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Bell numbers, determinants and series

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Abstract. In this article, we study Bell numbers and Uppuluri Carpenter numbers. We obtain various expressions and relations between them. These include polynomial recurrences and expressions as determinants of certain matrices of binomial coefficients.

Keywords. p-adic series; Bell numbers.

1. Introduction

Bell numbers, B_n [2] are defined as the total number of partition(s) of a set with n element(s) into non empty disjoint subset(s) (partition of the set). Complementary Bell numbers [6] or Uppuluri Carpenter numbers $\tilde{B}_n$ are defined as the numbers which count the excess of partitions of $\{1, 2, 3, \ldots, n\}$ into an even number of blocks over the number of partitions into an odd number of blocks. These numbers can also be characterized as the sum or alternating sum of Stirling numbers $S(n, k)$ of the second kind. More explicitly, we have the following formulae and table 1 for Bell numbers and complementary Bell numbers

$$B_n = \sum_{k=0}^{n} S(n, k) \qquad \text{and} \qquad \tilde{B}_n = \sum_{k=0}^{n} (-1)^k S(n, k).$$

The usual generating functions for Bell numbers and complementary Bell numbers are given by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \mathrm{e}^{e^t - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} \tilde{B}_n \frac{t^n}{n!} = \mathrm{e}^{-e^t + 1}$$

Stirling numbers can also be characterized in terms of falling factorials $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1)$ and rising factorials $x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1)$. In fact for a positive integer n the following equation holds:

$$x^n = \sum_{k=1}^{n} S(n, k) x^{\underline{k}}. \tag{1}$$

Table 1.

n	0	1	2	3	4	5	6	7	8	9	10
$\tilde{B}_n$	1	−1	0	1	1	−2	−9	−9	50	267	413
B_n	1	1	2	5	15	52	203	877	4140	21147	115975

In this article we show that Bell numbers can be characterized using a polynomial recurrence relation given by Dragovich [5]. Using these recurrence relations we have been able to obtain determinants with their values as Bell numbers. We have derived series converging to these polynomials and used the series to prove the well-known Dobinsky formula for Bell numbers. We have obtained a few other recurrence relations for those polynomials as well. These recurrence relation are similar to the recurrence relation for Bell numbers.

Most of the results that we have derived for Bell numbers also holds for complementary Bell numbers but with a suitable adjustment. The proof being similar in both the cases, we have included the proof in one case and just stated the result for the other case.

We assume for convenience that the empty product is 1 throughout the paper.

2. Bell numbers via polynomial recurrence

In Theorem 2.1 we show that Bell numbers can also be characterized using the recurrence on polynomials.

Theorem 2.1. *For a given positive integer n there is a unique polynomial $g_n(x)$ and a unique constant c_{n+1} satisfying the following recurrence relation*:

$$g_n(x)(x+1) + g_n(x-1) = x^n + (-1)^{n+1} c_{n+1}.$$

In fact $c_{n+1} = B_{n+1}$ is the $n+1$-th Bell number.

Proof. Noting

$$(-x)^{\underline{n}} = (-1)^n x(x+1)(x+2)\cdots(x+n-1),$$

it follows from eq. (1) that

$$x^{n-1} = \sum_{k=1}^{n} S(n,k)(-1)^{n-k} x^{\overline{k}}/x. \tag{2}$$

Taking $x = 1, 2, 3, \ldots, m$ successively in (2), multiplying the resultant equations for $x = j$ by $j!$ and finally summing up alternatively, we get

$$\begin{aligned} &- 1^{n-1}1! + 2^{n-1}2! - \cdots (-1)^m m^{n-1} m! \\ &\quad = -\sum_{k=1}^{n} S(n,k)(-1)^{n-k} 1^{\overline{k}} 1!/1 + \sum_{k=1}^{n} S(n,k)(-1)^{n-k} 2^{\overline{k}} 2!/2 + \cdots \\ &\qquad + (-1)^m \sum_{k=1}^{n} S(n,k)(-1)^{n-k} m^{\overline{k}} m!/m \end{aligned}$$

and the following observation:

$$j^{\overline{k}} \frac{j!}{j} = (j + k - 1)!$$

gives

$$\begin{aligned} &- 1^{n-1}1! + 2^{n-1}2! - 3^{n-1}3! + \cdots + (-1)^m m^{n-1} m! \\ &\quad = -\sum_{k=1}^{n} S(n,k)(-1)^{n-k} k! + \sum_{k=1}^{n} S(n,k)(-1)^{n-k}(k+1)! - \cdots \\ &\qquad + (-1)^m \sum_{k=1}^{n} S(n,k)(-1)^{n-k}(m+k-1)! \end{aligned} \tag{3}$$

For $m \geq n \geq 2$, we collect terms with $(m+1)!, (m+2)!, (m+3)!, \ldots, (m+n-1)!$ and observe that these terms can be written as

$$(-1)^{n-2} A_{n-2}(m+1)!, (-1)^{n-3} A_{n-3}(m+2)!, \ldots, A_0(m+n-1)!,$$

where A_i is given by

$$A_i = S(n,n) + S(n,n-1) + \cdots + S(n,n-i).$$

For $j \geq 2$, write $(m+j)!$ as

$$(m+j)! = (m+j)(m+j-1)\cdots(m+2)(m+1)!.$$

It follows that terms with $(m+1)!, (m+2)!, (m+3)!, \ldots, (m+n-1)!$ in eq. (3) after summing up can be written as $g_{n-1}(m)(m+1)!(-1)^m$. In fact $g_{n-1}(x)$ is a polynomial given by

$$\begin{aligned} g_{n-1}(x) = {} & (-1)^{n-2} A_{n-2} + (-1)^{n-3} A_{n-3}(x+2) + \cdots \\ & + A_0(x+2)\cdots(x+n-1). \end{aligned} \tag{4}$$

We separate out $g_{n-1}(m)(m+1)!(-1)^m$ from the right-hand side and assemble the coefficient of $-n!, (n+1)!, \ldots, (-1)^{m-n-1}m!$ and observing that the coefficient of each of these factorials is the sum

$$S(n,n) + S(n,n-1) + \cdots + S(n,1),$$

it follows that

$$\begin{aligned} &- 1^{n-1}1! + 2^{n-1}2! - \cdots + (-1)^m m^{n-1} m! \\ &\quad = B_n \sum_{i=n}^{m} (-1)^{i-n-1} i! + c_n + (-1)^m g_{n-1}(m)(m+1)!, \end{aligned} \tag{5}$$

where c_n is a constant depending on n. Since eq. (5) is true for every $m \geq n$ we can consider $m \geq n+1$, replace m by $m-1$ and then eliminate the constant c_n to get

$$g_{n-1}(m)(m+1) + g_{n-1}(m-1) = m^{n-1} + B_n(-1)^n.$$

Since there are infinitely many such m the required polynomial recurrence follows.

Table 2.

n	$g_n(x)$	$f_n(x)$
1	1	1
2	$-2+x$	x
3	$6-2x+x^2$	$-2+x^2$
4	$-21+7x-2x^2+x^3$	$3-3x+x^3$
5	$82-28x+8x^2-2x^3+x^4$	$4+6x-4x^2+x^4$
6	$-354+121x-36x^2+9x^3-2x^4+x^5$	$-30+5x+10x^2-5x^3+x^5$
7	$1671-570x+170x^2-45x^3+10x^4-2x^5+x^6$	$55-66x+4x^2+15x^3-6x^4+x^6$
8	$-8536+2911x-865x^2+230x^3-55x^4+11x^5-2x^6+x^7$	$126+175x-119x^2+21x^4-7x^5+x^7$
9	$46814-15968x+4742x^2-1254x^3+302x^4-66x^5+12x^6-2x^7+x^8$	$-1190+150x+416x^2-190x^3-8x^4+28x^5-8x^6+x^8$
10	$-273907+93433x-27757x^2+7332x^3-1753x^4+387x^5-78x^6$ $+13x^7-2x^8+x^9$	$3333-3273x-45x^2+834x^3-279x^4-21x^5$ $+36x^6-9x^7+x^9$
11	–	$4522+13248x-7007x^2-740x^3+1493x^4-385x^5$ $-40x^6+45x^7-10x^8+x^{10}$

The uniqueness assertion is easy to prove. □

Instead of considering the sum

$$-1^{n-1}1! + 2^{n-1}2! - \cdots + (-1)^m m^{n-1} m!,$$

let us consider the sum

$$1^{n-1}1! + 2^{n-1}2! + \cdots + m^{n-1} m!$$

and then proceed as above to obtain the following theorem.

Theorem 2.2. *For a given positive integer n there is a unique polynomial $f_n(x)$ and a unique constant a_n satisfying the following recurrence relation*

$$f_n(x)(x+1) - f_n(x-1) = x^n - a_n.$$

In fact

$$a_n = S(n+1, n+1) - S(n+1, n) + \cdots + (-1)^n S(n+1, 1) = \tilde{B}_{n+1} \cdot (-1)^{n+1}.$$

We include table 2 for the first few $g_n(x)$ and $f_n(x)$.

3. Bell numbers and determinant

Now, we try to solve the coefficients of the polynomials $g_n(x)$. We compare the constant term and the coefficients of $x, x^2, \ldots, x^{n-1}$ in the polynomial recurrence relation. This gives us a system of n linear equations with n unknowns namely the constant term and the coefficients of $x, x^2, \ldots, x^{n-1}$ in the recurrence relation. If we try to solve these equations by Cramer's method then we have to deal with the determinant

$$\begin{vmatrix} 2 & -1 & 1 & -1 & \cdots & (-1)^{n-1} \\ 1 & 2 & -\binom{2}{1} & \binom{3}{1} & \cdots & (-1)^{n-2}\binom{n-1}{1} \\ 0 & 1 & 2 & -\binom{3}{2} & \cdots & (-1)^{n-3}\binom{n-1}{2} \\ 0 & 0 & 1 & 2 & \cdots & \cdot \\ 0 & 0 & 0 & 1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \cdots & -\binom{n-1}{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{vmatrix}.$$

It is interesting to note that this determinant is the $(n+1)$-th Bell number. We prove the same in the next proposition.

PROPOSITION 3.1

The $n+1$-th Bell number is given by

$$B_{n+1} = \det(a_{ij})_{n \times n},$$

where

$$a_{ij} = \begin{cases} 2, & \text{for } i = j, \\ 1, & \text{for } i - j = 1, \\ 0, & \text{for } i - j > 1, \\ (-1)^{i+j}\binom{j-1}{i-1}, & \text{for } j > i. \end{cases}$$

Proof. We prove the proposition by induction.

For this purpose we use a recurrence relation for Bell numbers given by

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

Assume that for a given positive integer N, $B_{n+1} = \det(a_{ij})_{n\times n}$ holds for every $n \leq N$. Expanding the $(n+1)\times(n+1)$ determinant along the last column and using induction hypothesis, it is sufficient to show that

$$B_{n+2} = 1 + \binom{n}{1} B_2 + \binom{n}{2} B_3 + \cdots + \binom{n}{n-1} B_n + 2B_{n+1}$$

which can be shown to be true by using the recurrence relation for Bell numbers. □

We can proceed as above and consider the well-known recurrence relation

$$\tilde{B}_{n+1} = -\sum_{k=0}^{n} \binom{n}{k} \tilde{B}_k$$

for complementary Bell numbers or we can consider a slightly different form

$$a_{n+1} = \sum_{r=0}^{n-1} \binom{n}{r} (-1)^{n-r} a_r, \tag{6}$$

obtained by considering the identity

$$a_n = (-1)^{n+1} \tilde{B}_{n+1}.$$

Then the following proposition follows.

PROPOSITION 3.2

The $n+1$-th complementary Bell numbers, $\tilde{B}_{n+1}$ are given by

$$\tilde{B}_{n+1} = -\begin{vmatrix} 0 & 1 & -1 & 1 & \cdots & (-1)^n \\ 1 & 0 & \binom{2}{1} & -\binom{3}{1} & \cdots & (-1)^{n-1}\binom{n-1}{1} \\ 0 & 1 & 0 & \binom{3}{2} & \cdots & (-1)^{n-2}\binom{n-1}{2} \\ 0 & 0 & 1 & 0 & \cdots & \cdot \\ 0 & 0 & 0 & 1 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \cdots & \binom{n-1}{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{vmatrix}.$$

4. On series converging to $f_n(x)$, $g_n(x)$

Theorem 4.1. *Let S be a compact subset of $\mathbb{R} \setminus \{-1, 0, 1, 2, \ldots\}$. Then the sequences*

$$\frac{x^n + (-1)^{n+1} B_{n+1}}{x+1} - \frac{(x-1)^n + (-1)^{n+1} B_{n+1}}{(x+1)x} + \cdots$$

$$\frac{x^n - a_n}{x+1} + \frac{(x-1)^n - a_n}{(x+1)x} + \frac{(x-2)^n - a_n}{(x+1)x(x-1)} + \cdots$$

converge on S uniformly to $g_n(x)$, $f_n(x)$ respectively.

Proof. We first show that the series converges except for $x = -1, 0, 1, 2, 3, \ldots$.

Let

$$t_j(x) = \frac{(x-j)^n - a_n}{(x+1)x\cdots(x-j+1)},\ j \geq 1.$$

Then

$$\left|\frac{t_{j+1}(x)}{t_j(x)}\right| = \left|\frac{(x-j-1)^n - a_n}{\{(x-j)^n - a_n\}(x-j)}\right|.$$

For a given real number x,

$$\lim_{j\to\infty} \frac{(x-j-1)^n - a_n}{(x-j)^n - a_n} = 1.$$

So it is possible to choose large j such that

$$\left|\frac{(x-j-1)^n - a_n}{(x-j)^n - a_n}\right| \leq 2.$$

Also if we choose $j \geq 2|x|$, then

$$\frac{1}{|x-j|} \leq \frac{2}{j}.$$

Combining these two inequalities we have

$$\left|\frac{t_{j+1}(x)}{t_j(x)}\right| \leq \frac{4}{j}.$$

Using this inequality repeatedly one obtains

$$\left|\frac{t_{j+m}(x)}{t_j(x)}\right| \leq \frac{4^m}{(j+m-1)(j+m-2)\cdots j} \leq \frac{4^m}{8^{m-1}} \quad \text{for} \quad j \geq 8$$

and thus the given series can be compared with the series

$$\sum \frac{4^m}{8^{m-1}}$$

and hence the given series converges.

Now, for a given x using the recurrence relation for a polynomial it is easy to see that

$$f_n(x) - \frac{x^n - a_n}{x+1} - \cdots - \frac{(x-j)^n - a_n}{(x+1)\cdots(x-j+1)} = \frac{f_n(x-j-1)}{(x+1)\cdots(x-j+1)}.$$

However for a given x and n since f_n is a monic polynomial of degree $n-1$, we can choose a large j such that

$$|f_n(x-j-1)| \leq 2 \cdot j^{n-1}$$

and the fact $x - 1 < [x] \leq x$ about the greatest integer function gives

$$\frac{1}{[x]+j+1} < \frac{1}{x+j} \leq \frac{1}{[x]+j}$$

except for the case when one of these happens to be zero.

We consider $j \geq |x| + 2$. Then it is easy to see that

$$\left| \frac{f_n(x-j-1)}{(x+1)x\cdots(x-j+1)} \right| \leq \lambda \frac{2j^{n-1}}{(j-|m|)!},$$

where $m = [x] + 2$,

$$\lambda = \frac{1}{|(x+1)x\cdots(x-m-2)|},$$

a fixed quantity for a given x and j is chosen greater than $|m|$.

Hence the given series converges to $f_n(x)$.

The series converges uniformly on S because of the fact that a sequence of continuous function on a compact set converges uniformly to a continuous function. □

In the next proposition we derive a formula for a_n, B_n using the convergent series given in Theorem 4.1.

PROPOSITION 4.2

Numbers a_n and B_n as defined previously are given by

$$a_n = e[(-1)^n + (-2)^n/(-1) + (-3)^n/(-1)(-2) + \cdots],$$

$$B_{n+1} = e^{-1}[1^n/0! + 2^n/1! + 3^n/2! + \cdots]$$

Proof. It follows from Theorem 2.1, 2.2 that

$$f_n(-2) = (-1)^{n-1} + a_n$$

and

$$g_n(-2) = (-1)^n + (-1)^{n+1} B_{n+1}.$$

Also by Theorem 4.1 we have

$$f_n(-2) = \frac{(-2)^n - a_n}{(-1)} + \frac{(-3)^n - a_n}{(-1)(-2)} + \frac{(-4)^n - a_n}{(-1)(-2)(-3)} + \frac{(-5)^n - a_n}{(-1)(-2)(-3)(-4)} + \cdots,$$

$$g_n(-2) = \frac{(-2)^n + (-1)^{n+1} B_{n+1}}{(-1)} - \frac{(-3)^n + (-1)^{n+1} B_{n+1}}{(-1)(-2)} + \frac{(-4)^n + (-1)^{n+1} B_{n+1}}{(-1)(-2)(-3)} - \cdots.$$

Eliminating $f_n(-2)$ and $g_n(-2)$ from these equations one obtains the desired result. □

The formula for B_n given above is the well-known Dobinsky formula for Bell numbers [2].

Lemma 4.3. *For a given n all the elements of the set*

$$\{\sigma_n(-2), \sigma_n(-3), \sigma_n(-4), \ldots\}$$

are either transcendental numbers or rational numbers, where $\sigma_n(x)$ are defined by

$$\sigma_n(x) = \frac{x^n}{x+1} + \frac{(x-1)^n}{(x+1)x} + \frac{(x-2)^n}{(x+1)x(x-1)} + \cdots$$

for a real number x different from $-1, 0, 1, 2, \ldots$.

Proof. As in Theorem 4.1, it is easy to see that the series

$$\frac{x^n}{x+1} + \frac{(x-1)^n}{(x+1)x} + \frac{(x-2)^n}{(x+1)x(x-1)} + \cdots$$

is a convergent series except for $x = -1, 0, 1, 2, \ldots$.

It is clear that

$$\sigma_n(-m) = (-1)^n \left\{ -\frac{m^n}{m-1} + \frac{(m+1)^n}{(m-1)m} - \frac{(m+2)^n}{(m-1)m(m+1)} + \cdots \right\} \quad (7)$$

We have the formula for complementary Bell numbers [6]

$$\tilde{B}_{n+1} = e \sum_{k=0}^{\infty} (-1)^k \frac{k^{n+1}}{k!}.$$

We break this sum into two parts to obtain

$$\tilde{B}_n = e \sum_{j=0}^{m-1} (-1)^j j^{n+1}/j! + e \frac{(-1)^{m-1}}{(m-1)!} \left\{ -m^n + \frac{(m+1)^n}{m} - \cdots \right\}.$$

Hence using eq. (7) we have

$$\tilde{B}_n = e \sum_{j=0}^{m-1} (-1)^j j^{n+1}/j! + e \frac{(-1)^{m-1-n}}{(m-1)!} \sigma_n(-m).$$

Thus σ_n is a rational number iff $\tilde{B}_n$ is zero and in case it is not a rational number then it is a transcendental number. □

With a similar argument it can be shown that the following lemma is true.

Lemma 4.4. *For a non negative integer n each of the numbers $\sigma'_n(-2)$, $\sigma'_n(-3)$, $\sigma'_n(-4), \ldots$ is a transcendental number where $\sigma'_n(x)$ is defined as*

$$\sigma'_n(x) = \frac{x^n}{x+1} - \frac{(x-1)^n}{(x+1)x} + \frac{(x-2)^n}{(x+1)x(x-1)} - \cdots$$

for a real number x different from $-1, 0, 1, 2, \ldots$.

Lemma 4.5. *If a_n is nonzero then $\sigma_n(x)$ is not a rational function of x.*

Proof. If a_n is nonzero then $\sigma_n(x)$ possessess infinitely many residues namely at $x = -1, 0, 1, 2, \ldots$. □

Of course it is easy to see that

Lemma 4.6. *$\sigma_n'(x)$ is not a rational function of x for any non-negative integer n.*

4.1 *p-Adic series converging to $f_n(m)$, $g_n(m)$*

In Theorem 4.1 we have obtained two series in $\mathbb{R}$ converging to f_n, g_n. In the next theorem we obtain two series in $\mathbb{Z}_p$ converging to these polynomials in $\mathbb{Z}_p$, the ring of p-adic integers.

Theorem 4.7. *For a given m in $\mathbb{Z}_p$ both the series in*

$$f_n(m) = a_n\{1 + (m+2) + (m+2)(m+3) + \cdots\}$$
$$- (m+1)^n - (m+2)^n(m+2) - (m+3)^n(m+3)(m+2) - \cdots$$

converge in $\mathbb{Z}_p$ and the given equality holds.

Proof. For a p-adic integer m we can always choose another positive integer j such that

$$|m + j|_p$$

can be made less than any preassigned positive number. Using this fact it can shown that p-adic norms of the terms in these series can be made less than any preassigned positive quantity, a standard condition for convergence of p-adic series.

It is sufficient to show that the limit

$$\lim_{j\to\infty}\left|\begin{matrix} f_n(m) + (m+1)^n + (m+2)^n(m+2) + \cdots + (m+j)^n(m+2)\cdots(m+j) \\ -a_n\{1 + (m+2) + (m+2)(m+3) + \cdots + (m+2)\cdots(m+j)\} \end{matrix}\right|_p$$

is 0.

Using recurrence relation for $f_n(m)$ repeatedly the above limit turns out to be

$$\lim_{j\to\infty} |f_n(m+j)(m+2)(m+3)\cdots(m+j+1)|_p.$$

As $|f_n(m+j)|_p \leq 1$, the limit must be 0. □

COROLLARY 4.8

$$1^n 1! + 2^n 2! + 3^n 3! + \cdots = a_n\{1 + 2! + 3! + \cdots\} - f_n(0).$$

Proof. Proof follows by considering $m = 0$ in the preceding theorem. □

COROLLARY 4.9

If $a_n = 0$, then

$$-f_n(0) = 1^n 1! + 2^n 2! + 3^n 3! + \cdots.$$

Proof. Proof follows by considering $m = 0$ and $a_n = 0$ in the preceding theorem. □

PROPOSITION 4.10

Let p, q be two distinct primes. Assume that $\alpha = \sum_{n=0}^{\infty} n!$ is a p-adic irrational number. Then for any positive integer m, $\sum_{n=0}^{\infty} n^{q^m+1} n!$ is also a p-adic irrational number.

Proof. Since

$$\sum_{i=0}^{\infty} i^n i! = \sum_{i=0}^{\infty} (i+1)^n (i+1)!$$
$$= \sum_{i=0}^{\infty} (i+1)^{n+1} i!$$
$$= \sum_{i=0}^{\infty} \sum_{r=0}^{n+1} \binom{n+1}{r} i^r i!,$$

it follows that

$$\sum_{i=0}^{\infty} i^{n+1} i! = -n \sum_{i=0}^{\infty} i^n i! - \sum_{i=0}^{\infty} \sum_{r=0}^{n-1} \binom{n+1}{r} i^r i!.$$

Hence the relation $\sum_{i=0}^{\infty} i^n i! = a_n \alpha + b_n$ yields

$$a_{n+1}\alpha + b_{n+1} = -n(a_n\alpha + b_n) - \sum_{r=0}^{n-1} \binom{n+1}{r} (a_r\alpha + b_r).$$

Assuming α to be irrational for at least one prime and since α, 1 are linearly independent over $\mathbb{Q}$ comparing rational and irrational part of the preceding equality, we obtain

$$a_{n+1} = -na_n - \sum_{r=0}^{n-1} \binom{n+1}{r} a_r \tag{8}$$

and

$$b_{n+1} = -nb_n - \sum_{r=0}^{n-1} \binom{n+1}{r} b_r. \tag{9}$$

Taking $n = p^m$ and observing $a_1 = 0$, $\binom{p^m+1}{r} \equiv 0 \pmod{p}$ for $r = 2, 3, 4, \ldots, n-1$ and reducing (8) modulo p we have

$$a_{p^m+1} \equiv -1 \pmod{p}.$$

We conclude that a_{p^m+1} is a nonzero integer. Hence the proposition follows. □

The question of non vanishing of a_n has been studied extensively in [1,3,6,7]. Murty and Sumner [6] have shown that if $a_n = 0$ then the series

$$1^n 1! + 2^n 2! + 3^n 3! + \cdots$$

converges to an integer. Thus the result obtained by Murty and Sumner can be regarded as a special case of Theorem 4.7.

A similar approach to Theorem 4.7 would give

Theorem 4.11. *For a given m in $\mathbb{Z}_p$ both the series in*

$$g_n(m) = (-1)^{n+1}B_{n+1}\left\{1+\sum_{j=2}^{\infty}(-1)^{j+1}(m+2)(m+3)\cdots(m+j)\right\}$$
$$+(m+1)^n+\sum_{j=2}^{\infty}(-1)^{j+1}(m+2)(m+3)\cdots(m+j)(m+j)^n$$

converges in $\mathbb{Z}_p$ and the given equality holds.

COROLLARY 4.12

For a fixed prime p the p-adic sum

$$1^n 1! - 2^n 2! + 3^n 3! - 4^n 4! + \cdots + (-1)^{n+1}B_{n+1}\{1! - 2! + 3! - 4! + \cdots\}$$

is equal to $g_n(0)$.

Murty and Sumner [6] have suggested that the series

$$1! + 2! + 3! + 4! + \cdots$$

converges to an irrational number in $\mathbb{Q}_p$ and Dragovich [4] has shown that as p varies over all primes the series cannot converge to the same rational number in $\mathbb{Q}_p$. By following the approach of Dragovich [4] it can be shown that if the series

$$1! - 2! + 3! - 4! + \cdots$$

converges to rational numbers for all the primes then it should converge to at least two different rational numbers. By using the formula for $g_n(0)$ and the approach of Dragovich, it is clear that the series

$$1^n 1! - 2^n 2! + 3^n 3! - 4^n 4! + \cdots$$

does not converge to a fixed rational number for different primes p and a fixed n.

5. Another polynomial recurrence

In §2 we have given recurrence relation of $f_n(x)$ in term of $f_n(x-1)$. In this section we have recurrence for f_n in terms of $f_{n-1}, f_{n-2}, \ldots$. These recurrence relations are similar to the recurrence relation for B_n, a_n.

Theorem 5.1. *The polynomials $f_n(x), g_n(x)$ satisfies the recurrence relation*

$$f_{n+1}(x) = (x+1)^n - nf_n(x) - \sum_{r=1}^{n-1}\binom{n+1}{r}f_r(x),$$
$$g_{n+1}(x) = (x+1)^n - (n-2)g_n(x) - \sum_{r=1}^{n-1}\binom{n+1}{r}g_r(x).$$

Proof. We take $n = 1, 2, 3, \ldots, N$ in the recurrence relation given in Theorem 2.1 and so we would obtain N equations. We multiply the first equation by $\binom{N+1}{1}$, the second by

$\binom{N+1}{2}$ and so on up to $(N-1)$-th and we multiply N-th equation by N and $N+1$-th by 1 and then we sum these equations to obtain

$$\lambda_N(x)(x+1) - \lambda_N(x-1) = x^{N+1} + Nx^N - a_{N+1} - Na_N + \sum_{r=1}^{N-1} \binom{N+1}{r}(x^r - a_r),$$

where

$$\lambda_N(x) = f_{N+1}(x) + Nf_N(x) + \sum_{r=1}^{N-1} \binom{N+1}{r} f_r(x).$$

Using this recurrence relation for a_n given in eq. (8) it is easy to see that $-a_{N+1} - Na_N - \sum_{r=1}^{N-1} \binom{N+1}{r} a_r = 1$ and thus we have

$$\lambda_N(x)(x+1) - \lambda_N(x-1) = x^{N+1} + Nx^N + 1 + \sum_{r=1}^{N-1} \binom{N+1}{r} x^r.$$

Using the binomial expansion of $(x+1)^{N+1}$ above, the equation reduces to

$$\lambda_N(x)(x+1) - \lambda_N(x-1) = (x+1)^{N+1} - x^N. \tag{10}$$

However it is easy to see that there is a unique polynomial $\lambda_N(x) = (x+1)^N$ satisfying the recurrence relation in eq. (10). Hence the theorem follows. □

Using Fermat's theorem, it is easy to see from table 2 that for first few primes p, $f_{p+1}(j) \equiv j \pmod{p}$. We prove the same to be true for every prime p in the next lemma.

Lemma 5.2. *For a given prime p and an integer j,*

$$f_{p+1}(j) \equiv j \pmod{p}.$$

Proof. From the preceding theorem, using the fact that $\binom{p+1}{r} \equiv 0 \pmod{p}$ for an odd prime p and $2 \le r \le p-1$ we have

$$f_{p+1}(j) \equiv (j+1)^p - f_1(j).$$

Using Fermat's theorem and the fact that $f_1(j) = 1$, the lemma follows. □

It is interesting to see that the congruence for $g_n(x)$ as simple as that given in Lemma 5.2 is not visible from table 2 but we have the following lemma.

Lemma 5.3. *For an odd prime p,*

$$g_p(0) - 1 \equiv 1! - 2! + 3! - 4! + \cdots + p! \pmod{p}.$$

Proof. We would apply eq. (4) to show the lemma but before that we note the following about Stirling numbers for an odd prime p,

$$S(p+1, p+1) = 1, S(p+1, 2) \equiv 1 \pmod{p}$$

and

$$S(p+1, j) \equiv 0 \pmod{p} \text{ for } 3 \le j \le p.$$

These congruences follow from the congruence $S(p, j) \equiv 0 \pmod{p}$ for $2 \le j \le p-1$ and recurrence relation $S(n+1, j) = S(n, j-1) + jS(n, j)$ for Stirling numbers.

Thus

$$A_{p-1} \equiv 2 \pmod{p}$$

and

$$A_i \equiv 1 \pmod{p} \qquad \text{for } 0 \le i \le p-2.$$

It follows that

$$g_p(x) \equiv 2-(x+2)+(x+2)(x+3)-\cdots+(x+2)(x+3)\cdots(x+p) \pmod{p}.$$

Considering $x = 0$ the result follows. □

Absolute value of the alternating sum $1!-2!+3!-4!+\cdots+(-1)^{n-1}n!$ is referred to as alternating factorials. Zivkovic [8] in 1999 proved that there are only a finite number of alternating factorials that are also prime numbers.

The recurrence relation given in Theorem 5.1 gives f_n in terms of $f_{n-1}, f_{n-2}, f_{n-3}, \ldots$. However we have another recurrence relation which in our view is more simpler as far as coefficients of $f_n's$ are concerned.

Theorem 5.4. *f_n satisfies the recurrence relation*

$$f_n(x) = \sum_{r=2}^{n} \binom{n}{r} (-1)^{n-r} f_{r-1}(x) + \frac{x^n - (-1)^n}{x+1}$$

with initial condition $f_1(x) = 1$.

Proof. By Theorem 4.7 we have

$$\begin{aligned} f_n(m) = & a_n \sum_{j=1}^{\infty} (m+2)(m+3)\cdots(m+j) \\ & - \sum_{j=1}^{\infty} (m+j)^n (m+2)(m+3)\cdots(m+j). \end{aligned} \tag{11}$$

Using binomial theorem the second summation can be written as

$$\sum_{j=1}^{\infty} (m+2)(m+3)\cdots(m+j) \sum_{r=0}^{n} \binom{n}{r} (m+j+1)^r (-1)^{n-r}.$$

We rearrange summation symbols to obtain

$$\sum_{r=0}^{1} \binom{n}{r} (-1)^{n-r} \sum_{j=1}^{\infty} (m+2)(m+3)\cdots(m+j)(m+j+1)^r$$

$$+ \sum_{r=2}^{n} \binom{n}{r} \sum_{j=1}^{\infty} (m+2)(m+3)\cdots(m+j)(m+j+1)(m+j+1)^{r-1}(-1)^{n-r}.$$

By eq. (11),

$$\sum_{j=1}^{\infty}(m+2)(m+3)\cdots(m+j)(m+j+1)(m+j+1)^{r-1}$$

can be written in terms of $f_{r-1}, \ldots$ and thus eq. (11) can be written as

$$\begin{aligned} f_n(m) = a_n &\left\{\sum_{j=1}^{\infty}(m+2)(m+3)\cdots(m+j)\right\} \\ &- \sum_{r=0}^{1}\binom{n}{r}(-1)^{n-r}\sum_{j=1}^{\infty}(m+2)(m+3)\cdots(m+j)(m+j+1)^{r} \\ &- \sum_{r=2}^{n}\binom{n}{r}(-1)^{n-r}\left\{-f_{r-1}(m)-(m+1)^{r-1}\right\} \\ &- \sum_{r=2}^{n}\binom{n}{r}(-1)^{n-r}a_{r-1}\sum_{j=1}^{\infty}(m+2)\cdots(m+j). \end{aligned}$$

For a positive integer m, $f_n(m)$ is an integer whereas the right-hand side contains an infinite series which does not converge p-adically to a fixed rational number for different primes [4] and so the only possibility is coefficient of

$$\sum_{j=1}^{\infty}(m+2)(m+3)\cdots(m+j)$$

must be zero. This amounts to the same thing as saying

$$a_n - \sum_{r=0}^{1}\binom{n}{r}(-1)^{n-r} - \sum_{r=2}^{n}\binom{n}{r}(-1)^{n-r}a_{r-1} = 0 \tag{12}$$

and so

$$f_n(m) = \binom{n}{1}(-1)^{n-1} - \sum_{r=2}^{n}\binom{n}{r}(-1)^{n-r}[-f_{r-1}(m)-(m+1)^{r-1}].$$

Using binomial theorem the required polynomial recurrence follows for a positive integer m and thus for arbitrary real number x.

Equation (12) could also be derived from eq. (6). □

Similar result to Theorem 5.4 holds for g_n and we state it without proof in the next theorem.

Theorem 5.5. *$g_n(x)$ satisfies the recurrence relation*

$$g_n(x) = \sum_{r=2}^{n}\binom{n}{r}(-1)^{n+1-r}g_{r-1}(x) + \frac{x^n-(-1)^n}{x+1}$$

with the initial condition $g_1(x) = 1$.

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