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Multiplicity of summands in the random partitions of an integer

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Abstract. In this paper, we prove a conjecture of Yakubovich regarding limit shapes of 'slices' of two-dimensional (2D) integer partitions and compositions of n when the number of summands $m \sim An^{\alpha}$ for some $A > 0$ and $\alpha < \frac{1}{2}$. We prove that the probability that there is a summand of multiplicity j in any randomly chosen partition or composition of an integer n goes to zero asymptotically with n provided j is larger than a critical value. As a corollary, we strengthen a result due to Erdös and Lehner (*Duke Math. J.* **8** (1941) 335–345) that concerns the relation between the number of integer partitions and compositions when $\alpha = \frac{1}{3}$.

Keywords. Yakubovich conjecture; repeated summands; slices of Young diagrams.

1. Introduction

1.1 *Integer partitions*

Let $n \geq 1$ be any integer and let $n = a_1 + a_2 + \cdots + a_m$ for some $m \geq 1$ and some positive integers $\{a_i\}_{i=1}^{m}$. We define the set $\{a_1, \ldots, a_m\}$ to be a *partition* of n into m summands. Let $p(n)$ denote the total number of partitions of n without any restriction on the number of summands. By the Hardy–Ramanujan asymptotic formula [1] for $p(n)$, we have that

$$p(n) \sim \left(4n\sqrt{3}\right)^{-1} \mathrm{e}^{\frac{2\pi}{\sqrt{6}}\sqrt{n}}. \tag{1.1}$$

Throughout the paper, we write $a_n \sim b_n$ for two sequences a_n and b_n if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Analogous formulas have been derived in [5] for the number of partitions $p_m(n)$ of an integer n into m summands where $m = m_{A,\alpha}$ is related to n as

$$m \sim An^{\alpha} \tag{1.2}$$

for some positive constant A and $0 < \alpha < \frac{1}{2}$. Henceforth, unless otherwise mentioned, the integer m will always be related to n as in (1.2). The notion of randomness of an integer partition was first introduced in [4] to study the multiplicity of summands of a given partition. Suppose we define the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{n,m})$ where Ω denotes the set of all partitions of n into m summands, $\mathcal{F}$ is the collection of all subsets of Ω and for $\omega \in \Omega$, we let $\mathbb{P}_{n,m}(\omega) = \frac{1}{p_m(n)}$. If $B(n, m)$ denotes the event that there is a repeated summand in any such randomly chosen partition, then the main result in [4] states that $\mathbb{P}_{n,m}(B(n, m)) \to 0$ as $n \to \infty$ for $\alpha = \frac{1}{3}$. In other words, the probability that there is a summand of multiplicity two or larger in any randomly chosen partition of n into m summands is very small if $m \sim An^{\frac{1}{3}}$.

In [6] the above result has been generalized by considering limit shapes of slices of integer partitions. More precisely, let $q_k = q_{k,m,n}$ denote the number of summands of value k in any integer partition of n into m summands. For a positive integer j and $t \geq 0$, we define

$$\phi_j(t) = \sum_{k>t} \mathbf{1}(q_k = j), \tag{1.3}$$

where $\mathbf{1}(E)$ denotes the indicator function of the event E. Thus $\phi_j(t)$ denotes the number of summands larger than t that have multiplicity j. Our definition of $\phi_j(.)$ differs from [6] by a factor of j. In (1.2), we let $\alpha \geq \frac{1}{3}$ be such that

$$j_\alpha = \frac{1-\alpha}{1-2\alpha} \tag{1.4}$$

is an integer. We have the following result which is the second part of Theorem 2 of [6].

Theorem [6]. *Let $\epsilon > 0$ be fixed. For $1 \leq j < j_\alpha$, we have*

$$\mathbb{P}_{n,m}\left(\left|\frac{n^{j-1}}{m^{2j-1}}\phi_j\left(\frac{nt}{m}\right) - \frac{\mathrm{e}^{-jt}}{j}\right| > \epsilon\right) \longrightarrow 0$$

as $n \to \infty$. For $j > \frac{2-\alpha}{1-2\alpha}$ we have that

$$\mathbb{P}_{n,m}\left(\phi_j(t) > \epsilon\right) \longrightarrow 0$$

as $n \to \infty$.

For the range $j_\alpha \leq j \leq \frac{2-\alpha}{1-2\alpha}$, the limiting behaviour is stated as a conjecture which we prove as the following theorem.

Theorem 1. *Let $j \geq 1$ and $l \geq 0$ be fixed integers.*

(a) *If $j = j_\alpha$ and $s = \frac{A^{2j-1}\mathrm{e}^{-jt}}{j}$, then*

$$\mathbb{P}_{n,m}\left\{\phi_j\left(\frac{nt}{m}\right) = l\right\} \longrightarrow \frac{s^l}{l!}\mathrm{e}^{-s}$$

as $n \to \infty$.

(b) *If $j \geq j_\alpha + 1$, then for $\epsilon > 0$, we have*

$$\mathbb{P}_{n,m}\left(\phi_j(t) > \epsilon\right) \longrightarrow 0$$

as $n \to \infty$.

1.2 *Integer compositions*

Let $n \geq 1$ be any integer and let $n = a_1 + a_2 + \cdots + a_m$ for some $m \geq 1$ and some positive integers $\{a_i\}_{i=1}^m$. We define the m-tuple $(a_1, \ldots, a_m)$ to be a *composition* of n into m summands. Thus $(1, 1, 3)$ and $(3, 1, 1)$ are distinct compositions of the integer 5 into 3 summands. We define random compositions on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_{n,m})$

where $\tilde{\Omega}$ denotes the set of all compositions of n into m summands, $\tilde{\mathcal{F}}$ is the collection of all subsets of $\tilde{\Omega}$ and $\tilde{\mathbb{P}}_{n,m}(A)$ denotes the probability of occurrence of event A in the set of all compositions of n into m summands assuming each composition is equally likely. Analogous to $\phi_j(t)$ in (1.3), we define

$$\tilde{\phi}_j(t) = \sum_{k>t} \mathbf{1}(\tilde{q}_k = j)$$

with $\tilde{q}_k$ denoting the number of summands of value k in any composition of n into m summands. Letting j_α be as defined in (1.4), we have the following result which is Theorem 3 of [6].

Theorem [6]. *Let $\epsilon > 0$ be fixed. For $1 \leq j < j_\alpha$, we have*

$$\tilde{\mathbb{P}}_{n,m}\left(\left|\frac{n^{j-1}}{m^{2j-1}}\tilde{\phi}_j\left(\frac{nt}{m}\right) - \frac{\mathrm{e}^{-jt}}{j!j}\right| > \epsilon\right) \longrightarrow 0$$

as $n \to \infty$. For $j > \frac{2-\alpha}{1-2\alpha}$ we have that

$$\tilde{\mathbb{P}}_{n,m}(\tilde{\phi}_j(t) > \epsilon) \longrightarrow 0$$

as $n \to \infty$.

For the range $j_\alpha \leq j \leq \frac{2-\alpha}{1-2\alpha}$, the limiting behaviour is stated as a conjecture which we prove as the following theorem.

Theorem 2. *Let $j \geq 1$ and $l \geq 0$ be fixed integers.*

(a) *If $j \geq j_\alpha + 1$,*

$$\tilde{\mathbb{P}}_{n,m}\left(\tilde{\phi}_j(t) > \epsilon\right) \longrightarrow 0$$

as $n \to \infty$ for every $\epsilon > 0$.

(b) *If $j = j_\alpha$ and $\tilde{s} = \frac{A^{2j-1}\mathrm{e}^{-jt}}{j!j}$, then*

$$\tilde{\mathbb{P}}_{n,m}\left\{\tilde{\phi}_j\left(\frac{nt}{m}\right) = l\right\} \longrightarrow \mathrm{e}^{-\tilde{s}}\frac{\tilde{s}^l}{l!}$$

as $n \to \infty$.

The paper is organized as follows: In § 2 we prove Theorem 1 and in § 3 we prove Theorem 2. Finally, in § 4, we present the conclusion.

2. Proof of Theorem 1

In what follows, $\mathbb{Z}$ denotes the set of integers. For positive integers r and j, define $C_{r,j}$ to be the event that the number r occurs exactly j times in the partition of n into m summands. For any fixed integer $k \geq 1$ and a real number $t \geq 0$ we define $t_n = \frac{nt}{m}$,

$$\mathcal{A}(q) = \mathcal{A}_{n,k}(q) = \left\{(r_1, r_2, \ldots, r_k) \in \mathbb{Z}^k : t_n < r_1 < r_2 < \cdots < r_k \leq q\right\}$$

and

$$\tilde{A}(q) = \tilde{A}_{n,k}(q) = \left\{(r_1, r_2, \ldots, r_k) \in \mathbb{Z}^k : t_n < r_1, r_2, \ldots, r_k \leq q\right\}.$$

Let

$$S_{k,j} = S_{k,j}(t;n) = \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(\cap_{l=1}^k C_{r_l,j}). \tag{2.1}$$

To prove Theorem 1, it suffices to prove the following Proposition.

PROPOSITION 1

For $j \geq j_\alpha + 1$, we have that

$$S_{1,j}(0;n) \longrightarrow 0 \tag{2.2}$$

as $n \to \infty$. For $j = j_\alpha$ and for any fixed integer $k \geq 1$, we have that

$$S_{k,j_\alpha}(t;n) \longrightarrow \frac{s^k}{k!} \tag{2.3}$$

as $n \to \infty$, where s is as in Theorem 1.

Before we prove Theorem 1, we need the following result. The proof is analogous to the proof of Corollary 3 (pp. 34 of [3]).

Let $A_1, \ldots, A_n$ be any sequence of events. For a fixed $k \geq 1$, let

$$T_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \Pr(A_{i_1} A_{i_2} \ldots A_{i_k}).$$

For any fixed integers $l, l' \geq 1$, we have that

$$\sum_{i=l}^{2l'+l-1} (-1)^{i-l} \binom{i}{l} T_i \leq \Pr(\textit{exactly } l \textit{ of } A_1, \ldots, A_n \textit{ occur}) \leq \sum_{i=l}^{2l'+l} (-1)^{i-l} \binom{i}{l} T_i. \tag{2.4}$$

Proof of Theorem 1 (*assuming Proposition* 1).

(a) Let $j \geq j_\alpha + 1$ be fixed. From (1.3) we get that

$$\mathbb{P}_{n,m}(\phi_j(0) > 0) = \mathbb{P}_{n,m}(\cup_{r=1}^n C_{r,j}) \leq \sum_{r=1}^n \mathbb{P}_{n,m}(C_{r,j}) = S_{1,j}(0;n) \longrightarrow 0$$

as $n \to \infty$. In other words, the probability that a summand of multiplicity larger than j_α occurs in a partition of n into m summands converges to zero as $n \to \infty$.

(b) Fix two integers $l, l' \geq 1$ and let $j = j_\alpha$. From (1.3) we have that $\phi_j(t_n) = l$ if and only if exactly l of $C_{[t_n]+1,j}, \ldots, C_{n,j}$ occur. We use (2.4) to obtain that for any n,

$$\sum_{i=l}^{2l'+l-1} (-1)^{i-l} \binom{i}{l} S_{i,j}(t;n) \leq \mathbb{P}_{n,m}(\phi_j(t_n) = l) \leq \sum_{i=l}^{2l'+l} (-1)^{i-l} \binom{i}{l} S_{i,j}(t;n),$$

where $S_{i,j}(.;.)$ is as defined in (2.1). Allowing $n \to \infty$, we use Proposition 1 to obtain that

$$\sum_{i=l}^{2l'+l-1} (-1)^{i-l} \binom{i}{l} \frac{s^i}{i!} \leq \liminf_n \mathbb{P}_{n,m}(\phi_j(t_n) = l) \leq \limsup_n \mathbb{P}_{n,m}(\phi_j(t_n) = l) \leq \sum_{i=l}^{2l'+l} (-1)^{i-l} \binom{i}{l} \frac{s^i}{i!}.$$

Allowing $l' \to \infty$, we get that

$$\mathrm{e}^{-s} \frac{s^l}{l!} \leq \liminf_n \mathbb{P}_{n,m}(\phi_j(t_n) = l) \leq \limsup_n \mathbb{P}_{n,m}(\phi_j(t_n) = l) \leq \mathrm{e}^{-s} \frac{s^l}{l!}.$$

This proves (b) of Theorem 1. □

The rest of the section is devoted to the proof of Proposition 1. In what follows, we let $B_{r,j} = B_{r,j}(m,n)$ to be the event that the number r occurs at least j times in a partition of n into m summands. Let $\frac{1}{3} \leq \alpha < \frac{1}{2}$ be as in (1.2) and fix any $\beta \in (0,1)$ such that

$$\max\left(3\alpha - 1, \frac{\alpha}{2}\right) < \beta < \alpha. \tag{2.5}$$

Let $v_n = n^{1-\beta}$,

$$\begin{aligned} \beta_1 &= \beta + 1 - 3\alpha, \ \beta_2 = \alpha - \beta, \\ \beta_3 &= 2\beta - \alpha \quad \text{and} \quad \beta_0 = \min\left(\beta_1, \beta_2, \beta_3, \frac{1}{12}\right). \end{aligned} \tag{2.6}$$

Finally, choose $\theta < \frac{1-2\alpha}{\alpha}$ so that

$$\frac{m^{2+\theta}}{n} \longrightarrow 0 \tag{2.7}$$

as $n \to \infty$.

We use the following facts repeatedly in the proofs below. The positive integers d, $\{j_l\}_{l=1}^d$ and the positive numbers $\{\alpha_i\}_{i=1}^d$ are fixed. For all n sufficiently large, the following relations hold. The proofs are in the Appendix.

(A1) $\prod_{i=1}^d \left(1 + O\left(\frac{1}{n^{\alpha_i}}\right)\right)^{j_i} = 1 + O\left(\frac{1}{n^{\alpha_0}}\right)$, where $\alpha_0 = \min(\alpha_1, \alpha_2, \ldots, \alpha_d)$.

(A2) $\frac{1}{n^\beta} = O\left(\frac{1}{n^{\beta_0}}\right)$.

(A3) $\frac{1}{(n-j_1 r)^{\gamma}} = \frac{1}{n^{\gamma}}\left(1+O\left(\frac{1}{n^{\beta}}\right)\right) = \frac{1}{n^{\gamma}}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right)$ for any fixed $\gamma > 0$ and for all $r \leq j_2 v_n$.

(A4) $\frac{m}{n} = O\left(\frac{1}{m}\right) = O\left(\frac{m^2}{n}\right) = O\left(\frac{1}{n^{1-2\alpha}}\right) = O\left(\frac{1}{n^{\beta_0}}\right)$.

(A5) $\frac{m^{2j_\alpha - 1}}{n^{j_\alpha - 1}} = A^{2j_\alpha - 1}(1+o(1)) \leq 2A^{2j_\alpha - 1}$.

The proof of Proposition 1 follows from the following three lemmas.

Lemma 1. *Let $j \geq 1$ and $k \geq 1$ be any two fixed integers. We have that*

$$0 \leq \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}\left(\cap_{l=1}^{k} B_{r_l,j}\right) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}\left(\cap_{l=1}^{k} C_{r_l,j}\right) = O\left(\frac{m^2}{n}\right)^{k(j-j_\alpha)+1}.$$

Lemma 2. *Let $j \geq 1$ and $k \geq 1$ be any two fixed integers. We have that*

$$\sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}\left(\cap_{l=1}^{k} B_{r_l,j}\right) = \frac{1}{k!}\left(\frac{\mathrm{e}^{-jt}}{j}\right)^{k}\left(\frac{m^{2j-1}}{n^{j-1}}\right)^{k}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right).$$

Lemma 3. *Let $j \geq 1$ and $k \geq 1$ be any two fixed integers. We have that*

$$0 \leq \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}\left(\cap_{l=1}^{k} B_{r_l,j}\right) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}\left(\cap_{l=1}^{k} B_{r_l,j}\right) \leq \mathrm{e}^{-\frac{A}{8}n^{\beta_2}}.$$

Proof of Proposition 1 (*assuming Lemmas* 1–3). To prove (2.2), we let $k = 1$ and $t = 0$. Thus $t_n = \frac{nt}{m} = 0$ and $\mathcal{A}(q) = \tilde{\mathcal{A}}(q) = \{r : 1 \leq r \leq q\}$ where $\mathcal{A}(.)$ and $\tilde{\mathcal{A}}(.)$ are as defined in the equation preceding (2.1). Since $C_{r,j} \subseteq B_{r,j}$, we have that

$$\sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(C_{r,j}) = \sum_{1 \leq r \leq n} \mathbb{P}_{n,m}(C_{r,j}) \leq \sum_{1 \leq r \leq n} \mathbb{P}_{n,m}(B_{r,j}) = I_1 + I_2,$$

where $I_1 = \sum_{1 \leq r \leq v_n} \mathbb{P}_{n,m}(B_{r,j}) = \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(B_{r,j})$ and $I_2 = \sum_{v_n \leq r \leq n} \mathbb{P}_{n,m}(B_{r,j}) = \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(B_{r,j}) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(B_{r,j})$. From Lemma 2, we have that for sufficiently large n,

$$\begin{aligned} I_1 &= \frac{\mathrm{e}^{-tj}}{j}\left(\frac{m^{2j-1}}{n^{j-1}}\right)\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &\leq 2\frac{\mathrm{e}^{-tj}}{j}\left(\frac{m^{2j-1}}{n^{j-1}}\right) = 2\frac{\mathrm{e}^{-tj}}{j}\left(\frac{m^{2j_\alpha-1}}{n^{j_\alpha-1}}\right)\left(\frac{m^2}{n}\right)^{j-j_\alpha} \\ &\leq 4\frac{\mathrm{e}^{-tj}}{j}A^{2j_\alpha-1}\left(\frac{m^2}{n}\right)^{j-j_\alpha}. \end{aligned} \tag{2.8}$$

In the last inequality above, we have used (A5). Also, $\left(\frac{m^2}{n}\right)^{j-j_\alpha} = O\left(\frac{m^2}{n}\right)$ since $j \geq j_\alpha + 1$. We therefore have that

$$I_1 = O\left(\frac{m^2}{n}\right) \longrightarrow 0$$

as $n \to \infty$. From Lemma 3, we have that

$$I_2 \leq \mathrm{e}^{-\frac{An^{\beta_2}}{8}}.$$

From (2.1), we therefore have that

$$S_{1,j}(0; n) = \sum_{1 \leq r \leq n} \mathbb{P}_{n,m}(C_{r,j}) \leq I_1 + I_2 \longrightarrow 0$$

as $n \to \infty$. This proves (2.2).

To prove (2.3), we write $S_{k,j_\alpha} = \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(\cap_{l=1}^k C_{r_l,j_\alpha}) = S_1 - S_2 + S_3$ where $S_1 = \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j_\alpha})$,

$$S_2 = \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j_\alpha}) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k C_{r_l,j_\alpha})$$

and

$$S_3 = \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(\cap_{l=1}^k C_{r_l,j_\alpha}) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k C_{r_l,j_\alpha}).$$

From Lemma 2 and (A5) we have that

$$\begin{aligned} S_1 &= \frac{1}{k!}\left(\frac{\mathrm{e}^{-j_\alpha t}}{j_\alpha}\right)^k \left(\frac{m^{2j_\alpha - 1}}{n^{j_\alpha - 1}}\right)^k \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \frac{1}{k!}\left(\frac{\mathrm{e}^{-j_\alpha t}}{j_\alpha}\right)^k (A^{2j_\alpha - 1}(1 + o(1)))^k \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \frac{s^k}{k!}(1 + o(1))\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \longrightarrow \frac{s^k}{k!} \end{aligned}$$

as $n \to \infty$ where s is as defined in Theorem 1.

It suffices to show that $S_2 \longrightarrow 0$ and $S_3 \longrightarrow 0$ as $n \to \infty$. To estimate S_3 we use the fact that $C_{r,j} \subseteq B_{r,j}$ and have that

$$\begin{aligned} S_3 &= \sum_{\mathcal{A}(n)\backslash\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k C_{r_l,j}) \leq \sum_{\mathcal{A}(n)\backslash\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j}) \\ &= \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j}) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j}). \end{aligned} \tag{2.9}$$

From Lemma 3, we therefore have that $S_3 \leq \mathrm{e}^{-\frac{An^{\beta_2}}{8}} \longrightarrow 0$ as $n \to \infty$. Finally, letting $j = j_\alpha$ in Lemma 1, we have that $S_2 = O\left(\frac{m^2}{n}\right) \longrightarrow 0$ as $n \to \infty$. □

We prove Lemmas 1, 2 and 3 in that order.

Proof of Lemma 1. Let $k \geq 1$ and $y \geq 1$ be two integers and define $P_k(y)$ to be the number of partitions of y into less than or equal to k parts. We need the following result which is a theorem in pp. 2 of [2].

Theorem [2]. *Let $\epsilon > 0$ be given. We have that*

$$P_k(y) = \frac{1}{2\pi y}\exp\left(y^{\frac{1}{2}}g(u) + a(u) + O\left(y^{-\frac{1}{6}+\epsilon} + \frac{1}{k}\right)\right), \tag{2.10}$$

where $u = \frac{k}{\sqrt{y}}$,

$$\begin{aligned} g(u) &= \frac{2v}{u} - u\log(1-\mathrm{e}^{-v}), \\ a(u) &= \log\left(\frac{v}{u\sqrt{2}}\left(1-\mathrm{e}^{-v}-\frac{1}{2}u^2\mathrm{e}^{-v}\right)^{-1/2}\right) \end{aligned}$$

and $v = v(u)$ *is determined by*

$$u^2 = v^2\left(\int_0^v \frac{t}{\mathrm{e}^t - 1}\mathrm{d}t\right)^{-1}.$$

The proof of Lemma 1 is now obtained in three steps.

Step 1. We obtain a power series expansion for $g(.)$ for small u and derive uniform estimates for the remainder $O(.)$ term for various ranges of y (see eq. (2.13) below).

Step 2. We define a function $F(.,.)$ that is related to probability of the event $B_{r,j}$ and obtain an asymptotic expression for $F(r, j)$ and $\sum_r F(r, j)$ as r varies over a certain range.

Step 3. We convert sums involving the probabilities of the events $B_{r,j}$ into sums involving the function $F(.,.)$ to complete the proof of Lemma 1.

Step 1. By Comment 7 of [2], we know that there exists an $\eta > 0$ such that the function $v(u)$ is represented by a convergent power series in the interval $(0, \eta)$. By definition, we know that $v(.)$ is a even function of u. Choosing η sufficiently small, we then have that

$$v(u) = \sum_{k=1}^{J} a_k u^{2k} + O(u^{2J+2})$$

for all $0 < u < \eta$ and for some real constants a_k and any arbitrary integer $J \geq 1$. Also, by Comment 7, pp. 10 of [2], we have that $a_1 = 1$ and $a_2 = -\frac{1}{4}$. Thus

$$\frac{2v}{u} = 2u - \frac{u^3}{2} + \sum_{k=3}^{J} 2a_k u^{2k-1} + O(u^{2J+1}) \tag{2.11}$$

and

$$\begin{aligned}
e^{-v} &= \sum_{i=0}^{J}(-1)^i \frac{v^i}{i!} + O(v^{J+1}) \\
&= 1 - u^2 + \frac{3u^4}{4} + \sum_{k=3}^{J} b_k u^{2k} + O(u^{2J+2})
\end{aligned}$$

for all $0 < u < \eta$ and some real constants b_k. Using the expansion $\log(1-t) = -\sum_{i=1}^{2J} \frac{t^i}{i} + O(t^{2J+1}(1+|\log(1-t)|))$ for $0 < t < 1$, we then get

$$\begin{aligned}
\log(1-e^{-v}) &= \log\left(u^2 - \frac{3u^4}{4} - \sum_{k=3}^{J} b_k u^{2k} + O(u^{2J+2})\right) \\
&= 2\log u + \log\left(1 - \frac{3u^2}{4} - \sum_{k=3}^{J} b_k u^{2k-2} + O(u^{2J})\right) \\
&= 2\log u - \frac{3u^2}{4} + \sum_{k=3}^{J} c_k u^{2k-2} + O(u^{2J})
\end{aligned}$$

for some real constants c_k and for all $0 < u < \eta$. Substituting (2.11) and the above equation into the exact expression for $g(.)$ given in (2.10), we get that

$$g(u) = 2u\log\left(\frac{e}{u}\right) + \frac{u^3}{4} + \sum_{k=3}^{J} d_k u^{2k-1} + O(u^{2J+1}) \tag{2.12}$$

for some real constants d_k and for all $0 < u < \eta$. By Comment 7 of [2] we also have that $a(u) = O(u^4)$ for all $0 < u < \eta$ (our definition of $a(u)$ differs from that of [2] by an additional term of $\log 2\pi$).

To complete Step 1, we have the following result for $P_k(y)$ for k very close to m and as y varies in distinct ranges.

Let $j \geq 1$, $l \geq 0$ and $J \geq j_\alpha$ be fixed integers and for θ as in (2.7), let $\theta_0 = \min\left(2\theta, \frac{2+\theta}{12}, J\theta - 1\right)$. For $\epsilon = \frac{1}{12}$ and m as in (1.2), we have that

$$P_{m-l}(y) = \frac{1}{2\pi y}\exp\left((m-l)\log\left(\frac{ye^2}{(m-l)^2}\right) + \sum_{k=2}^{J} a_k \frac{(m-l)^{2k-1}}{y^{k-1}} + R\right) \tag{2.13}$$

for some real constants a_k and

$$R = \begin{cases} O\left(\frac{1}{n^{\beta_0}}\right), & \text{if } n - jv_n \leq y \leq n, \\ O\left(\frac{1}{m^{\theta_0}}\right), & \text{if } m^{2+\theta} \leq y \leq n - jv_n, \\ O\left(\frac{m}{(\log m)^J}\right), & \text{if } m^2\log m \leq y \leq m^{2+\theta}, \end{cases}$$

where the $O(\cdot)$ terms are all independent of y.

Proof of eq. (2.13). We prove for $l = 0$. Let $\{e_i\}$ be any sequence such that $\frac{m^2}{e_n} \longrightarrow 0$ as $n \to \infty$. For $e_n \leq y$ we have that

$$u = \frac{m}{\sqrt{y}} \leq \frac{m}{\sqrt{e_n}} \longrightarrow 0$$

as $n \to \infty$. Since $u < \eta$ for all n sufficiently large, the expansion for $g(u)$ given by (2.12) holds and $a(u) = O(u^4)$. Hence we have that

$$y^{\frac{1}{2}} g(u) + a(u) = m \log\left(\frac{ye^2}{m^2}\right) + \frac{m^3}{4y} + \sum_{k=3}^{J} a_k \frac{m^{2k-1}}{y^{k-1}} + R_1,$$

where $R_1 = O\left(\frac{m^{2J+1}}{y^J}\right) + O\left(\frac{m^4}{y^2}\right)$. Letting $\epsilon = \frac{1}{12}$ in (2.10) we then get that for $e_n \leq y$,

$$P_m(y) = \frac{1}{2\pi y} \exp\left(m \log\left(\frac{ye^2}{m^2}\right) + \sum_{k=2}^{J} a_k \frac{m^{2k-1}}{y^{k-1}} + R\right), \tag{2.14}$$

where

$$\begin{aligned} R &= R_1 + O\left(\frac{1}{y^{1/12}} + \frac{1}{m}\right) = O\left(\frac{m^{2J+1}}{y^J} + \frac{m^4}{y^2} + \frac{1}{y^{1/12}} + \frac{1}{m}\right) \\ &= O\left(R_{11} + R_{12} + R_{13} + \frac{1}{m}\right) \end{aligned} \tag{2.15}$$

and $R_{11} = \frac{m^{2J+1}}{e_n^J}$, $R_{12} = \frac{m^4}{e_n^2}$ and $R_{13} = \frac{1}{e_n^{1/12}}$. In (2.15) and henceforth, any $O(.)$ term is independent of the variable y. We consider three cases separately.

Case I. $e_n = n - jv_n$. We have that

$$\frac{m^2}{e_n} = \frac{m^2}{n}\left(1 - \frac{jv_n}{n}\right)^{-1} = \frac{m^2}{n}\left(1 + O\left(\frac{1}{n^\beta}\right)\right) \longrightarrow 0$$

as $n \to \infty$. Hence (2.14) holds and from (A3), we have that

$$R_{11} = \frac{m^{2J+1}}{(n - jv_n)^J} = \frac{m^{2J+1}}{n^J}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right).$$

Since $J \geq j_\alpha$, we therefore have that

$$\begin{aligned} \frac{m^{2J+1}}{n^J} &= \left(\frac{m^2}{n}\right)^{J-j_\alpha+1} \frac{m^{2j_\alpha-1}}{n^{j_\alpha-1}} \leq 2A^{2j_\alpha-1}\left(\frac{m^2}{n}\right)^{J-j_\alpha+1} \\ &\leq 2A^{2j_\alpha-1}\left(\frac{m^2}{n}\right) = O\left(\frac{m^2}{n}\right) = O\left(\frac{1}{n^{\beta_0}}\right) \end{aligned}$$

for sufficiently large n, where to obtain the first inequality in the first line we use (A5) and to obtain the last equality in the second line, we use (A4). Thus

$R_{11} = O\left(\frac{1}{n^{\beta_0}}\right)$. Analogously $R_{12} = \frac{m^4}{(n-jv_n)^2} = O\left(\frac{m^4}{n^2}\right) = O\left(\frac{1}{n^{\beta_0}}\right)$ and $R_{13} = \frac{1}{(n-jv_n)^{1/12}} = O\left(\frac{1}{n^{1/12}}\right) = O\left(\frac{1}{n^{\beta_0}}\right)$ by our choice of β_0 in (2.6). From (A4), we have that $\frac{1}{m} = O\left(\frac{1}{n^{\beta_0}}\right)$. Hence $R_{11} + R_{12} + R_{13} + \frac{1}{m} = O\left(\frac{1}{n^{\beta_0}}\right)$. This implies that R in (2.14) is $O\left(\frac{1}{n^{\beta_0}}\right)$. This proves (2.13) for the case $n - jv_n \leq y \leq n$.

Case II. $e_n = m^{2+\theta}$. We have that

$$\frac{m^2}{e_n} = \frac{1}{m^\theta} \longrightarrow 0$$

as $n \to \infty$. Hence (2.14) holds and we have $R_{11} = \frac{1}{m^{J\theta-1}}$, $R_{12} = \frac{1}{m^{2\theta}}$ and $R_{13} = \frac{1}{m^{(2+\theta)/12}}$. Hence $R_{11}+R_{12}+R_{13}+\frac{1}{m} = O\left(\frac{1}{m^{\theta_0}}\right)$ where $\theta_0 = \min(1, J\theta-1, 2\theta, \frac{2+\theta}{12}) = \min(J\theta - 1, 2\theta, \frac{2+\theta}{12})$ since $2\theta < 1$. Therefore $R = O\left(\frac{1}{m^{\theta_0}}\right)$ and this proves (2.13) for the case $m^{2+\theta} \leq y \leq n - jv_n$.

Case III. $e_n = m^2 \log m$. We have that

$$\frac{m^2}{e_n} = \frac{1}{\log m} \longrightarrow 0$$

as $n \to \infty$. Hence (2.14) holds and we have $R_{11} = \frac{m}{(\log m)^J}$, $R_{12} = \left(\frac{1}{\log m}\right)^2$ and $R_{13} = \frac{1}{m^{1/6}(\log m)^{1/12}}$. Hence $R_{11} + R_{12} + R_{13} + \frac{1}{m} = O\left(\frac{m}{(\log m)^J}\right)$. This implies that $R = O\left(\frac{m}{(\log m)^J}\right)$ and this proves (2.13) for the case $m^2 \log m \leq y \leq m^{2+\theta}$. □

Before we proceed to Step 2, we have the following result that is used frequently below. The proof is in the Appendix.

Let $j \geq 1$ and $l \geq 0$ be fixed integers. For all $r \leq jv_n$, we have

$$\frac{P_{m-l-1}(n-r)}{P_{m-l}(n-r)} = \frac{m^2}{n}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right), \tag{2.16}$$

where the $O(\cdot)$ term is independent of r.

Step 2. For positive integers j and r, we define

$$F(r, j) = F_{m,n}(r, j) = \frac{p_{m-j}(n-r)}{p_m(n)}, \tag{2.17}$$

where $p_m(n)$ denotes the number of partitions of n into m summands. We state and prove two results about the function $F(r, j)$ that are needed for the proof of Lemma 1.

Let $j \geq 1$ and $j_1 \geq 1$ be any two fixed integers. For n sufficiently large and $r \leq j_1 v_n$, we have

$$F(r, j) = \left(1 - \frac{r}{n}\right)^m \left(\frac{m^2}{n}\right)^j \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right), \tag{2.18}$$

where the $O(\cdot)$ term is independent of r.

Proof of (2.18). If $P_m(n)$ denotes the number of partitions of n into at most m summands, we have

$$p_m(n) = P_m(n) - P_{m-1}(n).$$

Letting $I_1 = I_1(r) = \frac{P_m(n-r)}{P_m(n)}$, $I_2 = I_2(r) = \frac{P_{m-j}(n-r)}{P_m(n-r)}$ and $I_3 = I_3(r) = \frac{\left(1 - \frac{P_{m-j-1}(n-r)}{P_{m-j}(n-r)}\right)}{\left(1 - \frac{P_{m-1}(n)}{P_m(n)}\right)}$, we therefore have from (2.17) that

$$F(r, j) = I_1(r) I_2(r) I_3(r). \tag{2.19}$$

We estimate I_1, I_2 and I_3 separately. To estimate $I_3(r)$, we have by (2.16) and (A4) that

$$\frac{P_{m-1}(n)}{P_m(n)} = \frac{m^2}{n}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) = O\left(\frac{1}{n^{\beta_0}}\right) \tag{2.20}$$

and for all $r \leq j_1 v_n$,

$$\frac{P_{m-j-1}(n-r)}{P_{m-j}(n-r)} = \frac{m^2}{n}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) = O\left(\frac{1}{n^{\beta_0}}\right).$$

Here and henceforth all $O(\cdot)$ terms are independent of r. Hence for all $r \leq j_1 v_n$, we have that

$$I_3(r) = 1 + O\left(\frac{1}{n^{\beta_0}}\right). \tag{2.21}$$

To estimate $I_2(r)$, we get from (5.1) that for all $r \leq j_1 v_n$,

$$\begin{aligned} I_2(r) &= \frac{P_{m-j}(n-r)}{P_m(n-r)} = \prod_{k=1}^{j} \frac{P_{m-k}(n-r)}{P_{m-k+1}(n-r)} \\ &= \prod_{k=1}^{j} \frac{m^2}{n}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) = \frac{m^{2j}}{n^j}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right). \end{aligned} \tag{2.22}$$

To obtain the last equality, we have used (A1).

We now estimate I_1. For all $r \le j_1 v_n$, we have from (2.13) that

$$
\begin{aligned}
I_1(r) &= \frac{P_m(n-r)}{P_m(n)} \\
&= \left(1-\frac{r}{n}\right)^{m-1} \exp\left(\sum_{k=2}^{j_\alpha} a_k \left(\frac{m^{2k-1}}{(n-r)^{k-1}} - \frac{m^{2k-1}}{n^{k-1}}\right)\right. \\
&\qquad \left. + O\left(\frac{1}{n^{\beta_0}}\right)\right). \qquad (2.23)
\end{aligned}
$$

For $k \ge 2$ and all $r \le j_1 v_n$, we have that

$$
\begin{aligned}
m^{2k-1}\left(\frac{1}{(n-r)^{k-1}} - \frac{1}{n^{k-1}}\right) &\le m^{2k-1}\left(\frac{1}{(n-j_1 v_n)^{k-1}} - \frac{1}{n^{k-1}}\right) \\
&= \frac{m^{2k-1}}{n^{k-1}}\left(\frac{n^{k-1}}{(n-j_1 v_n)^{k-1}} - 1\right) \\
&= \frac{m^{2k-1}}{n^{k-1}} O\left(\frac{1}{n^{\beta}}\right) \quad \text{(by (A3))}.
\end{aligned}
$$

Since $\frac{m^2}{n} < 1$ for n sufficiently large, we have that $\frac{m^{2k-1}}{n^{k-1}} = \frac{n}{m}\left(\frac{m^2}{n}\right)^k \le \frac{n}{m}\left(\frac{m^2}{n}\right)^2 = \frac{m^3}{n}$ for $k \ge 2$. Therefore

$$
\frac{m^{2k-1}}{n^{k-1}} O\left(\frac{1}{n^{\beta}}\right) \le \frac{m^3}{n} O\left(\frac{1}{n^{\beta}}\right) = O\left(\frac{n^{3\alpha}}{n^{1+\beta}}\right) = O\left(\frac{1}{n^{\beta_1}}\right).
$$

Since $\beta_0 \le \beta_1$, we have $O\left(\frac{1}{n^{\beta_1}}\right) = O\left(\frac{1}{n^{\beta_0}}\right)$ and therefore for all $k \ge 2$ and $r \le j_1 v_n$, we have

$$
m^{2k-1}\left(\frac{1}{(n-r)^{k-1}} - \frac{1}{n^{k-1}}\right) = O\left(\frac{1}{n^{\beta_0}}\right). \qquad (2.24)
$$

Substituting the above bound into (2.23), we have that for $r \le j_1 v_n$,

$$
\begin{aligned}
I_1(r) &= \left(1-\frac{r}{n}\right)^{m-1} \exp\left(O\left(\frac{1}{n^{\beta_0}}\right)\right) \\
&= \left(1-\frac{r}{n}\right)^{m} \left(1 + O\left(\frac{v_n}{n}\right)\right) \exp\left(O\left(\frac{1}{n^{\beta_0}}\right)\right) \\
&= \left(1-\frac{r}{n}\right)^{m} \left(1 + O\left(\frac{1}{n^{\beta}}\right)\right)\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\
&= \left(1-\frac{r}{n}\right)^{m} \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right). \qquad (2.25)
\end{aligned}
$$

To obtain the last equality, we use (A1). Substituting (2.25), (2.22) and (2.21) into (2.19) we have that

$$F(r,j) = \left(1-\frac{r}{n}\right)^m \frac{m^{2j}}{n^j}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right)^3.$$

To obtain (2.18) from the above equation, we use (A1). □

We complete Step 2 by proving the following result. Let $\tilde{\mathcal{A}}(\cdot)$ be as defined in the equation preceding (2.1).

For a fixed integer $k \geq 1$, let $j_1, j_2, \ldots, j_k$ be fixed positive integers and let $J = \sum_{l=1}^k j_l$. For all n sufficiently large, we have

$$\sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^k r_l j_l, J\right) = \frac{e^{-Jt}}{\prod_{l=1}^k j_l}\frac{m^{2J-k}}{n^{J-k}}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right). \tag{2.26}$$

Proof of (2.26). If $r_l \leq v_n$ for $1 \leq l \leq k$, we must have that $R = \sum_{l=1}^k r_l j_l \leq J v_n$ and therefore by (2.18), we have that

$$F\left(\sum_{l=1}^k r_l j_l, J\right) = \left(1-\frac{R}{n}\right)^m \left(\frac{m^2}{n}\right)^J\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right). \tag{2.27}$$

Here and henceforth, the $O(.)$ terms are independent of r_i, $1 \leq i \leq k$. For $R \leq J v_n$, we have

$$e^{-\frac{R}{n}} = 1-\frac{R}{n}+O\left(\frac{v_n^2}{n^2}\right) = 1-\frac{R}{n}+O\left(\frac{1}{n^{2\beta}}\right).$$

We therefore have

$$\begin{aligned}\left(1-\frac{R}{n}\right)^m &= \left(e^{-\frac{R}{n}}+O\left(\frac{1}{n^{2\beta}}\right)\right)^m = e^{-\frac{Rm}{n}}\left(1+e^{\frac{R}{n}}O\left(\frac{1}{n^{2\beta}}\right)\right)^m \\ &= e^{-\frac{Rm}{n}}\left(1+O\left(\frac{1}{n^{2\beta}}\right)\right)^m,\end{aligned}$$

where in the above equation, we use the fact that $e^{\frac{R}{n}}O\left(\frac{1}{n^{2\beta}}\right) \leq e^J O\left(\frac{1}{n^{2\beta}}\right) = O\left(\frac{1}{n^{2\beta}}\right)$. Since $2\beta > \alpha$, we have that $\left(1+O\left(\frac{1}{n^{2\beta}}\right)\right)^m = 1+O\left(\frac{m}{n^{2\beta}}\right) = 1+O\left(\frac{n^\alpha}{n^{2\beta}}\right) = 1+O\left(\frac{1}{n^{\beta_3}}\right)$. Thus from (2.27) we have

$$\begin{aligned}F\left(\sum_{l=1}^k r_l j_l, J\right) &= e^{-\frac{Rm}{n}}\left(\frac{m^2}{n}\right)^J\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right)\left(1+O\left(\frac{1}{n^{\beta_3}}\right)\right) \\ &= e^{-\frac{Rm}{n}}\left(\frac{m^2}{n}\right)^J\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right) \qquad \text{(by (A1))} \\ &= \prod_{l=1}^k e^{-\frac{j_l r_l m}{n}}\left(\frac{m^2}{n}\right)^{j_l}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right).\end{aligned}$$

For any $k \geq 1$ and any set of functions $h_j(.)$, $1 \leq j \leq k$, we have

$$\begin{aligned}
&\sum_{1\leq i_1,\ldots,i_k\leq n} h_1(i_1)h_2(i_2)\ldots h_k(i_k)\\
&\quad = \sum_{1\leq i_1\leq n}\sum_{1\leq i_2\leq n}\cdots\sum_{1\leq i_k\leq n} h_1(i_1)h_2(i_2)\ldots h_k(i_k)\\
&\quad = \sum_{1\leq i_1\leq n} h_1(i_1)\sum_{1\leq i_2\leq n} h_2(i_2)\ldots\sum_{1\leq i_k\leq n} h_k(i_k)\\
&\quad = \prod_{j=1}^{k}\left(\sum_{1\leq i\leq n} h_j(i)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^{k} r_l j_l,\, J\right) &= \sum_{\tilde{\mathcal{A}}(v_n)}\prod_{l=1}^{k} \mathrm{e}^{-\frac{j_l r_l m}{n}}\left(\frac{m^2}{n}\right)^{j_l}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right)\\
&= \prod_{l=1}^{k} J_l\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right),
\end{aligned} \tag{2.28}$$

where $J_l = \sum_{t_n<r\leq v_n} \mathrm{e}^{-\frac{j_l r m}{n}}\left(\frac{m^2}{n}\right)^{j_l}$.

Using the fact that $-\frac{jm}{n^{\beta}} - \frac{jm}{n} < -Dn^{\beta_2}$ for some positive constant D, we have that

$$\begin{aligned}
J_l &= \left(\frac{m^2}{n}\right)^{j_l}\frac{\mathrm{e}^{-\frac{j_l m}{n}(t_n+O(1))} - \mathrm{e}^{-\frac{j_l m}{n^{\beta}}-\frac{j_l m}{n}+O\left(\frac{m}{n}\right)}}{1-\mathrm{e}^{-\frac{j_l m}{n}}}\\
&= \left(\frac{m^2}{n}\right)^{j_l}\frac{\mathrm{e}^{-j_l t} + O\left(\mathrm{e}^{-Dn^{\beta_2}} + \frac{m}{n}\right)}{1-\mathrm{e}^{-\frac{j_l m}{n}}}\\
&= \left(\frac{m^2}{n}\right)^{j_l}\frac{\mathrm{e}^{-j_l t} + O\left(\mathrm{e}^{-Dn^{\beta_2}} + \frac{m}{n}\right)}{\frac{j_l m}{n} + O\left(\frac{m^2}{n^2}\right)}\\
&= \frac{\mathrm{e}^{-j_l t}}{j_l}\left(\frac{m^{2j_l-1}}{n^{j_l-1}}\right)\left(1+\mathrm{e}^{j_l t}\left(\mathrm{e}^{-Dn^{\beta_2}} + \frac{m}{n}\right)\right)\left(1+\frac{n}{m j_l}O\left(\frac{m^2}{n^2}\right)\right)^{-1}\\
&= \frac{\mathrm{e}^{-j_l t}}{j_l}\left(\frac{m^{2j_l-1}}{n^{j_l-1}}\right)\left(1+O\left(\frac{m}{n}\right)\right).
\end{aligned}$$

To obtain the last equality, we use

$$\begin{aligned}
&\left(1+\mathrm{e}^{j_l t}O\left(\mathrm{e}^{-Dn^{\beta_2}}\right)\right)\left(1+\frac{n}{m j_l}O\left(\frac{m^2}{n^2}\right)\right)^{-1}\\
&\quad = (1+O(\mathrm{e}^{-Dn^{\beta_2}}))\left(1+O\left(\frac{m}{n}\right)\right)^{-1}\\
&\quad = (1+O(\mathrm{e}^{-Dn^{\beta_2}}))\left(1+O\left(\frac{m}{n}\right)\right)\\
&\quad = 1+O(\mathrm{e}^{-Dn^{\beta_2}})+O\left(\frac{m}{n}\right)\\
&\quad = 1+O\left(\frac{m}{n}\right).
\end{aligned}$$

Substituting the above expression for J_l into (2.28), we therefore have that

$$\sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^{k} r_l j_l, J\right)$$
$$= \prod_{l=1}^{k} \frac{e^{-j_l t}}{j_l}\left(\frac{m^{2j_l-1}}{n^{j_l-1}}\right)\left(1+O\left(\frac{m}{n}\right)\right)^k\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right)$$
$$= \frac{e^{-Jt}}{\prod_{l=1}^{k} j_l}\left(\frac{m^{2J-k}}{n^{J-k}}\right)\left(1+O\left(\frac{m}{n}\right)\right)^k\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right).$$

To obtain (2.26) from the above equation, we use (A4) and (A1). □

Step 3.

Proof of Lemma 1. Let $k \geq 1$ be fixed and define

$$\Delta_n = \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) - \mathbb{P}_{n,m}(\cap_{l=1}^{k} C_{r_l,j}). \tag{2.29}$$

Since $C_{r,j} = B_{r,j} \setminus B_{r,j+1}$, we have that

$$0 \leq \Delta_n = \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) - \mathbb{P}_{n,m}(\cap_{l=1}^{k} B_{r_l,j} \cap \cap_{l=1}^{k} (B_{r_l,j+1})^c)$$
$$= \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^{k} B_{r_l,j} \cap (\cup_{w=1}^{k} B_{r_w,j+1}))$$
$$\leq \sum_{\mathcal{A}(v_n)} \sum_{w=1}^{k} \mathbb{P}_{n,m}(\cap_{l=1}^{k} B_{r_l,j} \cap B_{r_w,j+1})$$
$$= \sum_{\mathcal{A}(v_n)} \sum_{w=1}^{k} \mathbb{P}_{n,m}(\cap_{l=1,l\neq w}^{k} B_{r_l,j} \cap B_{r_w,j+1}).$$

For any fixed integers $j_1, \ldots, j_k$ and $r_1 < r_2 < \cdots < r_k$, we have that

$$\mathbb{P}_{n,m}\left(\cap_{l=1}^{k} B_{r_l,j_l}\right) = F\left(\sum_{l=1}^{k} r_l j_l, \sum_{l=1}^{k} j_l\right). \tag{2.30}$$

Hence

$$0 \leq \Delta_n \leq \sum_{\mathcal{A}(v_n)} \sum_{w=1}^{k} F\left(\sum_{l=1}^{k} r_l j + r_w, kj+1\right)$$
$$\leq \sum_{\tilde{\mathcal{A}}(v_n)} \sum_{w=1}^{k} F\left(\sum_{l=1}^{k} r_l j + r_w, kj+1\right)$$
$$= k \sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^{k} r_l j + r_1, kj+1\right), \tag{2.31}$$

where the last equality follows by symmetry. From (2.26), we have that

$$\sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^{k} r_l j + r_1, kj+1\right) = c_{k,j} \frac{m^{2kj+2-k}}{n^{kj+1-k}}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right),$$

where $c_{k,j} = \frac{e^{-(kj+1)t}}{j^{k-1}(j+1)}$. But, from (A5), we have that

$$\begin{aligned}\frac{m^{2kj+2-k}}{n^{kj+1-k}} &= \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \frac{m^2}{n} = \left(\frac{m^{2j_\alpha-1}}{n^{j_\alpha-1}}\right)^k \left(\frac{m^2}{n}\right)^{k(j-j_\alpha)+1} \\ &\le \left(2A^{2j_\alpha-1}\right)^k \left(\frac{m^2}{n}\right)^{k(j-j_\alpha)+1} = O\left(\frac{m^2}{n}\right)^{k(j-j_\alpha)+1}.\end{aligned} \tag{2.32}$$

This completes the proof of Lemma 1. □

Proof of Lemma 2. For fixed integers $k \ge 1$ and $q \ge 1$, define

$$\begin{aligned}\mathcal{D}(q) &= \mathcal{D}_{n,k}(q) \\ &= \{(r_1,r_2,\ldots,r_k) \in \mathbb{Z}^k : t_n < r_1,r_2,\ldots,r_k \le q \text{ and } r_i \ne r_j \text{ if } i \ne j\}.\end{aligned}$$

Further let $F(.,.)$ be as defined in (2.17). We first show that Lemma 2 follows from the two statements below that are proved later:

$$\sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j}) = \frac{1}{k!}\sum_{\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) \tag{2.33}$$

and

$$\sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) = \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k O\left(\frac{m}{n}\right). \tag{2.34}$$

For now we will assume that the above two statements hold. From (2.33) we have that

$$\begin{aligned}\sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_l,j}) &= \frac{1}{k!}\left(\sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right)\right. \\ &\quad \left. - \sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right)\right).\end{aligned}$$

We know by (2.26) that

$$\begin{aligned}\sum_{\tilde{\mathcal{A}}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) &= \frac{e^{-kjt}}{j^k}\frac{m^{2jk-k}}{n^{jk-k}}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \left(\frac{e^{-jt}}{j}\right)^k \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right).\end{aligned}$$

Hence from (2.34), we have that

$$\sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}\left(\cap_{l=1}^{k} B_{r_l,j}\right) = \frac{1}{k!}\left(\left(\frac{\mathrm{e}^{-jt}}{j}\right)^k \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right) - \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k O\left(\frac{m}{n}\right)\right) = \frac{1}{k!}\left(\frac{\mathrm{e}^{-jt}}{j}\right)^k \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \times R,$$

where

$$R = \left(1+O\left(\frac{1}{n^{\beta_0}}\right) - \left(\frac{j}{\mathrm{e}^{-jt}}\right)^k O\left(\frac{m}{n}\right)\right) = \left(1+O\left(\frac{1}{n^{\beta_0}}\right)+O\left(\frac{m}{n}\right)\right) = \left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right).$$

In obtaining the last equality we have used (A4). This completes the proof of Lemma 2. □

Proof of (2.33). For any two sets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \tilde{\mathcal{A}}(n)$, we have that

$$\sum_{\mathcal{V}_1\cup\mathcal{V}_2} F\left(\sum_{l=1}^{k} jr_l, kj\right) \leq \sum_{\mathcal{V}_1} F\left(\sum_{l=1}^{k} jr_l, kj\right) + \sum_{\mathcal{V}_2} F\left(\sum_{l=1}^{k} jr_l, kj\right) \quad (2.35)$$

with equality, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint. Letting $\mathcal{P}_k$ to be the set of all permutations of the elements of the set $\{1, 2, \ldots, k\}$, we have that

$$\mathcal{D}(v_n) = \cup_{\sigma\in\mathcal{P}_k} \mathcal{V}_\sigma,$$

where

$$\mathcal{V}_\sigma = \{(r_1, r_2, \ldots, r_k) : t_n < r_{\sigma(1)} < r_{\sigma(2)} < \cdots < r_{\sigma(k)} \leq v_n\}.$$

Also, if $\sigma, \sigma' \in \mathcal{P}_k$ and $\sigma \neq \sigma'$, we have that $\mathcal{V}_\sigma$ and $\mathcal{V}_{\sigma'}$ are disjoint. Hence from (2.35), we have that

$$\sum_{\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) = \sum_{\sigma\in\mathcal{P}_k}\sum_{\mathcal{V}_\sigma} F\left(\sum_{l=1}^{k} jr_l, kj\right).$$

By symmetry, for $\sigma \in \mathcal{P}_k$, we have

$$\sum_{\mathcal{V}_\sigma} F\left(\sum_{l=1}^{k} jr_l, kj\right) = \sum_{\mathcal{V}_{\sigma_0}} F\left(\sum_{l=1}^{k} jr_l, kj\right),$$

where σ_0 is the permutation such that $\sigma_0(i) = i$ for $1 \le i \le k$. But $\mathcal{V}_{\sigma_0} = \mathcal{A}(v_n)$ and the number of elements in $\mathcal{P}_k$ is $k!$. Hence

$$\sum_{\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) = k! \sum_{\mathcal{A}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right). \tag{2.36}$$

Finally, (2.33) follows from (2.30). □

Proof of (2.34). If $(r_1, \ldots, r_k) \in \tilde{\mathcal{A}}(v_n) \setminus \mathcal{D}(v_n)$ then we have that $r_a = r_b$ for some two distinct indices a and b. If $\mathcal{E}$ denotes the set of such distinct pairs, then $\mathcal{E}$ has cardinality $\frac{k(k-1)}{2}$. For $(a, b) \in \mathcal{E}$ define $\mathcal{G}_{ab} = \{(r_1, ..., r_k) : t_n < r_l \le v_n, 1 \le l \le k \text{ and } r_a = r_b\}$. Hence we have that

$$\tilde{\mathcal{A}}(v_n) \setminus \mathcal{D}(v_n) \subseteq \cup_{(a,b)\in\mathcal{E}} \mathcal{G}_{ab}. \tag{2.37}$$

Hence from (2.35), we get that

$$\sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) \le \sum_{(a,b)\in\mathcal{E}} \sum_{\mathcal{G}_{ab}} F\left(\sum_{l=1}^{k} jr_l, kj\right).$$

By symmetry, we have that

$$\sum_{\mathcal{G}_{ab}} F\left(\sum_{l=1}^{k} jr_l, kj\right) = \sum_{\mathcal{G}_{12}} F\left(\sum_{l=1}^{k} jr_l, kj\right).$$

Since $\mathcal{E}$ has cardinality $\frac{k(k-1)}{2}$, we have

$$\sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)} F\left(\sum_{l=1}^{k} jr_l, kj\right) \le \frac{k(k-1)}{2} \sum_{\mathcal{G}_{12}} F\left(\sum_{l=1}^{k} jr_l, kj\right). \tag{2.38}$$

But

$$\sum_{\mathcal{G}_{12}} F\left(\sum_{l=1}^{k} jr_l, kj\right) = \sum_{t_n < r_1, \ldots, r_{k-1} \le v_n} F\left(2jr_1 + \sum_{l=2}^{k-1} jr_l, kj\right).$$

From (2.26), we therefore have that

$$\begin{aligned} \sum_{\mathcal{G}_{12}} F\left(\sum_{l=1}^{k} jr_l, kj\right) &= \frac{\mathrm{e}^{-kjt}}{2j^{k-1}} \frac{m^{2kj-k+1}}{n^{kj-k+1}} \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \left(\frac{m}{n}\right) \frac{\mathrm{e}^{-kjt}}{2j^{k-1}} \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k O\left(\frac{m}{n}\right). \end{aligned}$$

This proves (2.34). □

Proof of Lemma 3. We first estimate $\mathbb{P}_{n,m}(B_{r,j})$ for the range $r \geq v_n$ and for a fixed integer $j \geq 1$.

Lemma 4. *For all n sufficiently large we have*

$$\mathbb{P}_{n,m}(B_{r,j}) \leq \begin{cases} \exp\left(-\frac{An^{\beta_2}}{4}\right), & \text{if } v_n \leq r \leq n - m^{2+\theta}, \\ \exp\left(-\frac{m}{4}\right), & \text{if } n - m^{2+\theta} \leq r \leq n - m^2 \log m, \\ \exp\left(-C(\alpha) m \log m\right), & \text{if } r \geq n - m^2 \log m, \end{cases} \tag{2.39}$$

where $C(\alpha) = \frac{1-2\alpha}{8\alpha}$.

Proof. We note that the event $B_{r,j}$ is contained in the event $B_{r,1} = B_{r,1}(m,n)$ where r occurs as a summand in the partition of n into m parts. We have that for all sufficiently large n,

$$\begin{aligned} \mathbb{P}_{n,m}(B_{r,j}) &\leq \mathbb{P}_{n,m}(B_{r,1}) = \frac{p_{m-1}(n-r)}{p_m(n)} \\ &= \frac{P_{m-1}(n-r) - P_{m-2}(n-r)}{P_m(n) - P_{m-1}(n)} \leq \frac{P_{m-1}(n-r)}{P_m(n) - P_{m-1}(n)} \\ &= \frac{P_{m-1}(n-r)}{P_{m-1}(n)} \frac{P_{m-1}(n)}{P_m(n)} \left(1 - \frac{P_{m-1}(n)}{P_m(n)}\right)^{-1} \\ &= \frac{P_{m-1}(n-r)}{P_{m-1}(n)} O\left(\frac{1}{n^{\beta_0}}\right)\left(1 - O\left(\frac{1}{n^{\beta_0}}\right)\right)^{-1} \quad \text{(by (2.20))} \\ &= \frac{P_{m-1}(n-r)}{P_{m-1}(n)} O\left(\frac{1}{n^{\beta_0}}\right)\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &\leq \frac{P_{m-1}(n-r)}{P_{m-1}(n)}. \end{aligned} \tag{2.40}$$

For $v_n \leq r \leq n - m^{2+\theta}$, we have from (2.13) that for all sufficiently large n,

$$\begin{aligned} \frac{P_{m-1}(n-r)}{P_{m-1}(n)} &= \frac{n}{n-r} \exp\left((m-1)\log\left(\frac{n-r}{n}\right)\right. \\ &\quad \left. + T(n-r) - T(n) + O\left(\frac{1}{m^{\theta_0}} + \frac{1}{n^{\beta_0}}\right)\right), \end{aligned}$$

where $T(y) = \sum_{k=2}^{J} a_k \frac{(m-1)^{2k-1}}{y^{k-1}}$, $\theta_0 = \min(J\theta - 1, 2\theta, \frac{2+\theta}{12})$ and θ is as defined in (2.7). Choosing J large enough so that $J\theta \geq 2$, we have that θ_0 is positive and therefore $\exp\left(O\left(\frac{1}{m^{\theta_0}} + \frac{1}{n^{\beta_0}}\right)\right) \leq 2$ for all sufficiently large n. Writing

$$\frac{n}{n-r}\exp\left((m-1)\log\left(\frac{n-r}{n}\right)\right) = \exp\left((m-2)\log\left(\frac{n-r}{n}\right)\right),$$

we therefore have that

$$\frac{P_{m-1}(n-r)}{P_{m-1}(n)} \leq 2\exp\left((m-2)\log\left(\frac{n-r}{n}\right) + T(n-r) - T(n)\right)$$

for all sufficiently large n. Since

$$\begin{aligned} T(n-r) - T(n) &= \sum_{k=2}^{J} a_k \left(\frac{(m-1)^{2k-1}}{(n-r)^{k-1}} - \frac{(m-1)^{2k-1}}{n^{k-1}} \right) \\ &\leq \sum_{k=2}^{J} |a_k| \left(\frac{(m-1)^{2k-1}}{(n-r)^{k-1}} - \frac{(m-1)^{2k-1}}{n^{k-1}} \right), \end{aligned} \tag{2.41}$$

we have

$$\frac{P_{m-1}(n-r)}{P_{m-1}(n)} \leq 2 \exp\left(W(n-r) - W(n)\right), \tag{2.42}$$

where $W(y) = (m-2)\log y + \sum_{k=2}^{J} |a_k| \frac{(m-1)^{2k-1}}{y^{k-1}}$. We have that

$$W'(y) = \frac{1}{y}\left(m - 2 - |a_2| \frac{(m-1)^3}{y} - \sum_{k=3}^{J} |a_k|(k-1) \frac{(m-1)^{2k-1}}{y^{k-1}} \right).$$

For $m^{2+\theta} \leq y \leq n - v_n$ we have that $\frac{(m-1)^2}{y} \leq \frac{m^2}{y} \leq \frac{1}{m^\theta} \longrightarrow 0$ as $n \to \infty$. Hence

$$\begin{aligned} \sum_{k=3}^{J} |a_k|(k-1) \frac{(m-1)^{2k-1}}{y^{k-1}} &= \frac{(m-1)^3}{y} \sum_{k=3}^{J} |a_k|(k-1) \left(\frac{(m-1)^2}{y} \right)^{k-2} \\ &\leq \frac{(m-1)^3}{y} \sum_{k=3}^{J} |a_k|(k-1) \left(\frac{1}{m^\theta} \right)^{k-2} \\ &= \frac{(m-1)^3}{y} O\left(\frac{1}{m^\theta} \right) \leq |a_2| \frac{(m-1)^3}{2y} \end{aligned} \tag{2.43}$$

for all n sufficiently large. Therefore

$$W'(y) \geq \frac{1}{y}\left(m - 2 - 3|a_2| \frac{(m-1)^3}{2y} \right)$$

for all n sufficiently large. For $m^{2+\theta} \leq y \leq n - v_n$ and n sufficiently large, we therefore have that

$$\begin{aligned} W'(y) &\geq \frac{1}{y}\left(m - 2 - 3|a_2| \frac{(m-1)^3}{2m^{2+\theta}} \right) \\ &\geq \frac{1}{y}\left(m - 2 - \frac{3|a_2|}{2} m^{1-\theta} \right) \geq \frac{m}{2y}. \end{aligned}$$

In obtaining the second inequality in the above equation, we have used $\frac{(m-1)^3}{m^{2+\theta}} \leq \frac{m^3}{m^{2+\theta}} = m^{1-\theta}$. In obtaining the third inequality, we have used the fact that $\frac{m^{1-\theta}}{m} \longrightarrow 0$ as $n \to \infty$ and hence $2 + \frac{3|a_2|}{2} m^{1-\theta} \leq \frac{m}{2}$ for n sufficiently large. For all n sufficiently large we therefore have that $W(y)$ is an increasing function and hence attains its maximum at $y = n - v_n$. From (2.42), we therefore have that

$$\frac{P_{m-1}(n-r)}{P_{m-1}(n)} \leq 2\exp\left(W(n-v_n) - W(n)\right).$$

To estimate $W(n-v_n) - W(n)$ we proceed as follows. We write $W(n-v_n) - W(n) = W_1 + W_2$ where $W_1 = (m-2)\log\left(1 - \frac{v_n}{n}\right)$ and

$$W_2 = \sum_{k=2}^{J} |a_k| \left(\frac{(m-1)^{2k-1}}{(n-v_n)^{k-1}} - \frac{(m-1)^{2k-1}}{n^{k-1}}\right).$$

From (2.24) we have that

$$\begin{aligned} W_2 &= \sum_{k=2}^{J} |a_k| \left(\frac{m^{2k-1}}{(n-v_n)^{k-1}} - \frac{m^{2k-1}}{n^{k-1}}\right)\left(\frac{m-1}{m}\right)^{2k-1} \\ &= \sum_{k=2}^{J} |a_k| O\left(\frac{1}{n^{\beta_0}}\right)\left(1 - \frac{1}{m}\right)^{2k-1} \\ &= \sum_{k=2}^{J} |a_k| O\left(\frac{1}{n^{\beta_0}}\right) = O\left(\frac{1}{n^{\beta_0}}\right). \end{aligned}$$

Also using the inequality $\log(1-x) < -x$ and the fact that $m \sim An^{\alpha}$, we have

$$W_1 = (m-2)\log\left(1 - \frac{v_n}{n}\right) < -\frac{(m-2)v_n}{n} < -\frac{A}{2}n^{\alpha-\beta} = -\frac{A}{2}n^{\beta_2}$$

for n sufficiently large. From the above estimates for W_1 and W_2, we therefore get that

$$\frac{P_{m-1}(n-r)}{P_{m-1}(n)} \leq 2\exp\left(-\frac{A}{2}n^{\beta_2} + O\left(\frac{1}{n^{\beta_0}}\right)\right) \leq e^{-\frac{A}{4}n^{\beta_2}}$$

for all n sufficiently large. This proves (2.39) for $v_n \leq r \leq n - m^{2+\theta}$.

To estimate $\mathbb{P}_{n,m}(B_{r,j})$ for $n - m^{2+\theta} \leq r \leq n - m^2 \log m$, we proceed as follows. We let $J = j_\alpha$ and have from (2.13) that

$$\begin{aligned} \frac{P_{m-1}(n-r)}{P_{m-1}(n)} &= \exp\left((m-2)\log\left(\frac{n-r}{n}\right) + T(n-r) - T(n)\right. \\ &\qquad \left. + O\left(\frac{m}{(\log m)^J}\right) + O\left(\frac{1}{n^{\beta_0}}\right)\right), \end{aligned} \tag{2.44}$$

where $T(y) = \sum_{k=2}^{J} a_k \frac{(m-1)^{2k-1}}{y^{k-1}}$. For $m^2 \log m \leq y \leq m^{2+\theta}$, we have that $\frac{(m-1)^2}{y} \leq \frac{m^2}{y} \leq \frac{1}{\log m} \longrightarrow 0$ as $n \to \infty$. Hence as in (2.43), we have that

$$\sum_{k=3}^{J} |a_k| \frac{(m-1)^{2k-1}}{y^{k-1}} \leq |a_2| \frac{(m-1)^3}{2y} \tag{2.45}$$

for all n sufficiently large and for $n - m^{2+\theta} \le r \le n - m^2 \log m$, we have

$$\begin{aligned} T(n-r) - T(n) &\le \sum_{k=2}^{J} |a_k| \left(\frac{(m-1)^{2k-1}}{(n-r)^{k-1}} - \frac{(m-1)^{2k-1}}{n^{k-1}} \right) \quad \text{by (2.41)} \\ &\le \sum_{k=2}^{J} |a_k| \frac{(m-1)^{2k-1}}{(n-r)^{k-1}} \\ &= |a_2| \frac{(m-1)^3}{(n-r)} + \sum_{k=3}^{J} |a_k| \frac{(m-1)^{2k-1}}{(n-r)^{k-1}} \\ &\le |a_2| \frac{(m-1)^3}{(n-r)} + |a_2| \frac{(m-1)^3}{2(n-r)} = 3|a_2| \frac{(m-1)^3}{2(n-r)}. \end{aligned}$$

From (2.44), we therefore have that for all n sufficiently large and for all $n - m^{2+\theta} \le r \le n - m^2 \log m$,

$$\frac{P_{m-1}(n-r)}{P_{m-1}(n)} \le \exp\left(V(n-r) + O\left(\frac{m}{(\log m)^J} \right) + O\left(\frac{1}{n^{\beta_0}} \right) \right), \qquad (2.46)$$

where $V(y) = (m-2)\log\left(\frac{y}{n}\right) + 3|a_2|\frac{(m-1)^3}{2y}$. We estimate $V'(y)$ as follows. For $m^2 \log m \le y \le m^{2+\theta}$, we have $\frac{(m-1)^3}{2y} \le \frac{(m-1)^3}{2m^2 \log m} \le \frac{m}{2 \log m}$. Hence

$$V'(y) = \frac{1}{y}\left(m - 2 - 3|a_2| \frac{(m-1)^3}{2y} \right) \ge \frac{1}{y}\left(m - 2 - 3|a_2| \frac{m}{2\log m} \right).$$

Since $\frac{\frac{m}{\log m}}{m} = \frac{1}{\log m} \longrightarrow 0$ as $n \to \infty$, we have that $2 + 3|a_2| \frac{m}{2 \log m} \le \frac{m}{2}$ for all n sufficiently large. Hence $V'(y) \ge \frac{m}{2y}$ for all n sufficiently large. In particular, $V(y)$ is an increasing function for all n sufficiently large. Hence

$$V(n-r) \le V(m^{2+\theta}) = (m-2)\log\left(\frac{m^{2+\theta}}{n} \right) + \frac{3|a_2|}{2} \frac{(m-1)^3}{m^{2+\theta}}.$$

By our choice of θ in (2.7), we have that $\log\left(\frac{m^{2+\theta}}{n}\right) < -\frac{1}{2}$ for all n sufficiently large. Also $\frac{(m-1)^3}{m^{2+\theta}} \le m^{1-\theta} < \frac{m}{8}$, for all sufficiently large n. We therefore have that

$$V(n-r) \le \frac{-(m-2)}{2} + \frac{m}{8} = 1 - \frac{3m}{8}$$

for all n sufficiently large. Since $1 + O\left(\frac{1}{n^{\beta_0}}\right) + O\left(\frac{m}{(\log m)^J}\right) \le \frac{m}{8}$ for all n sufficiently large, we have that

$$\begin{aligned} &V(n-r) + O\left(\frac{m}{(\log m)^J} \right) + O\left(\frac{1}{n^{\beta_0}} \right) \\ &\quad \le -\frac{3m}{8} + 1 + O\left(\frac{m}{(\log m)^J} \right) + O\left(\frac{1}{n^{\beta_0}} \right) \\ &\quad \le -\frac{3m}{8} + \frac{m}{8} = -\frac{m}{4} \end{aligned}$$

for all n sufficiently large. From (2.46), we therefore get (2.39) for $n - m^{2+\theta} \leq r \leq n - m^2 \log m$.

We now consider the range $r \geq n - m^2 \log m$. Since $P_m(n) \leq p(n)$ where $p(.)$ is given by (1.1), we have from (2.40) that

$$\mathbb{P}_{n,m}(B_{r,j}) \leq \frac{P_{m-1}(n-r)}{P_{m-1}(n)} \leq \frac{p(n-r)}{P_{m-1}(n)}.$$

To bound the numerator, we have from (1.1) that

$$p(n-r) \leq \frac{D}{(n-r)} \exp\left(2c\sqrt{n-r}\right) \leq D\exp(2c\sqrt{n-r})$$

for some positive constants c and D and for all $r \leq n-1$. Hence for all $n - r \leq m^2 \log m$, we have that

$$\mathbb{P}_{n,m}(B_{r,j}) \leq \frac{D\exp(2c\sqrt{n-r})}{P_{m-1}(n)} \leq \frac{D\exp(2cm\sqrt{\log m})}{P_{m-1}(n)}. \tag{2.47}$$

To bound the denominator, we let $J = j_\alpha$ and we have from (2.13) that

$$\begin{aligned} P_{m-1}(n) &= \frac{1}{2\pi n} \exp\left((m-1)\log\left(\frac{ne^2}{(m-1)^2}\right) + T(n) + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &\geq \frac{1}{4\pi n} \exp\left((m-1)\log\left(\frac{ne^2}{(m-1)^2}\right) + T(n)\right) \end{aligned} \tag{2.48}$$

for all n sufficiently large, where $T(\cdot)$ is as defined in (2.44). Since $\frac{(m-1)^2}{n} \leq \frac{m^2}{n} \longrightarrow 0$ as $n \to \infty$, we have that (2.45) holds with $y = n$. Therefore,

$$\begin{aligned} |T(n)| &= \left|\sum_{k=2}^{J} a_k \frac{(m-1)^{2k-1}}{n^{k-1}}\right| \leq \sum_{k=2}^{J} |a_k| \frac{(m-1)^{2k-1}}{n^{k-1}} \\ &= |a_2|\frac{(m-1)^3}{n} + \sum_{k=3}^{J} |a_k| \frac{(m-1)^{2k-1}}{n^{k-1}} \\ &\leq |a_2|\frac{(m-1)^3}{n} + |a_2|\frac{(m-1)^3}{2n} = 3|a_2|\frac{(m-1)^3}{2n} \end{aligned}$$

for all sufficiently large n. Since $m \sim m-1 \sim An^\alpha$ and $\alpha < \frac{1}{2}$, we have that $\frac{1}{\log\left(\frac{ne^2}{(m-1)^2}\right)} \sim \frac{1}{\log\left(\frac{ne^2}{m^2}\right)} \sim \frac{1}{\log(n^{1-2\alpha})} \longrightarrow 0$ as $n \to \infty$. Also, $\frac{(m-1)^2}{n} < \frac{m^2}{n} \longrightarrow 0$ as $n \to \infty$. Hence

$$\begin{aligned} \frac{|T(n)|}{(m-1)\log\left(\frac{ne^2}{(m-1)^2}\right)} &\leq \frac{3|a_2|\frac{(m-1)^3}{2n}}{(m-1)\log\left(\frac{ne^2}{(m-1)^2}\right)} \\ &= \frac{3|a_2|}{2}\frac{(m-1)^2}{n}\frac{1}{\log\left(\frac{ne^2}{(m-1)^2}\right)} \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. In particular,

$$(m-1)\log\left(\frac{ne^2}{(m-1)^2}\right) + T(n) \geq \frac{(m-1)}{2}\log\left(\frac{ne^2}{(m-1)^2}\right)$$

for all sufficiently large n. Also, we have that $\frac{(m-1)}{2}\log\left(\frac{ne^2}{(m-1)^2}\right) \sim \frac{m}{2}\log\left(\frac{ne^2}{m^2}\right) \sim \frac{m}{2}(1-2\alpha)\log n = \frac{1-2\alpha}{2\alpha} m\log(n^\alpha) \sim \frac{1-2\alpha}{2\alpha} m\log m$. Hence $\frac{(m-1)}{2}\log\left(\frac{ne^2}{(m-1)^2}\right) \geq 2C(\alpha)m\log m$ for all n sufficiently large, where $C(\alpha) = \frac{1-2\alpha}{8\alpha}$. Consequently,

$$(m-1)\log\left(\frac{ne^2}{(m-1)^2}\right) + T(n) \geq 2C(\alpha)m\log m$$

for all n sufficiently large. Substituting the above lower bound into (2.48) we have that

$$P_{m-1}(n) \geq \frac{1}{4\pi n}\exp\left(2C(\alpha)m\log m\right).$$

From (2.47), for all $r \geq n - m^2\log m$, we have that

$$\begin{aligned}\mathbb{P}_{n,m}(B_{r,j}) &\leq 4\pi n D\exp(2cm\sqrt{\log m} - 2C(\alpha)m\log m)\\ &\leq \exp((2c+1)m\sqrt{\log m} - 2C(\alpha)m\log m).\end{aligned}$$

For all m sufficiently large, we have that $(2c+1)m\sqrt{\log m} - 2C(\alpha)m\log m < -C(\alpha)m\log m$. Hence we have that for all $r \geq n - m^2\log m$,

$$\mathbb{P}_{n,m}(B_{r,j}) \leq \exp\left(-C(\alpha)m\log m\right).$$

We have proved (2.39) for $r \geq n - m^2\log m$. □

Proof of Lemma 3. We first have that

$$\begin{aligned}\tilde{\Delta}_n &= \sum_{\mathcal{A}(n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_i,j}) - \sum_{\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_i,j})\\ &= \sum_{\mathcal{A}(n)\setminus\mathcal{A}(v_n)} \mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_i,j}) \geq 0.\end{aligned}$$

Also, if $(r_1, \ldots, r_k) \in \mathcal{A}(n) \setminus \mathcal{A}(v_n)$, there exists some i, $1 \leq i \leq k$, so that $r_i > v_n$. By Lemma 4, we have that

$$\begin{aligned}\mathbb{P}_{n,m}(\cap_{l=1}^k B_{r_i,j}) &\leq \mathbb{P}_{n,m}(B_{r_i,j})\\ &\leq \max\left(\exp\left(-C(\alpha)m\log m\right), e^{-\frac{m}{4}}, e^{-\frac{An^{\beta_2}}{4}}\right).\end{aligned}$$

Since $m \sim An^\alpha$, we have that $\frac{n^{\beta_2}}{m\log m} = \frac{n^{\alpha-\beta}}{m\log m} \sim \frac{1}{An^\beta\log m} \to 0$ as $n \to \infty$. Since $\beta_2 = \alpha - \beta < \alpha$, we have that $\frac{n^{\beta_2}}{m} \longrightarrow 0$ as $n \to \infty$. Hence the right-hand side of the above equation is bounded above by $e^{-\frac{An^{\beta_2}}{4}}$ for all n sufficiently large. Since the cardinality of $\mathcal{A}(n) \setminus \mathcal{A}(v_n)$ is at most n^k, we have that

$$\tilde{\Delta}_n \leq \sum_{\mathcal{A}(n)\setminus\mathcal{A}(v_n)} e^{-\frac{A}{4}n^{\beta_2}} \leq n^k e^{-\frac{A}{4}n^{\beta_2}} \leq e^{-\frac{A}{8}n^{\beta_2}}.$$

This proves Lemma 3. □

As a result of the above theorem, we strengthen Lemma 3 of [4].

COROLLARY 5

If $p_m(n)$ denotes the number of partitions of n into m summands, then

$$p_m(n) \sim \frac{1}{m!}\binom{n-1}{m-1},$$

if and only if $m = o(n^{1/3})$.

3. Proof of Theorem 2

Let m be as in (1.2) with $\frac{1}{3} \leq \alpha < \frac{1}{2}$. We let α be such that j_α defined in (1.4) is an integer. For positive integers r and j, define $C_{r,j} = C_{r,j}(m,n)$ to be the event that the number r occurs exactly j times in the composition of n into m summands. For any fixed integer $k \geq 1$, we define $S_{k,j} = S_{k,j}(t;n)$ as in (2.1). We claim that Theorem 2 follows from the following Proposition.

PROPOSITION 2

For $j \geq j_\alpha + 1$, we have that

$$S_{1,j}(0;n) \longrightarrow 0 \tag{3.1}$$

as $n \to \infty$. For $j = j_\alpha$ and for any fixed integer $k \geq 1$, we have that

$$S_{k,j_\alpha}(t;n) \longrightarrow \frac{\tilde{s}^k}{k!} \tag{3.2}$$

as $n \to \infty$, where $\tilde{s}$ is as in Theorem 2.

Proof of Theorem 2 (assuming Proposition 2). The proof is analogous to the proof of Theorem 1. □

In the rest of the section, we prove Proposition 2. For a positive integer j, we define $B_{r,j} = B_{r,j}(m,n)$ to be the event that the number r occurs at least j times in a composition of n into m summands. Choose $\delta \in (0,1)$ such that

$$\frac{\alpha}{2} < \delta < \frac{1-\alpha}{2}$$

and define $v_n = n^{1-\delta}$,

$$\delta_1 = \delta+1-2\alpha,\ \ \delta_2 = \alpha-\delta,\ \ \delta_3 = 2\delta-\alpha \quad \text{and} \quad \delta_0 = \min(\delta_1, \delta_2, \delta_3). \tag{3.3}$$

The relations (A1) and (A5) continue to hold in the case of compositions. Also, for fixed integers $j_1, j_2 \geq 1$, we have

(B1) $\frac{1}{n^{\delta}} = O\left(\frac{1}{n^{\delta_0}}\right)$.

(B2) $\frac{1}{(n-j_1 r)^{\gamma}} = \frac{1}{n^{\gamma}}\left(1+O\left(\frac{1}{n^{\delta}}\right)\right) = \frac{1}{n^{\gamma}}\left(1+O\left(\frac{1}{n^{\delta_0}}\right)\right)$ for any fixed $\gamma > 0$ and for all $r \leq j_2 v_n$.

(B3) $\frac{m}{n} = O\left(\frac{1}{m}\right) = O\left(\frac{m^2}{n}\right) = O\left(\frac{1}{n^{1-2\alpha}}\right) = O\left(\frac{1}{n^{\delta_0}}\right)$.

The proofs are analogous to the corresponding proofs for (A2)–(A4).

Let $\mathcal{A}(.)$ be as defined in the equation preceding (2.1). As in the case of partitions, we claim that the proof of Proposition 2 follows from the following three lemmas.

Lemma 6. *Let* $j, k \geq 1$ *be any two fixed integers. We have that*

$$0 \leq \sum_{\mathcal{A}(v_n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) - \sum_{\mathcal{A}(v_n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} C_{r_l,j}) = O\left(\frac{m^2}{n}\right)^{k(j-j_\alpha)+1}.$$

Lemma 7. *Let* $j, k \geq 1$ *be any two fixed integers. We have that*

$$\sum_{\mathcal{A}(v_n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) = \frac{1}{k!}\left(\frac{\mathrm{e}^{-jt}}{j!j}\right)^{k}\left(\frac{m^{2j-1}}{n^{j-1}}\right)^{k}\left(1+O\left(\frac{1}{n^{\delta_0}}\right)\right).$$

Lemma 8. *Let* $j, k \geq 1$ *be any two fixed integers. We have that*

$$0 \leq \sum_{\mathcal{A}(n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) - \sum_{\mathcal{A}(v_n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) \leq \mathrm{e}^{-\frac{A}{8}n^{\delta_2}}.$$

Proof of Proposition 2 (*assuming Lemmas* 6–8). The proof is analogous to the proof of Proposition 1. To prove (3.1), we let $k = 1$ and $t = 0$ in (2.1). Thus $t_n = \frac{nt}{m} = 0$ and

$$\sum_{\mathcal{A}(n)} \tilde{\mathbb{P}}_{n,m}(C_{r,j}) = \sum_{1 \leq r \leq n} \tilde{\mathbb{P}}_{n,m}(C_{r,j}) \leq I_1 + I_2,$$

where I_1 and I_2 are as defined in the proof of Proposition 1 with $\mathbb{P}_{n,m}$ replaced by $\tilde{\mathbb{P}}_{n,m}$. Analogous to (2.8), we have from Lemma 7 that for n sufficiently large,

$$I_1 = \frac{\mathrm{e}^{-tj}}{j!j}\left(\frac{m^{2j-1}}{n^{j-1}}\right)\left(1+O\left(\frac{1}{n^{\delta_0}}\right)\right) \leq 4\frac{\mathrm{e}^{-tj}}{j!j}A^{2j_\alpha-1}\left(\frac{m^2}{n}\right)^{j-j_\alpha}.$$

Since $j \geq j_\alpha + 1$, we have $\left(\frac{m^2}{n}\right)^{j-j_\alpha} = O\left(\frac{m^2}{n}\right)$ and therefore

$$I_1 = O\left(\frac{m^2}{n}\right) \longrightarrow 0$$

as $n \to \infty$. From Lemma 8, we have that

$$I_2 \le \mathrm{e}^{-\frac{An^{\delta_2}}{8}}.$$

Hence we have that $I_1 + I_2 \longrightarrow 0$ as $n \to \infty$. This proves (3.1).

To prove (3.2), we write $S_{k,j_\alpha} = \sum_{\mathcal{A}_n} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} C_{r_l,j_\alpha}) = S_1 - S_2 + S_3$, where S_1, S_2 and S_3 are as defined in the proof of Proposition 1. From Lemma 7 and (A5) we have that

$$\begin{aligned} S_1 &= \frac{1}{k!}\left(\frac{\mathrm{e}^{-j_\alpha t}}{j_\alpha! j_\alpha}\right)^k \left(\frac{m^{2j_\alpha - 1}}{n^{j_\alpha - 1}}\right)^k \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \frac{\tilde{s}^k}{k!}(1 + o(1))\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \longrightarrow \frac{\tilde{s}^k}{k!} \end{aligned}$$

as $n \to \infty$ where $\tilde{s}$ is as defined in Theorem 2.

It suffices to show that $S_2 \longrightarrow 0$ and $S_3 \longrightarrow 0$ as $n \to \infty$. To estimate S_3 we use the fact that $C_{r,j} \subseteq B_{r,j}$. Analogous to (2.9), we therefore have that

$$S_3 \le \sum_{\mathcal{A}(n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) - \sum_{\mathcal{A}(v_n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}).$$

From Lemma 8, we have $S_3 \le \mathrm{e}^{-\frac{An^{\delta_2}}{8}} \longrightarrow 0$ as $n \to \infty$. Finally, letting $j = j_\alpha$ in Lemma 6, we have that $S_2 = O\left(\frac{m^2}{n}\right) \longrightarrow 0$ as $n \to \infty$. □

We prove Lemmas 6, 7 and 8 in that order.

Proof of Lemma 6. For positive integers $j \ge 1$ and $j + 1 \le r \le n - 1$, define the quantity $P_{m,n}(r, j)$ as

$$P_{n,m}(r, j) = \frac{\left(1 - \frac{m}{n}\right)\left(1 - \frac{m+1}{n}\right)\ldots\left(1 - \frac{m+r-j-1}{n}\right)}{\left(1 - \frac{j+1}{n}\right)\left(1 - \frac{j+2}{n}\right)\ldots\left(1 - \frac{r}{n}\right)}$$

and define for $r \ge j$,

$$t(r, j) = t_{n,m}(r, j) = \begin{cases} P_{n,m}(r, j) w_{n,m}(j), & \text{if } r \ge j + 1, \\ w_{n,m}(j), & \text{if } r = j, \end{cases}$$

where $w_{n,m}(j) = \prod_{i=1}^{j}\left(\frac{m-i}{n-i}\right)$.

The proof of Lemma 6 is now obtained in three steps.

Step 1. We obtain a relation between $\tilde{\mathbb{P}}_{n,m}$ and $t(.,.)$. and estimate $t(r, j)$ for a suitable range of r.

Step 2. We obtain a relation between probabilities of the events $B_{r,j}$ and the quantity $t(r, j)$ and obtain an asymptotic expression for $\sum_r t(r, j)$ as r varies over a certain range.

Step 3. We convert sums involving the probabilities of the events $B_{r,j}$ into sums involving the function $t(.,.)$ to complete the proof of Lemma 6.

Step 1. We have the following relation:

Let $k \geq 1$ be any fixed integer and let $j_0 = 0, j_1, \dots, j_k$ be fixed integers. Let $n = \sum_{i=1}^{m} X_i$ be a randomly chosen composition of n into m parts. For positive integers r_i, $1 \leq i \leq k$, let $R = \sum_{l=1}^{k} r_l j_l$ and $J = \sum_{l=1}^{k} j_l$ be such that $R \leq n-1$ and $J \leq m-1$. We have that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} \cap_{i=j_{l-1}+1}^{j_{l-1}+j_l} X_i = r_l) = t_{n,m}(R, J). \tag{3.4}$$

Proof of (3.4). Let $\mathcal{C}(n, m)$ denote the set of all compositions of n into m parts. We have (see [1]) that

$$\#\mathcal{C}(n, m) = \binom{n-1}{m-1}.$$

Suppose $\mathcal{C}_r(n, m)$ denotes the set of all compositions of n into m summands with $r \geq 1$ being the value of the first summand. The set $\mathcal{C}_r(n, m)$ has a one-to-one correspondence with the set of all compositions of $n - r$ into $m - 1$ summands. Therefore we have that

$$\#\mathcal{C}_r(n, m) = \binom{n-r-1}{m-2}.$$

Hence for $r_1 \geq 2$, we have

$$\begin{aligned} \tilde{\mathbb{P}}_{n,m}(X_1 = r_1) &= \frac{\#\mathcal{C}_{r_1}(n, m)}{\#\mathcal{C}(n, m)} = \frac{\binom{n-r_1-1}{m-2}}{\binom{n-1}{m-1}} \\ &= \left(\frac{m-1}{n-1}\right) \times P'_{m,n}(r_1), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} P'_{m,n}(r_1) &= \frac{(n-r_1-1)}{(n-2)} \cdots \frac{(n-r_1-m+2)}{(n-m+1)} \\ &= (n-m)\dots(n-r_1)\frac{(n-r_1-1)}{(n-2)} \cdots \\ &\quad \times \frac{(n-r_1-m+2)}{(n-m+1)} \frac{1}{(n-m)\dots(n-r_1)} \\ &= \frac{\left(1-\frac{m}{n}\right)\left(1-\frac{m+1}{n}\right)\dots\left(1-\frac{m+r_1-2}{n}\right)}{\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\dots\left(1-\frac{r_1}{n}\right)} \\ &= P_{m,n}(r_1, 1). \end{aligned}$$

For $r_1 = 1$, we have from (3.5) that

$$\tilde{\mathbb{P}}_{n,m}(X_1 = r_1) = \frac{m-1}{n-1} = w_{n,m}(1).$$

Thus

$$\tilde{\mathbb{P}}_{n,m}(X_1 = r_1) = t_{n,m}(r_1, 1) \tag{3.6}$$

We now proceed by induction on n. Since all compositions are equally likely, we have that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} \cap_{i=j_{l-1}+1}^{j_l+j_{l-1}} X_i = r_l) \;=\; \tilde{\mathbb{P}}_{n,m}(X_1 = r_1)\delta_{m,n}, \tag{3.7}$$

where

$$\delta_{m,n} = \tilde{\mathbb{P}}_{n-r_1,m-1}(\cap_{i=2}^{j_1} X_i = r_1 \cap \cap_{l=2}^{k} \cap_{i=j_{l-1}+1}^{j_l+j_{l-1}} X_i = r_l)$$

and $\cap_{i=2}^{j_1} X_i = r_1$ is taken to be empty if $j_1 = 1$. Letting $R' = r_1(j_1 - 1) + \sum_{l=2}^{k} r_l j_l$, we have by induction assumption that

$$\tilde{\mathbb{P}}_{n-r_1,m-1}(\cap_{i=2}^{j_1} X_i = r_1 \cap \cap_{i=j_{l-1}+1}^{j_l+j_{l-1}} X_i = r_l) = t_{n-r_1,m-1}(R', J-1).$$

From (3.6), we have that $\tilde{\mathbb{P}}_{n,m}(X_1 = r_1) = t_{n,m}(r_1, 1)$. Hence from (3.7) we have that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} \cap_{i=j_{l-1}+1}^{j_l+j_{l-1}} X_i = r_l) = t_{n,m}(r_1, 1) t_{n-r_1,m-1}(R', J-1).$$

Using the identity

$$t_{n,m}(r, 1) t_{n-r,m-1}(r', j') = t_{n,m}(r + r', j' + 1)$$

we get that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} \cap_{i=j_{l-1}+1}^{j_l+j_{l-1}} X_i = r_l) = t_{n,m}(r_1 + R', J) = t_{n,m}(R, J).$$

This proves the induction step. □

In what follows, we write $t_{n,m}(r, j)$ as $t(r, j)$. We complete Step 1 by estimating $t(r, j)$ for a suitable range of r.

Let $j, j_1 \geq 1$ be any two fixed integers. For all $r \leq j_1 v_n$, we have

$$t(r, j) = \mathrm{e}^{-\frac{rm}{n}} \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right), \tag{3.8}$$

where the $O(.)$ term is independent of r.

Proof of (3.8). We first let $r \geq j + 1$ and obtain that

$$\begin{aligned} \log\left(\frac{t(r, j)}{w_{m,n}}\right) &= -\sum_{k=2}^{r-j+1} \left(\log\left(1 - \frac{m+k-2}{n}\right) - \log\left(1 - \frac{k+j-1}{n}\right)\right), \\ &= -R_1 - R_2 \end{aligned} \tag{3.9}$$

where $R_1 = \sum_{k=2}^{r-j+1} \frac{(m+k-2)-(k+j-1)}{n}$ and $R_2 = \sum_{k=2}^{r-j+1} \sum_{l \geq 2} \frac{(m+k-2)^l-(k+j-1)^l}{ln^l}$. We estimate R_1 and R_2 separately. For all $r \leq j_1 v_n$, we have that

$$\begin{aligned} R_1 &= \frac{(m-j-1)(r-j)}{n} = \frac{mr}{n} - \frac{jm+(j+1)r-j(j+1)}{n} \\ &= \frac{mr}{n} + O\left(\frac{1}{n^\delta}\right). \end{aligned}$$

Here and henceforth all $O(.)$ terms are independent of r. To obtain the above equation, we use (B3) and get that

$$\begin{aligned} \frac{jm+(j+1)r-j(j+1)}{n} &\leq \frac{jm+(j+1)j_1v_n-j(j+1)}{n} \\ &= O\left(\frac{m}{n}\right) + O\left(\frac{v_n}{n}\right) \\ &= O\left(\frac{m}{n}\right) + O\left(\frac{1}{n^\delta}\right) = O\left(\frac{1}{n^\delta}\right). \end{aligned} \tag{3.10}$$

Also, we have

$$\begin{aligned} R_2 &= \sum_{k=2}^{r-j+1} \sum_{l \geq 2} \frac{(m+k-2)^l-(k+j-1)^l}{ln^l} \\ &= \sum_{k=2}^{r-j+1} \sum_{l \geq 2} \frac{(m-j-1)}{ln^l} \{(m+k-2)^{l-1}+(m+k-2)^{l-2}(k+j-1)+\cdots \\ &\qquad +(k+j-1)^{l-1}\} \\ &\leq \sum_{k=2}^{r-j+1} \sum_{l \geq 2} \frac{(m-j-1)}{n^l}(m+k-2)^{l-1} \\ &\leq \sum_{l \geq 2} \frac{(m-j-1)(r-j-1)}{n^l}(m+r-j-1)^{l-1} \leq \frac{mr}{n} \sum_{l \geq 2} \left(\frac{m+r}{n}\right)^{l-1} \\ &= \frac{mr}{n} \frac{m+r}{n} \left(1 - \frac{m+r}{n}\right)^{-1}. \end{aligned} \tag{3.11}$$

As in (3.10), we have that $\frac{m+r}{n} = O\left(\frac{1}{n^\delta}\right)$ for all $r \leq j_1 v_n$. Hence for all $r \leq j_1 v_n$, we have

$$\begin{aligned} \frac{m+r}{n}\left(1 - \frac{m+r}{n}\right)^{-1} &= O\left(\frac{1}{n^\delta}\right)\left(1 - O\left(\frac{1}{n^\delta}\right)\right)^{-1} \\ &= O\left(\frac{1}{n^\delta}\right)\left(1 + O\left(\frac{1}{n^\delta}\right)\right) \\ &= O\left(\frac{1}{n^\delta}\right). \end{aligned} \tag{3.12}$$

Also,

$$\frac{mr}{n} = O\left(\frac{mv_n}{n}\right) = O\left(\frac{n^{\alpha} n^{1-\delta}}{n}\right) = O\left(n^{\delta_2}\right).$$

Substituting the above two estimates into (3.11), we get

$$0 \le R_2 \le O(n^{\delta_2}) O\left(\frac{1}{n^{\delta}}\right) = O\left(\frac{1}{n^{\delta_3}}\right).$$

Substituting the estimates for R_1 and R_2 into (3.9) we have that

$$\begin{aligned} t(r, j) &= w_{n,m}(j) \mathrm{e}^{-R_1 - R_2} = \mathrm{e}^{-\frac{rm}{n}} w_{n,m}(j) \exp\left(O\left(\frac{1}{n^{\delta}}\right) + O\left(\frac{1}{n^{\delta_3}}\right)\right) \\ &= \mathrm{e}^{-\frac{rm}{n}} w_{n,m}(j) \exp\left(O\left(\frac{1}{n^{\delta_0}}\right)\right) \qquad \text{(by (B1))} \\ &= \mathrm{e}^{-\frac{rm}{n}} w_{n,m}(j) \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \end{aligned}$$

To evaluate $w_{m,n}(j)$ we have by definition that

$$\left(\frac{m-j}{n}\right)^j \le w_{n,m}(j) \le \left(\frac{m}{n-j}\right)^j.$$

We have from (B3) that

$$\begin{aligned} \left(\frac{m-j}{n}\right)^j &= \left(\frac{m}{n}\right)^j \left(1 - \frac{j}{m}\right)^j = \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{m}\right)\right)^j \\ &= \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{m}\right)\right) = \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \end{aligned}$$

Analogously, $\left(\frac{m}{n-j}\right)^j = \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right)$. Hence we have

$$w_{n,m}(j) = \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right).$$

Thus

$$t(r, j) = \mathrm{e}^{-\frac{rm}{n}} \left(\frac{m}{n}\right)^j \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right)^2.$$

To obtain (3.8) the above equation, we use (A1). □

Step 2. In the case of partitions, we had defined an analogous function F in (2.17) and were able to obtain a relation between $\mathbb{P}_{n,m}(\cap_{l=1}^{k} B_{r_l, j_l})$ and $F(.,.)$ as in (2.30). Using (2.30), we were able to convert sums regarding the probabilities of the events $B_{r,j}$ into sums involving the function F. In the case of compositions, no such exact relation exists. We therefore have the following result.

Lemma 9. *Let $k \geq 1$ be any fixed integer and let $j_1, j_2, \ldots, j_k$ be any fixed integers. For all n sufficiently large and for all $t_n \leq r_1 < r_2 < \cdots < r_k \leq v_n$, we have that*

$$\begin{aligned}\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^k B_{r_l, j_l}) &= \frac{m^J t(R, J)}{\prod_{l=1}^k j_l!}\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \\ &= \frac{1}{\prod_{l=1}^k j_l!} \mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m^2}{n}\right)^J \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right), \end{aligned} \tag{3.13}$$

where $R = \sum_{l=1}^k r_l j_l$ and $J = \sum_{l=1}^k j_l$.

Proof. Let $n = \sum_{i=1}^m X_i$ be a randomly chosen composition of n into m parts. Let $r_1 < r_2 < \cdots < r_k$ and suppose that the number r_i occurs at least j_i times for each $1 \leq i \leq k$. Letting $J = \sum_{l=1}^k j_l$, we define $\mathcal{S}_J$ to be the set of all subsets of $\{1, 2, \ldots, m\}$ that have J elements. We order the elements of $\mathcal{S}_J$ as $\{e_i\}_{1 \leq i \leq \tilde{m}_J}$, where

$$\tilde{m}_J = \frac{m(m-1)\ldots(m-J+1)}{J!} \leq \frac{m^J}{J!} \tag{3.14}$$

is the number of elements in $\mathcal{S}_J$. Let

$$\mathcal{T} = \left\{(p_1, \ldots, p_J) : \sum_{l=1}^J \mathbf{1}(p_l = r_i) = j_i, 1 \leq i \leq k\right\}.$$

For $e = \{l_1, \ldots, l_J\} \in \mathcal{S}_J$ and $\mathbf{p} = (p_1, \ldots, p_J)$ define

$$X(\mathbf{p}, e) = \{X_{l_1} = p_1, \ldots, X_{l_J} = p_J\}$$

and

$$A_e = \cup_{\mathbf{p} \in \mathcal{T}} X(\mathbf{p}, e).$$

Hence we have that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^k B_{r_l, j_l}) = \tilde{\mathbb{P}}_{n,m}(\cup_{1 \leq i \leq \tilde{m}_J} A_i), \tag{3.15}$$

where $A_i = A_{e_i}$. We obtain an upper bound and a lower bound for the above expression using the inclusion–exclusion principle.

For an upper bound, we have from (3.15) that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^k B_{r_l, j_l}) \leq \sum_{1 \leq i \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i). \tag{3.16}$$

For a fixed $e \in \mathcal{S}_J$ and distinct $\mathbf{p}, \mathbf{p}' \in \mathcal{T}$, we have that $X(\mathbf{p}, e)$ and $X(\mathbf{p}', e)$ are disjoint. Hence for a fixed i, we have $\tilde{\mathbb{P}}_{n,m}(A_i) = \sum_{\mathbf{p} \in \mathcal{T}} \tilde{\mathbb{P}}_{n,m}\left(X(\mathbf{p}, e_i)\right)$ and therefore

$$\sum_{1 \leq i \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i) = \sum_{1 \leq i \leq \tilde{m}_J} \sum_{\mathbf{p} \in \mathcal{T}} \tilde{\mathbb{P}}_{n,m}\left(X(\mathbf{p}, e_i)\right).$$

For $\mathbf{p} \in \mathcal{T}$ and $e \in \mathcal{S}_J$, we have from (3.4) that

$$\tilde{\mathbb{P}}_{n,m}(X(\mathbf{p}, e)) = t(R, J),$$

where $R = \sum_{l=1}^{k} r_l j_l \leq J v_n$. Hence

$$\sum_{1 \leq i \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i) = \sum_{1 \leq i \leq \tilde{m}_J} \sum_{\mathbf{p} \in \mathcal{T}} t(R, J) = \tilde{m}_J (\#\mathcal{T}) t(R, J),$$

where $\#\mathcal{T}$ denotes the number of elements in the set $\mathcal{T}$.

Since $\#\mathcal{T} = \frac{J!}{J_p}$, where $J_p = \prod_{l=1}^{k} j_l!$, we have from (3.14) that

$$\begin{aligned}
\tilde{m}_J(\#\mathcal{T}) &= \frac{m^J}{J_p} \prod_{i=1}^{J} \left(1 - \frac{i}{m}\right) = \frac{m^J}{J_p} \prod_{i=1}^{J} \left(1 + O\left(\frac{1}{m}\right)\right) \\
&= \frac{m^J}{J_p} \prod_{i=1}^{J} \left(1 + O\left(\frac{1}{m}\right)\right) \\
&= \frac{m^J}{J_p} \left(1 + O\left(\frac{1}{m}\right)\right) = \frac{m^J}{J_p} \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right).
\end{aligned}$$

To obtain the last equality, we use (B3). Also, since $R \leq J v_n$, the expression (3.8) for $t(R, J)$ holds. From (3.16), we have that

$$\begin{aligned}
\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l, j_l}) &\leq \sum_{1 \leq i \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i) \\
&= \frac{1}{\prod_{l=1}^{k} j_l!} e^{-\frac{Rm}{n}} \left(\frac{m^2}{n}\right)^J \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \qquad (3.17)
\end{aligned}$$

To find a lower bound for (3.15), we have by the inclusion–exclusion principle that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l, j_l}) \geq \sum_{1 \leq i \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i) - \sum_{1 \leq i < j \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j). \qquad (3.18)$$

We want to find an upper bound for the second summation in the above equation. We first write

$$\sum_{1 \leq i < j \leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j) = \sum_{i=1}^{\tilde{m}_J} \sum_{j=i+1}^{\tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j).$$

Let $1 \leq i \leq \tilde{m}_J$ be fixed. To evaluate the inner sum in the above expression, we write $\mathcal{I}_q$ to be the set of all $e_j \in \mathcal{S}_J$ such that $j \geq i + 1$ and the number of elements common to e_i and e_j is q. Since $q \leq J - 1$, we have

$$\sum_{j=i+1}^{\tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j) = \sum_{q=0}^{J-1} \sum_{e \in \mathcal{I}_q} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_e). \qquad (3.19)$$

For $e \in \mathcal{I}_q$, we have that

$$\begin{aligned}\tilde{\mathbb{P}}_{n,m}(A_i \cap A_e) &= \tilde{\mathbb{P}}_{n,m}((\cup_{\mathbf{p}\in\mathcal{T}} X(\mathbf{p}, e_i)) \cap (\cup_{\mathbf{p}'\in\mathcal{T}} X(\mathbf{p}', e)))\\ &\leq \sum_{\mathbf{p}\in\mathcal{T}} \sum_{\mathbf{p}'\in\mathcal{T}} \tilde{\mathbb{P}}_{n,m}(X(\mathbf{p}, e_i) \cap X(\mathbf{p}', e)).\end{aligned}$$

Since $e \in \mathcal{I}_q$, the event $X(\mathbf{p}, e_i) \cap X(\mathbf{p}', e)$ is either empty or can be written as $\cap_{l=1}^{2J-q}\{X_{i_l} = \tilde{p}_l\}$ for some distinct X_{i_l}'s and some integers $\tilde{p}_l$. Hence by (3.4) we have that

$$\tilde{\mathbb{P}}_{n,m}(X(\mathbf{p}, e_i) \cap X(\mathbf{p}', e)) \leq t(R', 2J - q),$$

where $R' = \sum_{l=1}^{2J-q} \tilde{p}_l$. Moreover, if we denote $\mathbf{p} = (p_1, \ldots, p_J)$ and $\mathbf{p}' = (p'_1, \ldots, p'_J)$, we have that $R = \sum_{l=1}^{J} p_l \leq R' \leq \sum_{l=1}^{J} p_l + \sum_{l=1}^{J} p'_l \leq 2Jv_n$. By (3.8), we therefore have that

$$\begin{aligned}t(R', 2J - q) &= \mathrm{e}^{-\frac{R'm}{n}} \left(\frac{m}{n}\right)^{2J-q} \left(1 + \frac{1}{n^{\beta_0}}\right)\\ &\leq 2\mathrm{e}^{-\frac{R'm}{n}} \left(\frac{m}{n}\right)^{2J-q} \leq 2\mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m}{n}\right)^{2J-q}\end{aligned}$$

for all sufficiently large n. Using the fact that $\#\mathcal{T} = \frac{J!}{J_p}$, we therefore have that

$$\tilde{\mathbb{P}}_{n,m}(A_i \cap A_e) \leq \sum_{\mathbf{p}\in\mathcal{T}} \sum_{\mathbf{p}'\in\mathcal{T}} 2\mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m}{n}\right)^{2J-q} = 2\left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m}{n}\right)^{2J-q}.$$

From (3.19), we get that

$$\sum_{j=i+1}^{\tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j) \leq 2\left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \sum_{q=0}^{J-1} n_q \left(\frac{m}{n}\right)^{2J-q},$$

where $n_q = \#\mathcal{I}_q$. Let $e \in \mathcal{S}_J$ be fixed. The number of elements $e' \in \mathcal{S}_J$ that have exactly q elements in common with e is $n_q = \begin{pmatrix} J \\ q \end{pmatrix}\begin{pmatrix} m-J \\ J-q \end{pmatrix} \leq 2^J m^{J-q}$. Hence

$$\begin{aligned}\sum_{j=i+1}^{\tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j) &\leq 2^{J+1}\left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \sum_{q=0}^{J-1} m^{J-q} \left(\frac{m}{n}\right)^{2J-q}\\ &= 2^{J+1}\left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \frac{1}{m^J} \sum_{q=0}^{J-1} \left(\frac{m^2}{n}\right)^{2J-q}\\ &= 2^{J+1}\left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \frac{1}{m^J} \left(\frac{m^2}{n}\right)^{J+1} \frac{1-\left(\frac{m^2}{n}\right)^J}{1-\frac{m^2}{n}}\\ &\leq 2^{J+2}\left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \frac{1}{m^J} \left(\frac{m^2}{n}\right)^{J+1}.\end{aligned}$$

In obtaining the last inequality, we have used the fact that $\frac{1-\left(\frac{m^2}{n}\right)^J}{1-\frac{m^2}{n}} \leq \frac{1}{1-\frac{m^2}{n}} \leq 2$ for sufficiently large n. We therefore have

$$\sum_{i=1}^{\tilde{m}_J} \sum_{j=i+1}^{\tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j) \leq \tilde{m}_J 2^{J+2} \left(\frac{J!}{J_p}\right)^2 \mathrm{e}^{-\frac{Rm}{n}} \frac{1}{m^J} \left(\frac{m^2}{n}\right)^{J+1}$$
$$= \mathrm{e}^{-\frac{Rm}{n}} \frac{\tilde{m}_J}{m^J} O\left(\frac{m^2}{n}\right)^{J+1} = \mathrm{e}^{-\frac{Rm}{n}} O\left(\frac{m^2}{n}\right)^{J+1}$$

by (3.14). From (3.17) and the above equation, we get that

$$\sum_{1\leq i\leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i) - \sum_{1\leq i<j\leq \tilde{m}_J} \tilde{\mathbb{P}}_{n,m}(A_i \cap A_j)$$
$$= \frac{1}{J_p}\mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m^2}{n}\right)^J \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) - \mathrm{e}^{-\frac{Rm}{n}} O\left(\frac{m^2}{n}\right)^{J+1}$$
$$= \frac{1}{J_p}\mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m^2}{n}\right)^J \times R'' ,$$

where

$$R'' = \left(1 + O\left(\frac{1}{n^{\delta_0}}\right) - J_p O\left(\frac{m^2}{n}\right)\right)$$
$$= \left(1 + O\left(\frac{1}{n^{\delta_0}}\right) + O\left(\frac{m^2}{n}\right)\right) = \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right)$$

by (B3). From (3.18), we therefore have

$$\tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^k B_{r_l,j_l}) \geq \frac{1}{J_p}\mathrm{e}^{-\frac{Rm}{n}} \left(\frac{m^2}{n}\right)^J \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right).$$

From the above equation and (3.17), we get (3.13). □

From the above result, it is intuitive that sums involving the probabilities of the events $B_{r,j}$ can be converted into sums involving $m^j t(r, j)$. We therefore have the following result. The proof is analogous to the proof of (2.26).

For a fixed integer $k \geq 1$, let $j_1, j_2, \ldots, j_k$ be positive integers and let $J = \sum_{l=1}^k j_l$ and $J_p = \prod_{l=1}^k j_l!$. For all sufficiently large n, we have

$$\sum_{t_n<r_1,r_2,\ldots,r_k\leq v_n} \frac{1}{J_p} m^J t\left(\sum_{l=1}^k r_l j_l, J\right) = \frac{\mathrm{e}^{-Jt}}{\prod_{l=1}^k j_l! j_l} \frac{m^{2J-k}}{n^{J-k}} \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \tag{3.20}$$

Proof of Lemma 6. The proof is analogous to the proof of Lemma 1. Defining Δ_n as in (2.29) and (2.30), we get that

$$0 \le \Delta_n = \sum_{\mathcal{A}(v_n)} \sum_{w=1}^{k} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1,l\neq w}^{k} B_{r_l,j} \cap B_{r_w,j+1}),$$

where $\mathcal{A}(.)$ is as defined in the equation preceding (2.1). For any fixed integers $j_1, \ldots, j_k$ and $r_1 < r_2 < \cdots < r_k$, we have from Lemma 9 that

$$\tilde{\mathbb{P}}_{n,m}\big(\cap_{l=1}^{k} B_{r_l,j_l}\big) = \frac{m^J}{J_p} t\left(\sum_{l=1}^{k} r_l j_l, J\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right),$$

where $J_p = (j!)^{k-1}(j+1)!$. This is analogous to (2.30) with $F(.,.)$ replaced by $\frac{m^J}{J_p} t(.,.)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right)$. Hence as in (2.31) we get that

$$\Delta_n \le k \sum_{t_n < r_1,\ldots,r_k \le v_n} \frac{1}{J_p} m^J t\left(\sum_{l=1}^{k} r_l j_l + r_1, kj+1\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right).$$

But from (3.20), we have that

$$\sum_{t_n < r_1,\ldots,r_k \le v_n} \frac{1}{J_p} m^J t\left(\sum_{l=1}^{k} r_l j_l + r_1, kj+1\right) = c_{k,j} \frac{m^{2kj+2-k}}{n^{kj+1-k}} \times \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right),$$

where $c_{k,j} = \frac{e^{-(kj+1)t}}{(j!j)^{k-1}(j+1)!(j+1)}$ and as in (2.32), we have that $\frac{m^{2kj+2-k}}{n^{kj+1-k}} = O\left(\frac{m^2}{n}\right)^{k(j-j_\alpha)+1}$. This completes the proof of Lemma 6. □

Proof of Lemma 7. Let $\tilde{\mathcal{A}}(.)$ and $\mathcal{D}(.)$ be as defined in the equations preceding (2.1) and (2.33), respectively. We claim that Lemma 7 follows from the following two results.

We have that

$$\sum_{\mathcal{A}(v_n)} \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) = \frac{1}{k!} \sum_{\mathcal{D}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} j r_l, kj\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \tag{3.21}$$

We have that

$$\sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} j r_l, kj\right) = \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k O\left(\frac{m}{n}\right). \tag{3.22}$$

Proof of Lemma 7 (*assuming* (3.21) *and* (3.22)). The proof is analogous to the proof of Lemma 2. From (3.20), we have that

$$\begin{aligned}\sum_{\tilde{\mathcal{A}}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} j r_l, kj\right) &= \frac{e^{-kjt}}{(j!j)^k} \frac{m^{2jk-k}}{n^{jk-k}} \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \\ &= \left(\frac{e^{-jt}}{j!j}\right)^k \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned}\sum_{\mathcal{A}(v_n)} \quad & \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) \\ &= \frac{1}{k!} \sum_{\mathcal{D}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} j r_l, kj\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \qquad \text{(by (3.21))} \\ &= \frac{1}{k!}\left(\sum_{\tilde{\mathcal{A}}(v_n)} - \sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)}\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \\ &= \frac{1}{k!}\left(\left(\frac{e^{-jt}}{j!j}\right)^k \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right)\right. \\ &\qquad \left. - \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k O\left(\frac{m}{n}\right)\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \quad \text{(by (3.23) and (3.22))} \\ &= \frac{1}{k!}\left(\frac{e^{-jt}}{j!j}\right)^k \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \times R, \end{aligned}$$

where

$$\begin{aligned} R &= \left(1 + O\left(\frac{1}{n^{\delta_0}}\right) - \left(\frac{j!j}{e^{-jt}}\right)^k O\left(\frac{m}{n}\right)\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \\ &= \left(1 + O\left(\frac{1}{n^{\delta_0}}\right) + O\left(\frac{m}{n}\right)\right)\left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) = \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right). \end{aligned}$$

In obtaining the last equation, we have used (B3) and (A1). □

Proof of (3.21). The proof is analogous to the proof of (2.33) with $F(.,.)$ replaced by $\frac{1}{(j!)^k} m^{kj} t(.,.)$. As in (2.36), we therefore get that

$$\sum_{\mathcal{D}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} j r_l, kj\right) = k! \sum_{\mathcal{A}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} j r_l, kj\right).$$

But, from (3.20), we have that for $(r_1, \dots, r_k) \in \mathcal{B}_n$, and $R = \sum_{l=1}^{k} jr_l$,

$$\begin{aligned}\frac{1}{(j!)^k} m^{kj} t(R, kj) &= \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right)^{-1} \\ &= \tilde{\mathbb{P}}_{n,m}(\cap_{l=1}^{k} B_{r_l,j}) \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right).\end{aligned}$$

This proves (3.21). □

Proof of (3.22). The proof is analogous to the proof of (2.34) with $F(.,.)$ replaced by $\frac{1}{(j!)^k} m^{kj} t(.,.)$. We define the sets $\mathcal{G}_{ij}$ as in the proof of (2.34). As in (2.38), we get that

$$\sum_{\tilde{\mathcal{A}}(v_n)\setminus\mathcal{D}(v_n)} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} jr_l, kj\right) \le \frac{k(k-1)}{2} \sum_{\mathcal{G}_{12}} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} jr_l, kj\right).$$

Since

$$\sum_{\mathcal{G}_{12}} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} jr_l, kj\right) = \sum_{t_n \le r_1,\dots,r_{k-1} \le v_n} \frac{1}{(j!)^k} m^{kj} t\left(2jr_1 + \sum_{l=2}^{k-1} jr_l, kj\right),$$

from (3.20), we have that

$$\begin{aligned}\sum_{\mathcal{G}_{12}} \frac{1}{(j!)^k} m^{kj} t\left(\sum_{l=1}^{k} jr_l, kj\right) &= c_{k,j} \frac{m^{2kj-k+1}}{n^{kj-k+1}} \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \\ &= c_{k,j} \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k \left(\frac{m}{n}\right) \left(1 + O\left(\frac{1}{n^{\delta_0}}\right)\right) \\ &= \left(\frac{m^{2j-1}}{n^{j-1}}\right)^k O\left(\frac{m}{n}\right),\end{aligned}$$

where $c_{k,j} = \frac{\mathrm{e}^{-kjt}}{2j^k(2j!)(j!)^{k-1}}$. This proves (3.22). □

Proof of Lemma 8. We first estimate $\tilde{\mathbb{P}}_{n,m}(B_{r,j})$ for the range $r \ge v_n$.

Lemma 10. *For all n sufficiently large and for all $r \ge v_n$, we have that*

$$\tilde{\mathbb{P}}_{n,m}(B_{r,j}) \le \mathrm{e}^{-C_4 n^{\delta_2}} \tag{3.24}$$

for some positive constant C_4.

Proof. Let $n = \sum_{i=1}^{m} X_i$ be a randomly chosen composition of n into m parts. We have from (3.4) that

$$\tilde{\mathbb{P}}_{n,m}(X_1 = r) = \prod_{i=2}^{r} \left(\frac{1 - \frac{m+i-2}{n}}{1 - \frac{i}{n}}\right) w_{m,n}(1).$$

For $i \geq 2$, we have that $\left(\frac{1-\frac{m+i-2}{n}}{1-\frac{i}{n}}\right) = 1 - \frac{(m-2)/n}{1-i/n} < 1 - \frac{m-2}{n}$. Also, since $m \leq n$, we have that $w_{m,n}(1) \leq 1$. For any r, we have that

$$\begin{aligned}
\tilde{\mathbb{P}}_{n,m}(X_1 = r) &\leq 2\left(1 - \frac{m-2}{n}\right)^{r-1} \\
&= 2\left(1 - \frac{m-2}{n}\right)^{r}\left(1 - \frac{m-2}{n}\right)^{-1} \\
&= 2\left(1 - \frac{m-2}{n}\right)^{r}\left(1 + O\left(\frac{m}{n}\right)\right) \\
&\leq 4\left(1 - \frac{m-2}{n}\right)^{r}.
\end{aligned}$$

Also, $B_{r,j} \subseteq B_{r,1} = \cup_{i=1}^{m}\{X_i = r\}$. For all $r \geq v_n$, we have that

$$\begin{aligned}
\tilde{\mathbb{P}}_{n,m}(B_{r,j}) &\leq \tilde{\mathbb{P}}_{n,m}(\cup_{i=1}^{m}\{X_i = r\}) \leq m\tilde{\mathbb{P}}_{n,m}(X_1 = r) \\
&\leq 4m\left(1 - \frac{m-2}{n}\right)^{r} \\
&\leq 4m\mathrm{e}^{-\frac{r(m-2)}{n}} \leq 4m\mathrm{e}^{-\frac{v_n(m-2)}{n}} = 4me^2\mathrm{e}^{-mn^{-\delta}}.
\end{aligned}$$

for all n sufficiently large. In the last inequality, we use $1 - x \leq \mathrm{e}^{-x}$ and in the third inequality we use $v_n \leq r \leq n$ and hence that $-\frac{r(m-2)}{n} = \frac{2r}{n} - \frac{rm}{n} \leq 2 - \frac{mv_n}{n} = 2 - mn^{-\delta}$. Since $m \sim An^{\alpha}$, we have that $-mn^{-\delta} < -C_5 n^{\delta_2}$ for some positive constant C_5 and all n sufficiently large. Hence, we have that for all n sufficiently large,

$$\tilde{\mathbb{P}}_{n,m}(B_{r,j}) \leq 4me^2\mathrm{e}^{-mn^{-\delta}} \leq 4me^2\mathrm{e}^{-C_5 n^{\delta_2}} \leq \mathrm{e}^{-C_4 n^{\delta_2}}$$

for some positive constant C_4 smaller than C_5. □

Proof of Lemma 8. If $(r_1, \ldots, r_k) \in \mathcal{A}(n) \setminus \mathcal{A}(v_n)$, there exists some i, $1 \leq i \leq k$, so that $r_i > v_n$. By Lemma 10, we have that

$$\tilde{\mathbb{P}}_{n,m}(\cap_{i=1}^{k} B_{r_i,j}) \leq \tilde{\mathbb{P}}_{n,m}(B_{r_i,j}) \leq \mathrm{e}^{-C_4 n^{\delta_2}}.$$

The rest of the proof is analogous to the proof of Lemma 3. We define $\tilde{\Delta}_n$ as in the proof of Lemma 3. Using the fact that the cardinality of $\mathcal{A}(n) \setminus \mathcal{A}(v_n)$ is at most n^k, as in the proof of Lemma 3, we have that

$$\tilde{\Delta}_n \leq \sum_{\mathcal{A}(n)\setminus\mathcal{A}(v_n)} \mathrm{e}^{-C_4 n^{\delta_2}} \leq n^k \mathrm{e}^{-C_4 n^{\delta_2}} \leq \mathrm{e}^{-C_6 n^{\delta_2}}$$

for some positive constant C_6 less than C_4. □

4. Conclusion

In this paper, we have proved a conjecture of Yakubovich regarding limit shapes of slices of partitions of an integer n when the number of summands $m \sim An^{\alpha}$ for some $\alpha < \frac{1}{2}$. We have proved that the probability that there exists a summand of multiplicity j in a randomly chosen partition or composition of an integer n goes to zero asymptotically with n provided j is larger than a critical value. As a corollary, we have strengthened a result of [4] concerning the repeatability of summands in a randomly chosen integer partition of n when $\alpha = \frac{1}{3}$.

5. Appendix

Proofs of (A2)–(A5).

(A2) Follows since $\beta_3 < \beta$, and hence $\beta_0 < \beta$.

(A3) For $r \le j_2 v_n$, we have

$$\begin{aligned}\frac{1}{(n-j_1 r)^\gamma} &= \frac{1}{n^\gamma}\left(1-\frac{j_1 r}{n}\right)^{-\gamma} = \frac{1}{n^\gamma}\left(1+O\left(\frac{v_n}{n}\right)\right)^{-\gamma}\\ &= \frac{1}{n^\gamma}\left(1+O\left(\frac{v_n}{n}\right)\right) = \frac{1}{n^\gamma}\left(1+O\left(\frac{1}{n^\beta}\right)\right)\\ &= \frac{1}{n^\gamma}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right).\end{aligned}$$

In obtaining the last equality, we have used (A1).

(A4) In the first inequality we use $\alpha < \frac{1}{2}$, in the second we use $\alpha \ge \frac{1}{3}$, in the third we use $m \sim An^\alpha$ and in the fourth we use $\beta_0 \le \beta_1 < 1-2\alpha$.

(A5) Follows since $m \sim An^\alpha$. □

Proof of (2.16). Let $J = j_\alpha$ and we obtain from (2.13) that

$$\frac{P_{m-l-1}(n-r)}{P_{m-l}(n-r)} = \exp\left(K_1 + K_2 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \tag{5.1}$$

for any fixed integer $l \ge 1$ and for all $r \le j v_n$ where

$$K_1 = (m-l-1)\log\left(\frac{(n-r)e^2}{(m-l-1)^2}\right) - (m-l)\log\left(\frac{(n-r)e^2}{(m-l)^2}\right)$$

and

$$K_2 = \sum_{k=2}^{j_\alpha} a_k \frac{(m-l-1)^{2k-1}-(m-l)^{2k-1}}{(n-r)^{k-1}}.$$

In (5.1) and henceforth, any $O(\cdot)$ term is independent of r. We evaluate K_2 first. For any integer $k \ge 2$, we have that

$$\begin{aligned}0 &\le (m-l)^{2k-1}-(m-l-1)^{2k-1} = \sum_{l_1=1}^{2k-1}\binom{2k-1}{l_1}(m-l-1)^{2k-1-l_1}\\ &\le m^{2k-2}\sum_{l_1=1}^{2k-1}\binom{2k-1}{l_1} = (2^{2k-1}-1)m^{2k-2}.\end{aligned}$$

Therefore

$$\begin{aligned} |K_2| &\leq \sum_{k=2}^{j_\alpha} |a_k| \frac{(m-l)^{2k-1} - (m-l-1)^{2k-1}}{(n-r)^{k-1}} \\ &\leq D \sum_{k=2}^{j_\alpha} (2^{2k-1} - 1) \frac{m^{2k-2}}{(n-r)^{k-1}} \\ &\leq D(2^{2j_\alpha - 1} - 1) \sum_{k=2}^{j_\alpha} \left(\frac{m^2}{n-r} \right)^{k-1}, \end{aligned}$$

where $D = \sup_{2 \leq k \leq j_\alpha} |a_k|$. By (A3), we have that $\left(\frac{m^2}{n-r}\right)^{k-1} = \left(\frac{m^2}{n}\right)^{k-1} \left(\frac{n}{n-r}\right)^{k-1} = \left(\frac{m^2}{n}\right)^{k-1} \left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right) \leq 2 \left(\frac{m^2}{n}\right)^{k-1}$ for all n sufficiently large. Hence by (A4) we have

$$|K_2| \leq 2D(2^{2j_\alpha - 1} - 1) \sum_{k=2}^{j_\alpha} \left(\frac{m^2}{n} \right)^{k-1} = O\left(\frac{m^2}{n}\right) = O\left(\frac{1}{n^{\beta_0}}\right).$$

Also, we have

$$\begin{aligned} K_1 &= \log\left(\frac{(m-l)^2}{(n-r)}\right) + (m-l-1)\log\left(\frac{m-l}{m-l-1}\right)^2 - 2 \\ &= \log\left(\frac{(m-l)^2}{(n-r)}\right) + 2(m-l-1)\log\left(1 + \frac{1}{m-l-1}\right) - 2. \end{aligned}$$

For a fixed $l \geq 1$ and m sufficiently large, we have

$$\log\left(1 + \frac{1}{m-l-1}\right) = \frac{1}{m-l-1} + O\left(\frac{1}{m-l-1}\right)^2.$$

Hence by (A4) we have

$$\begin{aligned} 2(m-l-1)\log\left(\frac{m-l}{m-l-1}\right) &= 2 + O\left(\frac{1}{m-l-1}\right) \\ &= 2 + O\left(\frac{1}{m}\right) = 2 + O\left(\frac{1}{n^{\beta_0}}\right). \end{aligned}$$

Thus

$$K_1 = \log\left(\frac{(m-l)^2}{(n-r)}\right) + O\left(\frac{1}{n^{\beta_0}}\right).$$

Substituting the estimates for K_1 and K_2 in (5.1), we get that

$$\begin{aligned} \frac{P_{m-l-1}(n-r)}{P_{m-l}(n-r)} &= \exp\left(\log\left(\frac{(m-l)^2}{(n-r)}\right) + O\left(\frac{1}{n^{\beta_0}}\right)\right) \\ &= \frac{(m-l)^2}{(n-r)} \exp\left(O\left(\frac{1}{n^{\beta_0}}\right)\right) = \frac{(m-l)^2}{(n-r)}\left(1 + O\left(\frac{1}{n^{\beta_0}}\right)\right). \end{aligned}$$

But

$$\frac{(m-l)^2}{n-r} = \frac{m^2}{n}\left(1-\frac{l}{m}\right)^2 \frac{n}{n-r}.$$

Also, by (A4), $\left(1-\frac{l}{m}\right)^2 = 1+O\left(\frac{1}{m}\right) = 1+O\left(\frac{1}{n^{\beta_0}}\right)$ and by (A3), $\frac{n}{n-r} = 1+O\left(\frac{1}{n^{\beta_0}}\right)$. Hence

$$\frac{P_{m-l-1}(n-r)}{P_{m-l}(n-r)} = \frac{m^2}{n}\left(1+O\left(\frac{1}{n^{\beta_0}}\right)\right)^3.$$

To obtain (5.1) from the above equation, we use (A1). □

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