\left(\lambda, \frac{hj}{k}\right), \\ s_{4,r}(h, k|\lambda) &= -4\sum_{j=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{hj}{2k}\right). \end{aligned}$$

Note that in the limiting case $\lambda = 0$, these sums reduce to the generalized Hardy–Berndt sums for odd $r > 1$ and $S_r(h, k|0) = S_r(h, k) + 4\bar{B}_r(0)$, $s_{3,r}(h, k|0) = s_{3,r}(h, k) + \bar{B}_r(0)$ and $s_{4,r}(h, k|0) = s_{4,r}(h, k) - 4\bar{B}_r(0)$ for even r. Also, we investigate these sums only for $r > 1$, since these can be written in terms of Hardy–Berndt sums for $r = 1$. Throughout this paper, we assume that h and k are coprime integers with $h, k > 0$ and r is a positive integer with $r > 1$.

We express these sums in terms of Dedekind sums $s_r(h,k|\lambda)$ as follows:

Theorem 1.1. *If k is odd, then*

$$s_{3,r}(h,k|\lambda) = 2s_r(h,k|\lambda) - 4s_r(2h,k|\lambda) + \lambda k^{1-r}\beta_r(k\lambda) \tag{1.6}$$

and if h is odd, then

$$s_{4,r}(h,k|\lambda) = -2^{3-r}s_r(h,k|2\lambda) + 8s_r(h,2k|\lambda) - 2^{2-r}k^{1-r}\beta_r(2k\lambda). \tag{1.7}$$

Theorem 1.2. *If $h+k$ is odd, then*

$$\begin{aligned} S_r(h,k|\lambda) &= -16s_r(h,k|\lambda) - 2^{3-r}s_r(h,k|2\lambda) + 2^{4-r}s_r(2h,k|2\lambda) \\ &\quad + 8s_r(h,2k|\lambda) - 2^{3-r}\lambda A_r(2h,k|2\lambda) + 2^{2-r}k^{1-r}\beta_r(2k\lambda), \end{aligned}$$

where

$$\begin{aligned} A_r(ph,k|p\lambda) &= \sum_{v=0}^{k-1}\bar{\beta}_r\left(p\lambda,\frac{phv}{k}\right) \\ &= \begin{cases} k^{1-r}\beta_r(pk\lambda), & (k,p)=1 \\ p(\frac{k}{p})^{1-r}\beta_r(k\lambda), & (k,p)=p. \end{cases} \end{aligned} \tag{1.8}$$

Also we the obtain the corresponding reciprocity laws for $(h,k)=1$.

Theorem 1.3. *If h is odd, the following reciprocity law holds:*

$$\begin{aligned} &kh^r s_{3,r}(k,h|2k\lambda) - hk^r 2^{r-2}s_{4,r}(h,k|h\lambda) \\ &= \sum_{j=0}^{r}\binom{r}{j}h^j\beta_j(2k\lambda)k^{r+1-j}\varepsilon_{r-j}(2h\lambda) + hk\beta_r(2hk\lambda). \end{aligned}$$

Theorem 1.4. *If $h+k$ is odd, the following reciprocity law holds:*

$$\begin{aligned} &2^{r-1}\{hk^r S_r(h,k|h\lambda) + kh^r S_r(k,h|k\lambda)\} \\ &= -rhk\sum_{j=0}^{r-1}\binom{r-1}{j}h^j k^{r-1-j}\varepsilon_j(2k\lambda)\varepsilon_{r-1-j}(2h\lambda) \\ &\quad + 2\lambda hkr(h+k+\delta+2r-2)\varepsilon_{r-1}(2hk\lambda) \\ &\quad + 2r\varepsilon_r(2hk\lambda) + 4hk\beta_r(2hk\lambda) \end{aligned}$$

where $\delta = h$ or k, according as k is or h is even, respectively.

Here $\varepsilon_n(\lambda,x)$ is the n-th generalized (or degenerate) Euler polynomial defined by [10]

$$\frac{2}{(1+\lambda t)^{\mu}+1}(1+\lambda t)^{x\mu} = \sum_{n=0}^{\infty}\varepsilon_n(\lambda,x)\frac{t^n}{n!} \tag{1.9}$$

for $\lambda \neq 0$ and $\mu\lambda = 1$. For $x=0$, $\varepsilon_n(\lambda,0) = \varepsilon_n(\lambda)$ is called the generalized Euler number and for $\lambda \to 0$, $\varepsilon_n(0,x) = E_n(x)$ which is the ordinary Euler polynomial.

In addition, we show that the reciprocity formulas are valid when $(h, k) = q$ is odd.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $D(h, k|\lambda)$ be denoted by

$$D(h, k|\lambda) = \sum_{j=0}^{k-1} \bar{\beta}_1\left(\lambda, \frac{j}{k} - \frac{1}{2}\right) \bar{\beta}_r\left(\lambda, \frac{hj}{k}\right).$$

Then, for odd k

$$\begin{aligned} D(2h, k|\lambda) - s_r(2h, k|\lambda) &= \sum_{j=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{2hj}{k}\right) \\ &\quad \times \left\{\bar{\beta}_1\left(\lambda, \frac{j}{k} - \frac{1}{2}\right) - \bar{\beta}_1\left(\lambda, \frac{j}{k}\right)\right\} \\ &= -\frac{1}{2}\sum_{j=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{2hj}{k}\right) \\ &\quad + \sum_{j=0}^{\frac{k-1}{2}} \bar{\beta}_r\left(\lambda, \frac{2hj}{k}\right), \end{aligned} \tag{2.1}$$

where we used

$$\left[\frac{j}{k} - \frac{1}{2}\right] = \begin{cases} -1, & 0 \le j \le \dfrac{k-1}{2}, \\ 0, & \dfrac{k+1}{2} \le j \le k-1. \end{cases}$$

Applying (1.5) to the first sum in (2.1), we get

$$D(2h, k|\lambda) - s_r(2h, k|\lambda) = \sum_{j=0}^{\frac{k-1}{2}} \bar{\beta}_r\left(\lambda, \frac{2hj}{k}\right) - \frac{1}{2}k^{1-r}\beta_r(k\lambda). \tag{2.2}$$

Similarly we can show that

$$D(2h, k|\lambda) + s_r(2h, k|\lambda) = s_r(h, k|\lambda) + \frac{\lambda}{2}k^{1-r}\beta_r(k\lambda) \tag{2.3}$$

for odd k.

On the other hand, we have

$$\begin{aligned} s_{3,r}(h, k|\lambda) &= \sum_{j \text{ even}} \bar{\beta}_r\left(\lambda, \frac{hj}{k}\right) - \sum_{j \text{ odd}} \bar{\beta}_r\left(\lambda, \frac{hj}{k}\right) \\ &= 2\sum_{j=0}^{\frac{k-1}{2}} \bar{\beta}_r\left(\lambda, \frac{2hj}{k}\right) - k^{1-r}\beta_r(k\lambda). \end{aligned} \tag{2.4}$$

By combining (2.2), (2.3) and (2.4), we get (1.6).

For odd h, with the help of (1.5), we have

$$\begin{aligned} s_r(h,2k|\lambda)+D(h,2k|\lambda) &= \sum_{j=0}^{2k-1}\bar{\beta}_1\left(2\lambda,\frac{j}{k}\right)\bar{\beta}_r\left(\lambda,\frac{hj}{2k}\right) \\ &= \sum_{j=0}^{k-1}\bar{\beta}_1\left(2\lambda,\frac{j}{k}\right)\bar{\beta}_r\left(\lambda,\frac{hj}{2k}\right) \\ &\quad +\sum_{j=0}^{k-1}\bar{\beta}_1\left(2\lambda,\frac{j}{k}\right)\bar{\beta}_r\left(\lambda,\frac{hj}{2k}+\frac{1}{2}\right) \\ &= 2^{1-r}s_r(h,k|2\lambda). \end{aligned} \tag{2.5}$$

Similarly, we have

$$D(h,2k|\lambda)-s_r(h,2k|\lambda)=-\frac{(2k)^{1-r}}{2}\beta_r(2k\lambda)-\frac{1}{4}s_{4,r}(h,k|\lambda) \tag{2.6}$$

for odd h. Thus (1.7) holds from (2.5) and (2.6). □

Proof of Theorem 1.2. For odd $h+k$, because of

$$S_r(h,k|\lambda)=-s_{4,r}(h+k,k|\lambda)$$

and (1.7), we get

$$S_r(h,k|\lambda)=2^{3-r}s_r(h,k|2\lambda)-8s_r(h+k,2k|\lambda)+2^{2-r}k^{1-r}\beta_r(2k\lambda). \tag{2.7}$$

To complete the proof we need following theorem which will be proved later.

Theorem 2.1. *For a prime number p, we have*

$$\begin{aligned} \sum_{m=0}^{p-1}s_r(h+mk,pk|\lambda) &= p^{1-r}s_r(h,k|p\lambda)-p^{1-r}s_r(ph,k|p\lambda) \\ &\quad +ps_r(h,k|\lambda)+\left(\frac{p-1}{2}\right)p^{1-r}\lambda A_r(ph,k|p\lambda), \end{aligned}$$

where $A_r(ph,k|p\lambda)$ is given in (1.8).

Putting $p=2$ in Theorem 2.1 and combining with (2.7) we have the desired result. □

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. From Theorem 1.1, we have

$$(r+1)\{kh^r s_{3,r}(k,h|2k\lambda)-hk^r2^{r-2}s_{4,r}(h,k|h\lambda)\}=S_1+S_2+S_3,$$

where

$$S_1 = 2(r+1)\{kh^r s_r(k,h|k2\lambda) + hk^r s_r(h,k|h2\lambda)\},$$

$$S_2 = -2(r+1)\{(2k)h^r s_r(2k,h|2k\lambda) + (2k)^r h s_r(h,2k|h\lambda)\},$$

$$S_3 = (r+1)\{2\lambda k^2 h \beta_r(2hk\lambda) + hk\beta_r(2hk\lambda)\}.$$

With the help of (1.2) with $x = y = 0$ and $d = 1$, we get

$$S_1 = 2\sum_{j=0}^{r+1}\binom{r+1}{j} h^j \beta_j\,(2k\lambda)\, k^{r+1-j}\beta_{r+1-j}\,(2h\lambda)$$
$$+ 2\lambda hk(r+1)(h+k+2r-2)\beta_r\,(2hk\lambda) + 2r\beta_{r+1}\,(2hk\lambda)$$

and

$$S_2 = -2\sum_{j=0}^{r+1}\binom{r+1}{j} h^j \beta_j\,(2k\lambda)\,(2k)^{r+1-j}\beta_{r+1-j}\,(h\lambda)$$
$$- 2\lambda hk(r+1)(h+2k+2r-2)\beta_r\,(2hk\lambda) - 2r\beta_{r+1}\,(2hk\lambda)\,.$$

Thus, we obtain

$$(r+1)\{kh^r s_{3,r}(k,h|2k\lambda) - hk^r 2^{r-2} s_{4,r}(h,k|h\lambda)\}$$
$$= 2\sum_{j=0}^{r+1}\binom{r+1}{j} h^j \beta_j\,(2k\lambda)\, k^{r+1-j}\{\beta_{r+1-j}\,(2h\lambda) - 2^{r+1-j}\beta_{r+1-j}\,(h\lambda)\}$$
$$+ hk(r+1)\beta_r\,(2hk\lambda)\,. \tag{3.1}$$

From (1.4) and (1.9) for $x = 0$, we have

$$\beta_r(2\lambda) - 2^r \beta_r(\lambda) = \frac{r}{2}\varepsilon_{r-1}(2\lambda) \tag{3.2}$$

for $r \geq 1$. Substituting (3.2) in (3.1), the desired result follows. □

Proof of Theorem 1.4. From Theorem 1.2, we have

$$(r+1)2^{r-3}\{hk^r S_r(h,k|h\lambda) + kh^r S_r(k,h|k\lambda)\} = T_1 + T_2 + T_3 + T_4 + T_5,$$

where

$$T_1 = -2^{r+1}(r+1)\{hk^r s_r(h,k|h\lambda) + kh^r s_r(k,h|k\lambda)\},$$

$$T_2 = -(r+1)\{hk^r s_r(h,k|2h\lambda) + kh^r s_r(k,h|2k\lambda)\},$$

$$T_3 = (r+1)\{(2h)k^r s_r(2h,k|2h\lambda) + (2h)^r k s_r(k,2h|k\lambda)\},$$

$$T_4 = (r+1)\{h(2k)^r s_r(h,2k|h\lambda) + (2k)h^r s_r(2k,h|2k\lambda)\},$$

$$T_5 = (r+1)\{-h^2k^r\lambda A_r(2h,k|2h\lambda)$$
$$-k^2h^r\lambda A_r(2k,h|2k\lambda)+hk\beta_r(2hk\lambda)\}.$$

Again with the help of (1.2) with $x=y=0$ and $d=1$, we get

$$T_1 = -2^{r+1}\sum_{j=0}^{r+1}\binom{r+1}{j}h^j\beta_j(k\lambda)\,k^{r+1-j}\beta_{r+1-j}(h\lambda)$$
$$-2^r\lambda hk(r+1)(h+k+2r-2)\beta_r(hk\lambda)-r2^{r+1}\beta_{r+1}(hk\lambda),$$

$$T_2 = -\sum_{j=0}^{r+1}\binom{r+1}{j}h^j\beta_j(2k\lambda)\,k^{r+1-j}\beta_{r+1-j}(2h\lambda)$$
$$-\lambda hk(r+1)(h+k+2r-2)\beta_r(2hk\lambda)-r\beta_{r+1}(2hk\lambda),$$

$$T_3 = \sum_{j=0}^{r+1}\binom{r+1}{j}(2h)^j\beta_j(k\lambda)\,k^{r+1-j}\beta_{r+1-j}(2h\lambda)$$
$$+\lambda hk(r+1)(2h+k+2r-2)\beta_r(2hk\lambda)+r\beta_{r+1}(2hk\lambda),$$

$$T_4 = \sum_{j=0}^{r+1}\binom{r+1}{j}h^j\beta_j(2k\lambda)\,(2k)^{r+1-j}\beta_{r+1-j}(h\lambda)$$
$$+\lambda hk(r+1)(h+2k+2r-2)\beta_r(2hk\lambda)+r\beta_{r+1}(2hk\lambda).$$

Therefore, we obtain

$$(r+1)2^{r-3}\{hk^rS_r(h,k|h\lambda)+kh^rS_r(k,h|k\lambda)\}$$
$$=\sum_{j=0}^{r+1}\binom{r+1}{j}h^jk^{r+1-j}\{2^j\beta_j(k\lambda)-\beta_j(2k\lambda)\}$$
$$\times\{\beta_{r+1-j}(2h\lambda)-2^{r+1-j}\beta_{r+1-j}(h\lambda)\}$$
$$+\lambda hk(r+1)(h+k+2r-2)\{\beta_r(2hk\lambda)-2^r\beta_r(hk\lambda)\}$$
$$+r\{\beta_{r+1}(2hk\lambda)-2^{r+1}\beta_{r+1}(hk\lambda)\}$$
$$-k^2h^r\lambda(r+1)A_r(2k,h|2k\lambda)-h^2k^r\lambda(r+1)A_r(2h,k|2h\lambda)$$
$$+\lambda hk(r+1)(h+k)\beta_r(2hk\lambda)+hk(r+1)\beta_r(2hk\lambda), \tag{3.3}$$

where $A_r(2h, k|2h\lambda)$ is given in (1.8). By using (3.2), the right-hand side of (3.3) becomes

$$-\frac{r(r+1)hk}{4}\sum_{j=0}^{r-1}\binom{r-1}{j}h^j k^{r-1-j}\varepsilon_j(2k\lambda)\varepsilon_{r-1-j}(2h\lambda)+\lambda hk(r+1)$$
$$\times(h+k+2r-2)\frac{r}{2}\varepsilon_{r-1}(2hk\lambda)+\frac{r(r+1)}{2}\varepsilon_r(2hk\lambda)+\lambda(r+1)$$
$$\times\{hk(h+k)\beta_r(2hk\lambda)-h^2k^r A_r(2h,k|2h\lambda)-k^2h^r A_r(2k,h|2k\lambda)\}$$
$$+hk(r+1)\beta_r(2hk\lambda).$$

On the other hand, for odd k and even h, we have

$$hk(h+k)\beta_r(2hk\lambda)-h^2k^r A_r(2h,k|2h\lambda)-k^2h^r A_r(2k,h|2k\lambda)$$
$$=hk^2\frac{r}{2}\varepsilon_{r-1}(2hk\lambda)$$

and for odd h and even k, we have

$$hk(h+k)\beta_r(2hk\lambda)-h^2k^r A_r(2h,k|2h\lambda)-k^2h^r A_r(2k,h|2k\lambda)$$
$$=h^2k\frac{r}{2}\varepsilon_{r-1}(2hk\lambda).$$

Thus, the proof is completed. □

4. Further results

First we state following proposition which we need in the proof of Theorem 2.1.

PROPOSITION 4.1

For any positive integers q, we have

$$s_r(qh,qk|\lambda)=s_r(h,k|\lambda)+\frac{\lambda(q-1)}{2}k^{1-r}\beta_r(k\lambda).$$

Proof. From the definition of $s_r(h,k|\lambda)$ and (1.5), we have

$$\begin{aligned}s_r(qh,qk|\lambda) &= \sum_{j=0}^{qk-1}\bar{\beta}_1\left(\lambda,\frac{j}{qk}\right)\bar{\beta}_r\left(\lambda,\frac{qhj}{qk}\right)\\ &= \sum_{v=0}^{q-1}\sum_{\rho=0}^{k-1}\bar{\beta}_1\left(\lambda,\frac{vk+\rho}{qk}\right)\bar{\beta}_r\left(\lambda,\frac{h(vk+\rho)}{k}\right)\\ &= \sum_{\rho=0}^{k-1}\bar{\beta}_1\left(q\lambda,\frac{\rho}{k}\right)\bar{\beta}_r\left(\lambda,\frac{h\rho}{k}\right).\end{aligned}$$

Using the fact that $\beta_1(\lambda, x) = x + \frac{\lambda}{2} - \frac{1}{2}$ (see [12]), we get

$$s_r(qh, qk|\lambda) = \sum_{\rho=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{h\rho}{k}\right) \bar{\beta}_1\left(\lambda, \frac{\rho}{k}\right) + \frac{\lambda(q-1)}{2} \sum_{\rho=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{h\rho}{k}\right).$$

Since $(h, k) = 1$, from (1.5), we obtain

$$\sum_{\rho=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{h\rho}{k}\right) = \sum_{\rho=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{\rho}{k}\right) = k^{1-r} \bar{\beta}_r(k\lambda, 0) = k^{1-r} \beta_r(k\lambda) \tag{4.1}$$

which completes the proof. □

Proof of Theorem 2.1. From the definition of $s_r(h, k|\lambda)$, we have

$$\begin{aligned}
\sum_{m=0}^{p-1} s_r(h+mk, pk|\lambda) &= \sum_{m=0}^{p-1} \sum_{j=0}^{pk-1} \bar{\beta}_1\left(\lambda, \frac{j}{pk}\right) \bar{\beta}_r\left(\lambda, \frac{(h+mk)j}{pk}\right) \\
&= \sum_{v=0}^{k-1} \sum_{\mu=0}^{p-1} \sum_{m=0}^{p-1} \bar{\beta}_1\left(\lambda, \frac{vp+\mu}{pk}\right) \\
&\quad \times \bar{\beta}_r\left(\lambda, \frac{h(vp+\mu)}{pk} + \frac{m\mu}{p}\right) \\
&= \sum_{v=0}^{k-1} \sum_{\mu=1}^{p-1} \sum_{m=0}^{p-1} \bar{\beta}_1\left(\lambda, \frac{vp+\mu}{pk}\right) \\
&\quad \times \bar{\beta}_r\left(\lambda, \frac{h(vp+\mu)}{pk} + \frac{m\mu}{p}\right) \\
&\quad + \sum_{v=0}^{k-1} \sum_{m=0}^{p-1} \bar{\beta}_1\left(\lambda, \frac{v}{k}\right) \bar{\beta}_r\left(\lambda, \frac{hv}{k}\right).
\end{aligned} \tag{4.2}$$

Applying (1.5) to sum over m in (4.2), the right-hand side becomes

$$\begin{aligned}
&p^{1-r} \sum_{v=0}^{k-1} \sum_{\mu=1}^{p-1} \bar{\beta}_1\left(\lambda, \frac{vp+\mu}{pk}\right) \bar{\beta}_r\left(p\lambda, \frac{ph(vp+\mu)}{pk}\right) + ps_r(h, k|\lambda) \\
&= p^{1-r} \sum_{v=0}^{k-1} \sum_{\mu=0}^{p-1} \bar{\beta}_1\left(\lambda, \frac{vp+\mu}{pk}\right) \bar{\beta}_r\left(p\lambda, \frac{ph(vp+\mu)}{pk}\right) \\
&\quad - p^{1-r} \sum_{v=0}^{k-1} \bar{\beta}_1\left(\lambda, \frac{v}{k}\right) \bar{\beta}_r\left(p\lambda, \frac{phv}{k}\right) + ps_r(h, k|\lambda).
\end{aligned}$$

By the fact that

$$\sum_{v=0}^{k-1}\sum_{\mu=0}^{p-1}\bar{\beta}_1\left(\lambda,\frac{vp+\mu}{pk}\right)\bar{\beta}_r\left(p\lambda,\frac{ph(vp+\mu)}{pk}\right)$$
$$=s_r\left(ph,pk|p\lambda\right)+\frac{\lambda(1-p)}{2}pk^{1-r}\beta_r\left(pk\lambda\right),$$
$$\sum_{v=0}^{k-1}\bar{\beta}_1\left(\lambda,\frac{v}{k}\right)\bar{\beta}_r\left(p\lambda,\frac{phv}{k}\right)$$
$$=s_r(ph,k|p\lambda)+\frac{\lambda(1-p)}{2}\sum_{v=0}^{k-1}\bar{\beta}_r\left(p\lambda,\frac{phv}{k}\right)$$

and Proposition 4.1, we have

$$\begin{aligned}\sum_{m=0}^{p-1}s_r(h+mk,pk|\lambda) &= p^{1-r}s_r(h,k|p\lambda)\\ &\quad -p^{1-r}s_r(ph,k|p\lambda)+ps_r(h,k|\lambda)\\ &\quad +\frac{\lambda\left(p-1\right)}{2}p^{1-r}A_r\left(ph,k|p\lambda\right),\end{aligned}$$

where

$$\begin{aligned}A_r\left(ph,k|p\lambda\right) &= \sum_{v=0}^{k-1}\bar{\beta}_r\left(p\lambda,\frac{phv}{k}\right)\\ &= \begin{cases} k^{1-r}\beta_r\left(pk\lambda\right), & (k,p)=1,\\ p\left(\dfrac{k}{p}\right)^{1-r}\beta_r\left(k\lambda\right), & (k,p)=p\end{cases}\end{aligned}$$

which completes the proof. □

By the following proposition, we can show that the reciprocity theorems are satisfied not only in the case $(h,k)=1$ but also in the case $(h,k)=q$ be odd.

PROPOSITION 4.2

Let q be positive integer. If $(h+k)$ is odd, then

$$S_r(qh,qk|\lambda)=\begin{cases}2q(2k)^{1-r}\beta_r(2k\lambda), & \text{if } q \text{ is even},\\ S_r(h,k|\lambda)+\dfrac{2(q-1)}{(2k)^{r-1}}\beta_r(2k\lambda), & \text{if } q \text{ is odd}.\end{cases}$$

If k is odd, then

$$s_{3,r}(qh,qk|\lambda)=\begin{cases}0, & \text{if } q \text{ is even},\\ s_{3,r}(h,k|\lambda), & \text{if } q \text{ is odd}.\end{cases}$$

If h is odd, then

$$s_{4,r}(qh, qk|\lambda) = \begin{cases} -2q(2k)^{1-r}\beta_r(2k\lambda), & \text{if } q \text{ is even} \\ s_{4,r}(h,k|\lambda) - \dfrac{2(q-1)}{(2k)^{r-1}}\beta_r(2k\lambda), & \text{if } q \text{ is odd.} \end{cases}$$

Proof. By the definition of $S_r(h, k|\lambda)$, we have

$$\begin{aligned} \frac{1}{4}S_r(qh, qk|\lambda) &= \sum_{j=0}^{qk-1} \bar{\beta}_r\left(\lambda, \frac{(h+k)j}{2k}\right) \\ &= \sum_{n=0}^{k-1}\sum_{m=0}^{q-1} \bar{\beta}_r\left(\lambda, \frac{(h+k)n}{2k} + \frac{m}{2}\right). \end{aligned} \tag{4.3}$$

Let q be even. Then

$$\begin{aligned} \frac{1}{4}S_r(qh, qk|\lambda) &= \frac{q}{2}\sum_{n=0}^{k-1}\sum_{m=0}^{1} \bar{\beta}_r\left(\lambda, \frac{(h+k)n}{2k} + \frac{m}{2}\right) \\ &= \frac{q}{2}2^{1-r}\sum_{n=0}^{k-1} \bar{\beta}_r\left(2\lambda, \frac{(h+k)n}{k}\right) = \frac{q}{2}(2k)^{1-r}\beta_r(2k\lambda). \end{aligned}$$

If q is odd, from (4.3), we obtain similarly

$$\begin{aligned} \frac{1}{4}S_r(qh, qk|\lambda) &= \frac{(q-1)}{2}\sum_{n=0}^{k-1}\sum_{m=0}^{1} \bar{\beta}_r\left(\lambda, \frac{(h+k)n}{2k} + \frac{m}{2}\right) \\ &\quad + \sum_{n=0}^{k-1} \bar{\beta}_r\left(\lambda, \frac{(h+k)n}{2k} + \frac{q-1}{2}\right) \\ &= \frac{(q-1)}{2}(2k)^{1-r}\beta_r(2k\lambda) + \frac{1}{4}S_r(h, k|\lambda). \end{aligned}$$

Other statements can be shown in a similar way. □

With the help of Theorem 1.3, Theorem 1.4 and Proposition 4.2 we have following:

COROLLARY 4.3

Let $(h, k) = 1$ with $h, k > 0$ and q be odd. If h is odd, then

$$\begin{aligned} &kh^r s_{3,r}(qk, qh|qk2\lambda) - hk^r 2^{r-2} s_{4,r}(qh, qk|qh\lambda) \\ &= \sum_{j=0}^{r}\binom{r}{j} h^j \beta_j(2kq\lambda)\, k^{r+1-j}\varepsilon_{r-j}(2hq\lambda) + qhk\beta_r(2hk\lambda) \end{aligned}$$

and if $(h+k)$ *is odd, then*

$$
\begin{aligned}
&2^{r-1}\{hk^r S_r(hq, kq|hq\lambda) + kh^r S_r(kq, hq|kq\lambda)\}\\
&\quad = -rhk\sum_{j=0}^{r-1}\binom{r-1}{j}h^j k^{r-1-j}\varepsilon_j(2kq\lambda)\varepsilon_{r-1-j}(2hq\lambda)\\
&\qquad + 2q\lambda hkr(h+k+\delta+2r-2)\varepsilon_{r-1}(2hkq\lambda)\\
&\qquad + 2r\varepsilon_r(2hkq\lambda) + 4qhk\beta_r(2hk\lambda),
\end{aligned}
$$

where $\delta = h$ *or* k, *according as* k *is or* h *is even, respectively.*

The analogous of Theorem 2.1 may be given for a new generalization of the Hardy–Berndt sums as follows:

Theorem 4.4. *Let* p *be a prime number. If* k *is odd, then*

$$
\begin{aligned}
&\sum_{m=0}^{p-1} s_{3,r}(h+mk, pk|\lambda)\\
&\quad = \begin{cases} 2k^{1-r}\beta_r(k\lambda) - (2k)^{1-r}\beta_r(2k\lambda), & \text{if } p=2,\\ p^{1-r}\left\{s_{3,r}(h,k|p\lambda) - s_{3,r}(ph,k|p\lambda)\right\} + ps_{3,r}(h,k|\lambda), & \text{if } p>2. \end{cases}
\end{aligned} \tag{4.4}
$$

If h *is odd, then*

$$
\begin{aligned}
\sum_{m=0}^{p-1} s_{4,r}(h+2mk, pk|\lambda) = {} & p^{1-r}\{s_{4,r}(ph, pk|p\lambda)\\
& - s_{4,r}(ph, k|p\lambda)\} + ps_{4,r}(h,k|\lambda)
\end{aligned} \tag{4.5}
$$

and if $h+k$ *is odd, then*

$$
\begin{aligned}
&\sum_{m=0}^{p-1} S_r(h+2mk, pk|\lambda)\\
&\quad = \begin{cases} -p^{1-r}\left\{s_{4,r}(ph, pk|p\lambda) - s_{4,r}(ph, k|p\lambda)\right\} & \\ \qquad - ps_{4,r}(h,k|\lambda), & \text{if } p=2\\ p^{1-r}\{S_r(ph, pk|p\lambda) - S_r(ph,k|p\lambda)\} & \\ \qquad + pS_r(h,k|\lambda), & \text{if } p>2. \end{cases}
\end{aligned} \tag{4.6}
$$

Proof. From the definition of $s_{3,r}(h,k|\lambda)$, we have

$$\begin{aligned}\sum_{m=0}^{p-1} s_{3,r}(h+mk,pk|\lambda) &= \sum_{m=0}^{p-1}\sum_{j=0}^{pk-1}(-1)^j \bar{\beta}_r\left(\lambda,\frac{(h+mk)j}{pk}\right)\\ &= \sum_{v=0}^{k-1}\sum_{\mu=1}^{p-1}\sum_{m=0}^{p-1}(-1)^{vp+\mu}\\ &\quad\times \bar{\beta}_r\left(\lambda,\frac{h(vp+\mu)}{pk}+\frac{m\mu}{p}\right)\\ &\quad+\sum_{v=0}^{k-1}\sum_{m=0}^{p-1}(-1)^{vp}\bar{\beta}_r\left(\lambda,\frac{hvp}{pk}\right). \end{aligned} \tag{4.7}$$

Applying (1.5) to sum over m in (4.7), we get

$$\begin{aligned}\sum_{m=0}^{p-1} s_{3,r}(h+mk,pk|\lambda) &= p^{1-r}\sum_{v=0}^{k-1}\sum_{\mu=1}^{p-1}(-1)^{vp+\mu}\bar{\beta}_r\left(p\lambda,\frac{ph(vp+\mu)}{pk}\right)\\ &\quad+p\sum_{v=0}^{k-1}(-1)^{vp}\bar{\beta}_r\left(\lambda,\frac{hv}{k}\right)\\ &= p^{1-r}\sum_{v=0}^{k-1}\sum_{\mu=0}^{p-1}(-1)^{vp+\mu}\bar{\beta}_r\left(p\lambda,\frac{ph(vp+\mu)}{pk}\right)\\ &\quad-p^{1-r}\sum_{v=0}^{k-1}(-1)^{vp}\bar{\beta}_r\left(p\lambda,\frac{phv}{k}\right)\\ &\quad+p\sum_{v=0}^{k-1}(-1)^{vp}\bar{\beta}_r\left(\lambda,\frac{hv}{k}\right)\\ &= p^{1-r}s_{3,r}(ph,pk|p\lambda)-p^{1-r}B+pA, \end{aligned}$$

where

$$A=\sum_{v=0}^{k-1}(-1)^{vp}\bar{\beta}_r\left(\lambda,\frac{hv}{k}\right)=\begin{cases}k^{1-r}\beta_r(k\lambda), & \text{if } p=2,\\ s_{3,r}(h,k|\lambda), & \text{if } p>2\end{cases}$$

and

$$B=\sum_{v=0}^{k-1}(-1)^{vp}\bar{\beta}_r\left(p\lambda,\frac{phv}{k}\right)=\begin{cases}k^{1-r}\beta_r(2k\lambda), & \text{if } p=2,\\ s_{3,r}(ph,k|p\lambda), & \text{if } p>2\end{cases}$$

which completes the proof of (4.4). The proofs of (4.5) and (4.6) follow by using the arguments in the proof of (4.4). □

For the limiting case $\lambda = 0$ and odd integer $r > 1$, from Theorem 4.4 and Proposition 4.2, we have following results.

COROLLARY 4.5

Let p be a prime number. If k is odd, then

$$\sum_{m=0}^{p-1} s_{3,r}(h + mk, pk) = (p^{1-r} + p)s_{3,r}(h, k) - p^{1-r} s_{3,r}(ph, k) \tag{4.8}$$

for $p > 2$ and the sum on the left-hand side in (4.8) is zero for $p = 2$. If h is odd, then

$$\sum_{m=0}^{p-1} s_{4,r}(h + 2mk, pk) = p^{1-r}\{s_{4,r}(ph, pk) - s_{4,r}(ph, k)\} + ps_{4,r}(h, k)$$

and if $(h + k)$ is odd, then

$$\sum_{m=0}^{p-1} S_r(h + 2mk, pk)$$
$$= \begin{cases} -2s_{4,r}(h, k), & \text{if } p = 2, \\ p^{1-r}\{S_r(ph, pk) - S_r(ph, k)\} + pS_r(h, k), & \text{if } p > 2. \end{cases}$$

Acknowledgement

This work is supported by the Akdeniz University Scientific Research Projects Unit.

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