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Process convergence of self-normalized sums of i.i.d. random variables coming from domain of attraction of stable distributions

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Abstract. In this paper we show that the continuous version of the self-normalized process $Y_{n,p}(t) = S_n(t)/V_{n,p} + (nt - [nt])X_{[nt]+1}/V_{n,p}, 0 < t \leq 1; p > 0$ where $S_n(t) = \sum_{i=1}^{[nt]} X_i$ and $V_{(n,p)} = (\sum_{i=1}^{n} |X_i|^p)^{1/p}$ and X_i i.i.d. random variables belong to $DA(\alpha)$, has a non-trivial distribution iff $p = \alpha = 2$. The case for $2 > p > \alpha$ and $p \leq \alpha < 2$ is systematically eliminated by showing that either of tightness or finite dimensional convergence to a non-degenerate limiting distribution does not hold. This work is an extension of the work by Csörgő *et al.* who showed Donsker's theorem for $Y_{n,2}(\cdot)$, i.e., for $p = 2$, holds iff $\alpha = 2$ and identified the limiting process as a standard Brownian motion in sup norm.

Keywords. Domain of attraction; process convergence; self-normalized sums; stable distributions.

1. Introduction

Limit theorems plays a fundamental role in probability. Various forms of limit theorems, like the strong laws of large numbers, the central limit thoerems, the law of iterated logrithm and the laws of large deviations are celebrated results in this field. However restrictive assumptions like the finiteness of moments up to a certain order or the existence of the moment generating function in a neighbourhood of zero are necessary conditions for proving these theorems. Also the choice of the normalizing factor involves the standard deviation, which is typically unknown in many statistical applications. What is done instead is to estimate the unknown parameters by a sequence of random variables (like the sample standard deviation in the Student's t statistic). The normalizing factor is random in this case. To see whether the above mentioned limit laws hold with random normalization is a fruitful area of research that has yielded many interesting results in the last two decades. For example, it has been shown in [11] that even under much less assumptions an analogy of the law of iterated logarithm holds under randomized normalization. The same thing can be shown in case of laws of large and moderate deviations (see [18]).

The study of the asymptotics of the self-normalized sums are also interesting. Logan *et al*. [14] first showed the asymptotics of the self-normalized sums where the variables belong to the domain of attraction of a stable distribution. In [10], it has been shown that limiting distribution of the self-normalized sums converges to *normal* if and only if the constituent random variables are from the domain of attraction of a normal distribution (henceforth denoted as DAN). Hence they conclude the same for t-statistics. Csörgő, Szyszkowicz and Wang [4] proved a functional (process) convergence result in sup norm for suitably scaled products of the self-normalized sums (with L_2 normalization as in [10]). They also showed that the result holds if and only if constituent random variables are from DAN. Basak and Dasgupta [1] showed the convergence of a suitably scaled process to an Ornstein Uhlenbeck process. There also the constituent variables come from DAN. The aim of this paper is to show that the only case when the asymptotic distribution of the self-normalized process is non-trivial is when the norming index p is equal to the index of stability α which equals 2 (for definition, see §2).

This paper is organized as follows. Section 2 contains definitions and a preliminary result that is used throughout. Section 3 contains the main result of this paper together with a few remarks. Sections 4 and 5 show convergence of finite dimensional distribution of the self-normalized process and tightness result respectively for various choices of p and α. We show that the only case when the limiting distribution will be non-trivial (which turns out to be Brownian according to [4]) is when $p = \alpha = 2$. Section 6 contains some conclusion.

2. Definition and preliminaries

Let $\{X_i\}$ be a sequence of i.i.d. random variables. We intend to study the convergence of the process determined at time t by

$$Y_{n,p}(t) = \frac{S_n(t)}{V_{n,p}} + (nt - [nt])\frac{X_{[nt]+1}}{V_{n,p}}, \quad 0 < t < 1, \quad p > 0 \tag{2.1}$$

where the process $S_n(.)$ and $V_{n,p}$ are defined as $S_n(t) = \sum_{i=1}^{[nt]} X_i$ and $V_{n,p} = (\sum_{i=1}^{n} |X_i|^p)^{1/p}$ where X_i's belong to the domain of attraction of a α-stable family denoted by DA(α) and $[x]$ is the largest integer less than or equal to x. We prove process convergence by showing finite dimensional convergence and tightness. We state a lemma which is in the spirit of [9]. For an alternate proof, see Appendix 2.

Lemma 1. *If* $X \in$ DA(α), *then* $Y = sgn(X)|X|^{\alpha/2} \in$ DAN.

We also quote a theorem due to [4] that will be used later.

Theorem 1. *The following statements are equivalent*:

(1) $EX = 0$ *and* X *is in the domain of attraction of the normal law.*
(2) $S_{[nt_0]}/V_{n,2} \to N(0, t_0)$ *for* $t_0 \in (0, 1]$.
(3) $S_{[nt]}/V_n \to W(t)$ *on* $(D[0, 1], \rho)$, *where* ρ *is the sup-norm metric for functions in* $D[0, 1]$, *and* $\{W(t), 0 < t < 1\}$ *is a standard Wiener process.*
(4) *On an appropriate probability space for* $X, X_1, X_2, \ldots$ *we can construct a standard Wiener process* $\{W(t), 0 < t < \infty\}$ *such that*

$$\sup_{0\le t\le 1} |S_{[nt]}/V_{n,2} - W(nt)/\sqrt{n}| = o_p(1). \tag{2.2}$$

3. Main result

Let X_i be i.i.d. symmetric observations from the domain of attraction of a α-stable distribution and $\{Y_{n,p}(\cdot)\}$ as defined in (2.1). Then we have the following theorem:

Theorem 2. *$Y_{n,p}(t)$ converges weakly to Brownian motion in $C[0, 1]$, if and only if $p = \alpha = 2$.*

Proof. In § 4 we show that for $0 < p < \alpha \leq 2$ and $0 < p = \alpha < 2$ the finite dimensional distributions converge in probability to a degenerate distribution at zero. A non-trivial limiting distribution exists if $p > \alpha$ and $p = \alpha = 2$. In § 5 we show that the sequence $\{S_n/V_{n,p}\}$ of self-normalized sums is tight iff $0 < p \leq \alpha \leq 2$. The only case where we have both tightness and finite dimensional convergence is $p = \alpha = 2$. The limiting distribution of the sequence for this choice of p and α was identified by [10] as *normal*. Applying Prohorov's theorem we have the distributional convergence to the Wiener process. The convergence in the sup norm metric follows directly from (2.2) of the above theorem by [4]. □

In [4], Csörgő, Szyszkowicz and Wang are interested in the process $S_{[nt]}/V_{n,p}$ which is in $D([0, 1])$. However we are interested in the process $Y_{n,p}(t)$ which is in $C([0, 1])$. But from the definition of $Y_{n,p}(t)$,

$$|Y_{n,p}(t) - S_{[nt]}/V_{n,p}| = |(nt - [nt])X_{[nt]+1}|/V_{n,2} \leq |X_{[nt]+1}|/V_{n,p}.$$

If $p = \alpha = 2$ then, by Darling [7], we have that $\max_{1\leq i\leq n} |X_i|/(\sum_{i=1}^{n} X_i^2)^{1/2} \xrightarrow{P} 0$. So $|Y_{n,p}(t) - S_{[nt]}/V_{n,p}| \xrightarrow{P} 0$. This implies that $Y_{n,p}(t)$ takes the same limiting distribution as $S_{[nt]}/V_{n,p}$ which is *normal*.

In [16], Račkauskas and Suquet obtained the limiting distribution of the *adaptive* self-normalized process. However, in that paper the norming index p was 2. To the best of our knowledge, where norming index is different from 2 has not been studied for the adaptive self-normalized processes.

4. Convergence of finite dimensional distributions

To get the process convergence we first need to examine the convergence of finite-dimensional distributions, i.e., for $0 < t_1 < t_2 < \cdots < t_k$, $k \geq 1$ we want to examine the convergence of the random vector $(Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))$ as $n \to \infty$. We will do this for $p < \alpha$, $p = \alpha$ and $p > \alpha$ separately.

4.1 *Case* 1. $p < \alpha$

Since $X_i \in \mathrm{DA}(\alpha)$, by SLLN, $V_{n,p}/n^{1/p}$ converges to a positive constant, say, $k(\alpha, p)$.

Now, for $X_i \in \mathrm{DA}(\alpha)$, $S_n/(n^{1/\alpha}h(n))$ converges in distribution to a $S(\alpha)$ random variable, where h is a slowly varying function of n. Since $p < \alpha$, $S_n/n^{1/p} = n^{(1/\alpha)-(1/p)} S_n/n^{1/\alpha} \to 0$, in probability, as $n \to \infty$. Thus, $S_n/V_{n,p} = \frac{S_n/n^{1/p}}{V_{n,p}/n^{1/p}} \to 0$, in probability, as $n \to \infty$. Therefore, the joint distribution would converge to a degenerate one, in this case.

4.2 *Case* 2. $p = \alpha$

Here we assume that X_i is symmetric and belongs to DA(α).

Lemma 2. *For $V_{n,\alpha}$ defined as in § 2, $V_{n,\alpha} \geq V_{n,1} \geq V_{n,\beta}$ if $\alpha \leq 1 \leq \beta \leq 2$.*

Proof. We use the inequality for $a > 0, b > 0$, and $\alpha \leq 1,\ 2 \geq \beta \geq 1$,

$$\begin{aligned} a^\alpha + b^\alpha &\geq (a+b)^\alpha \quad \text{and} \quad (a+b)^\beta \geq a^\beta + b^\beta \\ &\Rightarrow (a^\alpha + b^\alpha)^{\beta/\alpha} \geq (a+b)^\beta \geq (a^\beta + b^\beta) \\ &\Rightarrow (a^\alpha + b^\alpha)^{1/\alpha} \geq (a^\beta + b^\beta)^{1/\beta}. \end{aligned}$$

Now, take $\alpha = 1$ and $\beta \geq 1$ and then $\alpha \leq 1$ and $\beta = 1$ to get

$$(a^\alpha + b^\alpha)^{1/\alpha} \geq (a+b) \geq (a^\beta + b^\beta)^{1/\beta}.$$

Also, for $1 \leq \beta \leq 2$, it follows that $1 \leq 2/\beta \leq 2$. Hence,

$$(a^\beta + b^\beta)^{2/\beta} \geq (a^{\beta(2/\beta)} + b^{\beta(2/\beta)}) = (a^2 + b^2) \Rightarrow (a^\beta + b^\beta)^{1/\beta} \geq (a^2 + b^2)^{1/2}.$$

The case for n positive numbers can be shown in the same manner. Thus, combining above, we get

$$V_{n,2} \leq V_{n,\beta} \leq V_{n,1} \leq V_{n,\alpha}.$$ □

Now, we show that the self-normalized sum for $p = \alpha$ converges to degenerate distribution as well.

Theorem 3. *If $p = \alpha \leq 1$, $\lim_{n\to\infty} \operatorname{Var}(\frac{S_n}{V_{n,p}}) = 0$.*

Proof. Note that

$$\begin{aligned} E\left(\frac{\sum X_i}{V_{n,\alpha}}\right)^2 &= \sum E\left(\frac{X_i^2}{V_{n,\alpha}^2}\right) + \sum_{(i,j):i\neq j} E\left(\frac{X_i X_j}{V_{n,\alpha}^2}\right) \\ &= \sum_i E\left(\frac{X_i^2}{V_{n,\alpha}^2}\right) + \sum_i E\left(\sum_{j\neq i} X_i E\left(\frac{X_j}{V_{n,\alpha}^2} \mid X_i, i \neq j\right)\right) \\ &= \sum_i E\left(\frac{X_i^2}{V_{n,\alpha}^2}\right), \end{aligned} \tag{4.1}$$

the second term vanishes since

$$\frac{X_j}{V_{n,\alpha}^2} = -\frac{X_j}{V_{n,\alpha}^2} \quad \text{in distribution.}$$

We now use the fact that $V_{n,\alpha} \geq V_{n,2}$ implies that $(\sum_{i=1}^n X_i^2/V_{n,\alpha}^2) \leq (\sum_{i=1}^n X_i^2/V_{n,2}^2) = 1$. Hence if we could show that $(\sum_{i=1}^n X_i^2/V_{n,\alpha}^2) \to 0$ in probability. Then by dominated convergence theorem (DCT), we have the result. Observe that, for $\alpha \leq 1$, from Lemma 1,

$Y_i = sgn(X_i)|X_i|^{\alpha/2} \in \mathrm{DAN}$. From [10], we have that $E\Big(\frac{Y_i^4}{(\sum |Y_i|^2)^2}\Big) = E\Big(\frac{X_i^{2\alpha}}{(\sum |X_i|^{\alpha})^2}\Big) = o(\frac{1}{n})$. Thus,

$$E\left(\frac{\sum_{i=1}^n X_i^2}{(V_{n,\alpha})^2}\right)^{\alpha} \le E\left(\frac{\sum_{i=1}^n |X_i|^{2\alpha}}{(\sum_{i=1}^n |X_i|^{\alpha})^2}\right) = o(1) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $(\sum_{i=1}^n X_i^2/V_{n,\alpha}^2) \to 0$ in probability, as it goes to zero in α-th mean. Therefore, by DCT we conclude the proof. □

We now proceed to prove the result for $p = \alpha > 1$.

Lemma 3. *If* $X \in \mathrm{DA}(\alpha)$, *then* $\frac{\sum X_i^2}{V_{n,\alpha}^2} \xrightarrow{P} 0$.

Proof. Observe that, if $X_i \in \mathrm{DAN}$ then by [15], $\max_{1\le i\le n} \frac{|X_i|}{V_{n,2}} \xrightarrow{P} 0$. Because, if $X_i \in \mathrm{DA}(\alpha)$ then $Y_i = sgn(X_i)|X_i|^{\alpha/2} \in \mathrm{DAN}$ by Lemma 1. Therefore

$$\max_{1\le i\le n} \frac{|Y_i|}{(\sum Y_i^2)^{1/2}} \xrightarrow{P} 0 \Leftrightarrow \max_{1\le i\le n} \frac{|X_i|^{\alpha/2}}{(\sum |X_i|^{\alpha})^{1/2}} \xrightarrow{P} 0$$
$$\Leftrightarrow \max_{1\le i\le n} \frac{|X_i|}{(\sum |X_i|^{\alpha})^{1/\alpha}} \xrightarrow{P} 0 \Leftrightarrow \max_{1\le i\le n} \frac{|X_i|^2}{V_{n,\alpha}^2} \xrightarrow{P} 0. \tag{4.2}$$

Again, from [7,9], since $X_i \in \mathrm{DA}(\alpha)$, one gets $|X_i|^2 \in \mathrm{DA}(\alpha/2)$. Define $Y_n^* = \max_{1\le i\le n} X_i^2$. Hence, for $\epsilon, \eta > 0$ choose $\delta = \frac{\epsilon}{K_\eta}$ where K_η is chosen so that $P(\sum X_i^2/Y_n^* > K_\eta) < \eta/2$. (This is possible since by [7] $Y_n^*/\sum X_i^2$ has a limiting distribution and hence tight.)

$$P\left(\frac{\sum X_i^2}{V_{n,\alpha}^2} > \epsilon\right) \le P\left(\frac{\sum X_i^2}{V_{n,\alpha}^2} > \epsilon,\ \frac{Y_n^*}{V_{n,\alpha}^2} > \delta\right) + P\left(\frac{\sum X_i^2}{V_{n,\alpha}^2} > \epsilon,\ \frac{Y_n^*}{V_{n,\alpha}^2} \le \delta\right)$$
$$\le P\left(\frac{Y_n^*}{V_{n,\alpha}^2} > \delta\right) + P\left(\frac{\sum X_i^2}{Y_n^*}\frac{Y_n^*}{V_{n,\alpha}^2} > \epsilon,\ \frac{Y_n^*}{V_{n,\alpha}^2} \le \delta\right)$$
$$\le P\left(\frac{Y_n^*}{V_{n,\alpha}^2} > \delta\right) + P\left(\frac{\sum X_i^2}{Y_n^*} > \frac{\epsilon}{\delta}\right).$$

Choose n_0 sufficiently large so that the first probability is less than $\eta/2$. By the choice of δ we have the second probability less than $\eta/2$ which implies that

$$P\left(\sum X_i^2/V_{n,\alpha}^2 > \epsilon\right) < \eta \quad \text{for } n \ge n_0.$$

Hence the lemma is proved. □

Theorem 4. *Let* $1 < p = \alpha < 2$, *and* X_i*'s are symmetric and* $X_i \in \mathrm{DA}(\alpha)$. *Then* $\lim_{n\to\infty} \mathrm{Var}(\frac{S_n}{V_{n,p}}) = 0$.

Note that $\mathrm{Var}\left(\frac{S_n}{V_{n,p}}\right) = E\left(\frac{S_n}{V_{n,p}}\right)^2$ by symmetry of X_i and also $E\left(\frac{S_n}{V_{n,p}}\right)^2 = E\left(\frac{\sum X_i^2}{V_{n,p}^2}\right)$ by (4.1) in the proof of Theorem 3.

$$\begin{aligned} V_{n,\alpha} &\geq V_{n,2} \quad \text{for } 0 < \alpha \leq 2 \\ &\Rightarrow \frac{\sum X_i^2}{(\sum |X_i|^\alpha)^{2/\alpha}} \leq \frac{\sum X_i^2}{\sum X_i^2} = 1. \end{aligned}$$

Hence, by Lemma 3 and applying bounded convergence theorem,

$$\lim_{n\to\infty} E\left(\frac{\sum X_i^2}{(\sum |X_i|^\alpha)^{2/\alpha}}\right) = 0.$$

Remark 1. For $X_i \in \mathrm{DA}(\alpha)$ symmetric, we, in fact, showed in Theorems 3 and 4 that $(S_n/V_{n,p}) \to 0$ in probability, for $0 < p = \alpha < 2$. Using the same technique, it is immediate that for any fixed $0 \leq t \leq 1$, $(S_{[nt]}/V_{n,p}) \to 0$, in probability, for $0 < p = \alpha < 2$ as well. The result for k dimension can be obtained from the above result. Note that the joint distribution of $\left(\frac{S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_2]}}{V_{n,p}}, \ldots, \frac{S_{[nt_k]}}{V_{n,p}}\right)$ can be obtained from the joint distribution of $\left(\frac{S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_2]}-S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_3]}-S_{[nt_2]}}{V_{n,p}}, \ldots, \frac{S_{[nt_k]}-S_{[nt_{k-1}]}}{V_{n,p}}\right)$ by a linear transformation. We next show that the joint distribution of the latter converges to zero. Write $S_1 = \frac{S_{[nt_1]}}{V_{n,p}}$, $S_2 = \frac{S_{[nt_2]}-S_{[nt_1]}}{V_{n,p}}$ and $S_k = \frac{S_{[nt_k]}-S_{[nt_{k-1}]}}{V_{n,p}}$. Now consider the varinace of any linear combination of $V(a_1S_1 + a_2S_2 + \cdots + a_kS_k)$ where a_i's are any arbitrary constants. Due to independence, the cross product term vanishes and by Theorem 4 the variances are zero which implies that any linear combination tends in probability to zero. Therefore $\phi_{S_1,S_2,\ldots,S_k}(a_1, a_2, \ldots, a_k) \to 1$, where $\phi_{S_1,S_2,\ldots,S_k}$ is the characteristic function. Applying continuity theorem we therefore have that the limiting joint distribution of $(S_1, S_2, \ldots, S_k)$ and hence $(S_{[nt_1]}/V_{n,p}, S_{[nt_2]}/V_{n,p}, \ldots, S_{[nt_k]}/V_{n,p})$ is degenerate at 0.

4.3 *Case* 3. $p > \alpha$

The aim of this subsection is to find the limiting joint characteristic function of the process $Y_{n,p}(t)$ at time points $0 < t_1 < t_2 < \cdots < t_k < 1$. Defining $m_i = [nt_i]$ $\forall i = 1, 2, \ldots, k$ we find the limiting joint characteristic function of $\mathbf{S}_1 := (S_{m_1}/n^{1/\alpha}, (S_{m_2} - S_{m_1})/n^{1/\alpha}, \ldots, (S_{m_k} - S_{m_{k-1}})/n^{1/\alpha}, V_{n,p}^p/n^{p/\alpha})$. Applying a transformation one can obtain the limiting joint distribution of $\mathbf{S} := \left(\frac{S_{m_1}/n^{1/\alpha}}{V_{n,p}/n^{1/\alpha}}, \frac{S_{m_2}/n^{1/\alpha}}{V_{n,p}/n^{1/\alpha}}, \ldots, \frac{S_{m_k}/n^{1/\alpha}}{V_{n,p}/n^{1/\alpha}}\right)$. Also since

$$\begin{aligned} E\left(|Y_{n,p}(t_1) - S_{[nt_1]}/V_{n,p}|^2\right) &= E\left((nt_1 - [nt_1]^2|X_{[nt_1]}|^2/V_{n,p}^2)\right) \\ &\leq E\left(|X_{[nt_1]}^2/V_{n,p}^2|\right) \\ &\leq E\left(|X_{[nt_1]}^2/V_{n,2}^2|\right) \ \forall p \leq 2 \\ &= \frac{1}{n} \text{ since } [nt_1] < n, \end{aligned}$$

the two vectors $(Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))$ and $\mathbf{S}$ are asymptotically negligible. To prove that the finite dimensional distribution of the process $Y_{n,p}(\cdot)$ exists it therefore suffices to show the existence of the limiting characteristic function of $\mathbf{S}_1$, say $\phi_{\mathbf{S}_1}(u_1, u_2, \ldots, u_k, s)$.

To find the required characteristic function we proceed along the same lines as in [14]. Note that for appropriately chosen constants a_n, $a_n S_n$ and $a_n^p V_{n,p}^p$ have the same limiting distribution as that in the case when X_i belongs to the stable distribution (also see p. 208 of [13]). So we may and do assume that X_i's belong to stable distributions (having density $g(\cdot)$) which satisfies $x^{\alpha+1} g(x) \to\ r$ as $x \to \infty$ and $|x|^{\alpha+1} g(x) \to l$ as $x \to -\infty$ with $r + l > 0$.

Lemma 4. $\mathbf{S}_1$ *converges in distribution to a random vector whose characteristic function is given by*

$$\exp\left(\sum_{i=1}^{k} \int \left[\exp\{i u_i y (t_i)^{1/\alpha} + i s |y|^p (t_i)^{p/\alpha})\} - 1\right] \frac{K(y)}{y^{\alpha+1}} \mathrm{d}y\right)$$
$$\times \lim_{m_k, n \to \infty} E\left(\mathrm{e}^{il|X|^p/n^{p/\alpha}}\right)^{n-m_k}, \tag{4.3}$$

where $K(y) = \begin{cases} r, & \text{if } y > 0, \\ l, & \text{if } y < 0. \end{cases}$

Proof. We adopted a proof of Logan *et al*. [14] and Lai *et al*. [13] and deferred it to Appendix 3. □

Remark 2. The second limit is the limit of the characteristic function of $\frac{1}{n^{p/\alpha}} \sum_{i=1}^{n-m_k} |X_i|^p$ where X_i's are identical and independently distributed as a stable distribution with index α. Using the fact that $\frac{n-m_k}{n} \to 1 - t_k$ and $|X|^p$ is stable with index p/α, by Slutsky's lemma we have that $\frac{1}{n^{p/\alpha}} \sum_{i=1}^{n-m_k} |X_i|^p \xrightarrow{D} (1 - t_k)|X|^{p/\alpha}$. Hence by Levy's continuity theorem the last limit exists and we have shown that the limiting characteristic function on the left-hand side of equation (4.3) exists for $p > \alpha$. (We have not identified the limiting distribution. For identification one can see the procedure followed in [14]).

Remark 3. For $p = \alpha = 2$, the finite dimensional distribution of $Y_{n,p}(t)$ can be obtained by using the fact that $\frac{S_n}{\sqrt{nl(n)}} \xrightarrow{D} N(0, 1)$ and $\frac{1}{nl(n)} V_{n,2}^2 \xrightarrow{P} 1$ for a slowly varying function $l(\cdot)$ (see [10]). Applying the same argument as above we see that the distribution of $(Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))$ can be obtained from the distribution of $\left(\frac{S_{[nt_1]}}{V_{n,p}}, \frac{S_{[nt_2]} - S_{[nt_1]}}{V_{n,p}}, \ldots, \frac{S_{[nt_k]} - S_{[nt_{k-1}]}}{V_{n,p}}\right)$ by a linear transformation. Now the components in the latter are uncorrelated and hence

$$\frac{S_{[nt_1]}}{V_{n,p}} = \frac{\sqrt{[nt_1] l([nt_1])}}{\sqrt{nl(n)}} \frac{\frac{1}{\sqrt{[nt_1] l([nt_1])}} S_{[nt_1]}}{\frac{1}{\sqrt{nl(n)}} V_{n,p}} \xrightarrow{D} \sqrt{t_1} N(0, 1)$$

(by using Slutsky's theorem and the fact that $l(\cdot)$ is a slowly varying function). Same thing can be done for $\frac{S_{[nt_2]}-S_{[nt_1]}}{V_{n,p}}$ and the limiting distribution in that case will be $\sqrt{t_2-t_1}N(0,1)$. If $t_i < t_j$ then $\text{Cov}\left(\frac{S_{[nt_i}}{V_{n,p}}, \frac{S_{[nt_j]}}{V_{n,p}}\right) = \text{Cov}\left(\frac{S_{[nt_i]}}{V_{n,p}}, \frac{S_{[nt_j]}-S_{[nt_i]}+S_{[nt_i]}}{V_{n,p}}\right) = V\left(\frac{S_{[nt_i]}}{V_{n,p}}\right) = t_i = \min(t_i, t_j)$. Since the Jacobian of the transformation is one the finite dimensional distribution of $(Y_{n,p}(t_1), Y_{n,p}(t_2), \ldots, Y_{n,p}(t_k))$ is a multivariate normal distribution with dispersion matrix $((v_{i,j}))$ given by

$$v_{i,j} = \begin{cases} t_i, & \text{if } i = j \\ \min(t_i, t_j), & \text{otherwise.} \end{cases}$$

In fact, the above finite dimensional convergence follows from Theorem 1 since the self-normalized sums is converging in probability to the Wiener motion properly scaled in the sup norm metric.

5. Tightness

Theorem 5. *The process* $\{Y_{n,p}(\cdot)\}$ *is tight iff* $p \le \alpha \le 2$.

We first prove the *if* part and then the *only if* part.

'If' part. The process $Y_{n,p}(\cdot)$ is tight if $p \le \alpha \le 2$.

Proof. From Theorem 7.3 of [3] the process $Y_{n,p}(\cdot)$ is tight iff $Y_{n,p}(0)$ is tight and for all ϵ and η positive, $\exists\delta$ $(0 < \delta < 1)$ such that $\lim_{n\to\infty} P(\omega_{Y_{n,p}}(\delta) \ge \epsilon) = 0$ where $\omega_X(\delta) = \sup_{t-s<\delta} |X(t) - X(s)|$ is the modulas of continuity of the process $X(\cdot)$. Also from eq. (7.11) of [3], for a process $X(\cdot)$, an arbitrary probability P and for any $\epsilon > 0$, $\delta > 0$, we have

$$P(\omega_X(\delta) \ge 3\epsilon) \le \sum_{i=1}^{v} P\left(\sup_{t_{i-1}<s<t_i} |X(s) - X(t_{i-1})| \ge \epsilon\right)$$

for any partition $0 = t_0 < t_1 < t_2 < \ldots < t_v = 1$ such that $\min_{1<i<v}(t_i - t_{i-1}) \ge \delta$.

Take partition $t_i = m_i/n$ where $0 = m_0 < m_1 < \cdots < m_v = n$. By the definition of the process in (2.1) we have that $\sup_{t_{i-1}<s<t_i} |Y_{n,p}(s) - Y_{n,p}(t)| = \max_{m_{i-1}<k<m_i} \frac{|S_k - S_{m_{i-1}}|}{V_{n,p}}$. Therefore,

$$P(\omega_{Y_{n,p}}(\delta) \ge 3\epsilon) \le \sum_{i=1}^{v} P\left[\max_{m_{i-1}<k<m_i} |S_k - S_{m_{i-1}}| \ge \epsilon V_{n,p}\right].$$

The sequence $\{S_n\}$ is stationary and hence the above is same as

$$\sum_{i=1}^{v} P\left[\max_{k<m_i-m_{i-1}} |S_k| > \epsilon V_{n,p}\right].$$

Choose $m_i = mi$ where m is an integer satisfying $m = \lceil n\delta \rceil$ and $v = \lceil n/m \rceil$. With this choice $v \to 1/\delta < 2/\delta$. Therefore for sufficiently large n,

$$\begin{aligned} P(\omega(Y_{n,p}, \delta) \geq 3\epsilon) &\leq vP\left(\max_{k\leq m} |S_k|/V_{n,p} > \epsilon\right) \\ &\leq (2/\delta)P\left(\max_{k\leq m} |S_k|/V_{n,p} > \epsilon\right). \end{aligned}$$

For fixed $n \geq 1$, define a finite filtration by $\mathcal{F}_{k,n} = \sigma\{\frac{X_1}{V_{n,p}}, \frac{X_2}{V_{n,p}}, \ldots, \frac{X_k}{V_{n,p}}\}$, $k = 1, 2, \ldots, n$. Then we have that $S_k/V_{n,p}$ is a martingale with respect to the filtration $\mathcal{F}_{k,n}$ (see Appendix 1) and the ratio

$$\begin{aligned} V_{m,p}/V_{n,p} &= \left(\frac{\sum_{i=1}^m |X_i|^p}{\sum_{i=1}^n |X_i|^p}\right)^{1/p} \\ &= \left(\frac{m}{n}\right)^{1/p}\left(\frac{h(m)}{h(n)}\right)^{2/p}\left(\frac{\frac{1}{mh^2(m)}\sum_{i=1}^m |X_i|^p}{\frac{1}{nh^2(n)}\sum_{i=1}^n |X_i|^p}\right)^{1/p} \\ &= \left(\frac{m}{n}\right)^{1/p}\left(\frac{h(m)}{h(n)}\right)^{1/p}\left(\frac{\frac{1}{mh^2(m)}\sum_{i=1}^m Y_i^2}{\frac{1}{nh^2(n)}\sum_{i=1}^n Y_i^2}\right)^{1/p}, \\ &\quad \text{putting } Y_i = |X_i|^{p/2}. \end{aligned} \tag{5.1}$$

Now $Y_i \in$ DAN from Lemma 1. From eq. (3.4) of [10] we have that if $Y_i \in$ DAN, then $\frac{\sum_{i=1}^n X_i^2}{nh^2(n)} \xrightarrow{P} 1$. Now

$$\begin{aligned} \frac{h(m)}{h(n)} &= \frac{h(\lceil n\delta \rceil)}{h(n)} = \frac{h(n\delta - x_n)}{h(n)} \quad \text{for some } 0 < x_n < 1 \\ &= \frac{h(n(\delta - \frac{x_n}{n}))}{h(n)}. \end{aligned} \tag{5.2}$$

Fox fixed δ, $\delta - \frac{x_n}{n}$ lies in some compact interval and from Theorem 1.1 of [17] we have that the convergence of $\frac{L(\lambda x)}{L(x)}$ to one is uniform (with respect to λ) for λ lying in any compact interval. Hence $\left(\frac{h(m)}{h(n)}\right)^{2/p}$ converges to 1. Since $\frac{\lceil n\delta \rceil}{n} \to \delta$ as $n \to \infty$, applying Slutsky's lemma we have that

$$V_{m,p}/V_{n,p} \xrightarrow{P} \delta^{1/p}.$$

Therefore,

$$\frac{1}{\delta}P\left(\max_{k\leq m} |S_k|/V_{n,p} > \epsilon\right) = \frac{1}{\delta}P\left(\max_{k\leq m} \frac{|S_k|}{V_{m,p}}\frac{V_{m,p}}{V_{n,p}} > \epsilon\right).$$

Writing $X_m = \max_{k \le m} \frac{|S_k|}{V_{m,p}}$ and $Y_m = \frac{V_{m,p}}{V_{n,p}}$, we have

$$\begin{aligned}
\frac{1}{\delta} P\big(\max_{k<m} |S_k|/V_{n,p} > \epsilon\big) &= \frac{1}{\delta} P(X_m Y_m > \epsilon) \\
&= \frac{1}{\delta}\big\{P\big(X_m Y_m > \epsilon, Y_m > 2\delta^{1/\delta}\big) \\
&\quad + P\big(X_m Y_m > \epsilon, Y_m < 2\delta^{1/\delta}\big)\big\} \\
&\le \frac{1}{\delta}\big\{P\big(X_m Y_m > \epsilon, Y_m < 2\delta^{1/p}\big) \\
&\quad + P\big(Y_m > 2\delta^{1/p}\big)\big\} \\
&\le \frac{1}{\delta}\big\{P\big(X_m > \epsilon/2\delta^{1/p}\big) + P\big(Y_m > 2\delta^{1/p}\big)\big\} \\
&\le \frac{1}{\delta}\big\{P(X_m > \epsilon/2\delta^{1/p}) + \eta\big\} \\
&\quad \text{(choosing sufficiently large } m \\
&\quad \text{such that } P(Y_m > 2\delta^{1/p}) < \eta) \\
&\le \frac{1}{\delta}\big\{4\delta^{2/p}/\epsilon^2 V(S_m/V_{m,p}) + \eta\big\} \\
&\quad \text{(by Doob's inequality} \\
&\quad \text{for nonnegative submartingales)} \\
&= (4\delta^{\gamma}/\epsilon^2) V(S_m/V_{m,p}) + \eta/\delta, \\
&\quad \text{for some } \gamma > 0. \qquad (5.3)
\end{aligned}$$

Now, for $p \le \alpha < 2$ or $p < \alpha = 2$, $\mathrm{Var}(S_m/V_{m,p})$ tends to zero (see §4.1, §4.2). Since $m \to \infty$ (since $m = \lceil n\delta \rceil$) we have that the right-hand side in (5.3) can be made arbitrarily small. Hence the lemma is proven.

For the case $p = \alpha = 2$, the lemma holds by [10] since it has been shown that the self-normalized sums converges to the normal distribution for $p = \alpha = 2$.

Before proving the 'only if' part we need the following lemma.

Lemma 5. *$\{Y_{n,p}(\cdot)\}$ is tight* $\Rightarrow \max_{1 \le i \le n} \frac{|X_i|}{V_{n,p}} \xrightarrow{P} 0$.

Proof. We use an equivalent condition of tightness given in Theorem 4.2 of [3]. A process is tight iff $\forall \epsilon > 0, \forall \eta > 0, \exists n_0$ and $0 < \delta < 1$ such that

$$P\left(\sup_{|t-s|<\delta} |Y_{n,p}(s) - Y_{n,p}(t)| \ge \epsilon\right) \le \eta \quad \forall t \in [0,1]. \tag{5.4}$$

Assume that the hypothesis is true, which means that for every $\epsilon, \eta > 0, \exists\, 0 < \delta < 1$ such that (5.4) holds. Choose n_0 sufficiently large so that $\frac{1}{n} < \delta \;\; \forall n > n_0$. Then we have

$$P\left(\sup_{|t-s|<\frac{1}{n}} |Y_{n,p}(t) - Y_{n,p}(s)| > \epsilon\right) < P\left(\sup_{|t-s|<\delta} |Y_{n,p}(t) - Y_{n,p}(s)| > \epsilon\right).$$

Now by definition of the process $Y_{n,p}(\cdot)$,

$$\sup_{|t-s|<\frac{1}{n}} |Y_{n,p}(t) - Y_{n,p}(s)| = \max_{1\le i\le n} \frac{|X_i|}{V_{n,p}} \forall t \in [0,1]$$
$$\Rightarrow P\left(\max_{1\le i\le n} \frac{|X_i|}{V_{n,p}} > \epsilon\right) < P\left(\sup_{|t-s|<\delta} |Y_{n,p}(t) - Y_{n,p}(s)| > \epsilon\right)$$
$$\Rightarrow P\left(\max_{1\le i\le n} \frac{|X_i|}{V_{n,p}} > \epsilon\right) < \eta \ \forall n > n_0, \text{by hypothesis.}$$
□

Remark 4. The converse is not necessarily true. To see this assume that $\max_{1\le i\le n} \frac{|X_i|}{V_{n,p}} \xrightarrow{P} 0$. Assume that there exists a δ_1 such that (5.4) holds. Given such a $\delta_1 > 0$, for any integer m we can get an n such that $\frac{m}{n} < \delta_1$. Then for such a m, n we have $|Y_{n,p}(t) - Y_{n,p}(s)| \le (\max_{1\le i\le n} \sum_{j=1}^{m})|X_{i+j}|/(V_{n,p})$. But the hypothesis does not guarantee that the right-hand side converges to zero in probability.

We use the above lemma to prove the necessary part in the following lemma.

'*Only if*' *part*. For $2 \ge p > \alpha$ the process is not tight.

Proof. For $2 \ge p > \alpha$ observe that

$$\max_{1\le i\le n} \frac{|X_i|}{V_{n,p}} \xrightarrow{P} 0 \Leftrightarrow \left(\max_{1\le i\le n} \frac{|X_i|}{V_{n,p}}\right)^p \xrightarrow{P} 0 \Leftrightarrow \max_{1\le i\le n} \frac{|X_i|^p}{\sum |X_i|^p} \xrightarrow{P} 0.$$

But $|X_i|^p \in DA(\gamma)$, where $\gamma = \frac{\alpha}{p} < 1$, for which Darling (Theorem 5.1 of [7]) says that if $Y_i \in DA(\gamma)$ where $\gamma < 1$ then $\max_{1\le 1\le n} \frac{|Y_i|}{\sum |Y_i|}$ converges in distribution to a non-degenerate random variable G whose characteristic function is identified in the same paper. Thus, $\max_{1\le i\le n} \frac{|X_i|^p}{\sum |X_i|^p}$ does not go to zero in probability. Hence, $\max_{1\le i\le n} \frac{X_i}{V_{n,p}}$ cannot converge to zero in probability and therefore from Lemma 5 the process cannot be tight. □

6. Conclusion

The study of self-normalized sums has seen a recent upsurge following the works of [6,10,14] and [18]. Results for functional convergence was shown only by [4] where the random variable were from the domain of attraction of a stable(α) distribution.

This paper deals with the same type of random variables but with norming index $p \in (0, 2]$. Although it is almost intuitive that the norming index p has something to do with the stability index α the relation between them has not been explored in the past. Csörgő *et al.* [4] kept the value of the norming index p fixed at 2 and compared with various choices of α. This paper, to our knowledge, seems to be the first one where we simultaneously change p and α. Here, using simple tools of tightness and finite dimensional convergence, we show that the only non-trivial case is iff $p = \alpha = 2$. The '*if*' part was shown by Giné *et al*. [10] and Csörgő *et al*. [4]. Our paper shows the *'only if'* part.

To proceed further a rate of convergence would be important. A non-uniform Berry Essen bound was given in [2], when the random variables are from DAN, and a bound using Saddlepoint approximation was proved in [12]. Although the process convergence is for $p = \alpha = 2$, Logan *et al.* [14] have shown that the self-normalized sequence can converge for $p > \alpha$. Using their techniques we have shown in §4.3 what the possible limiting chractetristic distribution would look like. From our personal communication with Qi-Man Shao we have learned about an unpublished result on limiting finite-dimensional distribution of $((S_{[nt_1]}/V_{n,p}, \ldots, S_{[nt_k]}/V_{n,p}),\ p > \alpha)$ where they have shown the limiting joint distribution as a mixture of Poisson-type distribution using the technique of Csörgő and Horvath [5]. The rate of convergence for this case has not been explored to our knowledge.

Appendices

Appendix 1

Let us introduce the Radmacher variables $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ where $P(\epsilon_i = 1) = P(\epsilon = -1) = \frac{1}{2}$. Since X_i is symmetric about zero the distribution of X_i is the same as $X_i^* := X_i\epsilon_i$ and the distribution of S_n is the same as the distribution of $\sum_{i=1}^n X_i\epsilon_i =: S_n^*$. Then

$$\begin{aligned}
E\left(\frac{S_{k+1}}{V_{n,p}}|\mathcal{F}_{k,n}\right) &= E\left(\frac{S_{k+1}^*}{V_{n,p}}\right)\\
&= E\left(E\left(\frac{S_k^* + X_{k+1}^*}{V_{n,p}}|\epsilon_i \quad, i = 1, \ldots, k\right)|\mathcal{F}_{k,n}\right)\\
&= E\left(\frac{S_k^*}{V_{n,p}} + E\left(\frac{X_{k+1}^*}{V_{n,p}}|\epsilon_i \quad, i = 1 \ldots, k\right)|\mathcal{F}_{k,n}\right)\\
&= E\left(\frac{S_k}{V_{n,p}}|\mathcal{F}_{k,n}\right) = \frac{S_k}{V_{n,p}}.
\end{aligned}$$

Appendix 2

To prove the lemma we need the following characterization:

$$Y \in \text{DAN} \quad \text{iff} \lim_{y\to\infty} \frac{y^2 P(|Y| > y)}{E(Y^2 I(|Y| < y))} = 0,$$

see [4].

We show that the random variable Y satisfies the necessary and sufficient condition. Now,

$$\begin{aligned}
y^2 P(|Y| > y) &= y^2 P(|X|^{\frac{\alpha}{2}} > y)\\
&= y^2 P(|X| > y^{\frac{2}{\alpha}})\\
&= y^2 h(y^{\frac{2}{\alpha}})(y^{\frac{2}{\alpha}})^{-\alpha}\\
&= h(y^{\frac{2}{\alpha}}) \ \ (\text{since if } X \in \text{DA}(\alpha),\ \ P(|X| > x)\\
&= x^{-\alpha} h(x) \text{ for slowly varying } h(\cdot)).
\end{aligned}$$

And,

$$\begin{aligned} E(Y^2 I(|Y| < y)) &= E(|X|^\alpha I(|X|^{\alpha/2} \le y)) \\ &= E(|X|^\alpha I(|X| \le y^{2/\alpha})) \\ &= \int_0^{y^{2/\alpha}} z^\alpha \mathrm{d}F_{|X|}(z) \\ &= \int_0^{y^{2/\alpha}} \left(\int_0^z \alpha t^{\alpha-1} \mathrm{d}t \right) \mathrm{d}F_{|X|}(z). \end{aligned}$$

Applying Fubini's theorem and interchanging the order of integration we get

$$\begin{aligned} E(Y^2 I(|Y| < y)) &= \int_0^{y^{2/\alpha}} \alpha \int_t^{y^{2/\alpha}} \mathrm{d}F_{|X|}(z) t^{\alpha-1} \mathrm{d}t \\ &= \alpha \int_0^{y^{2/\alpha}} P(t < |X| \le y^{2/\alpha}) t^{\alpha-1} \mathrm{d}t \\ &= \alpha \int_0^{y^{2/\alpha}} P(|X| > t) t^{\alpha-1} \mathrm{d}t \\ &\quad - \alpha \int_0^{y^{2/\alpha}} P(|X| > y^{2/\alpha}) t^{\alpha-1} \mathrm{d}t \\ &= \alpha \int_0^{y^{2/\alpha}} \frac{h(t)}{t} \mathrm{d}t - h(y^{2/\alpha}). \end{aligned}$$

Hence,

$$\lim_{y\to\infty} \frac{y^2 P(|Y| > y)}{E(Y^2 I(|Y| < y))} = 1 \Big/ \left(\alpha \lim_{y\to\infty} \frac{1}{h(y^{2/\alpha})} \int_0^{y^{2/\alpha}} \frac{h(t)}{t} \mathrm{d}t - 1 \right).$$

Now from Karamata's theorem (see, for example, [8]) for a slowly varying $h(\cdot)$

$$\begin{aligned} \lim_{x\to\infty} \frac{\int_0^x h(t)/t \,\mathrm{d}t}{h(x)} &= \infty \\ \Rightarrow \lim_{y\to\infty} \frac{y^2 P(|Y| > y)}{E(Y^2 I(|Y| < y))} &= 0. \end{aligned}$$

Appendix 3

The required characteristic function is

$$\phi_{\mathbf{S}_1}(u_1, u_2, \ldots, u_k, s) = E\left(\exp\left\{ i \frac{u_1}{n^{1/\alpha}} S_{m_1} + i \frac{u_2}{n^{1/\alpha}} (S_{m_2} - S_{m_1}) + \cdots + i \frac{u_k}{n^{1/\alpha}} (S_{m_k} - S_{m_{k-1}}) + i \frac{s}{n^{p/\alpha}} V_{n,p}^p \right\} \right)$$

$$
\begin{aligned}
&= E\left(\exp\left\{i\frac{u_1}{n^{1/\alpha}}S_{m_1} + i\frac{u_2}{n^{1/\alpha}}(S_{m_2} - S_{m_1})\right.\right.\\
&\quad + \cdots + i\frac{u_k}{n^{1/\alpha}}(S_{m_k} - S_{m_{k-1}})\\
&\quad + i\frac{s}{n^{p/\alpha}}(V^p_{n,p} - V^p_{m_k,p} + V^p_{m_k,p} - V^p_{m_{k-1},p}\\
&\quad \left.\left. + \cdots + V^p_{m_2,p} - V^p_{m_1,p} + V^p_{m_1,p})\right\}\right)\\
&= E\left(\exp\left\{i\left[\frac{u_1}{n^{1/\alpha}}S_{m_1} + \frac{s}{n^{p/\alpha}}V^p_{m_1,p}\right]\right.\right.\\
&\quad + i\left[\frac{u_2}{n^{1/\alpha}}(S_{m_2} - S_{m_1}) + \frac{s}{n^{p/\alpha}}(V^p_{m_2,p} - V^p_{m_1,p})\right]\\
&\quad \left.\left. + \cdots + \frac{is}{n^{p/\alpha}}(V^p_{n,p} - V^p_{m_k,p})\right\}\right)
\end{aligned}
$$

Due to independence and identical distribution of X's we have

$$
E\left[\exp\left\{i\frac{u_1}{n^{1/\alpha}}S_{m_1} + i\frac{s}{n^{p/\alpha}}V_{m_1,p}\right\}\right] = E^{m_1}\left[\exp\left\{iu_1\frac{X}{n^{1/\alpha}} + is\left(\frac{|X|}{n^{1/\alpha}}\right)^p\right\}\right]
$$

and

$$
\begin{aligned}
&E\left[\exp\left\{i\frac{u_k}{n^{1/\alpha}}(S_{m_k} - S_{m_{k-1}}) + i\frac{s}{n^{p/\alpha}}(V_{m_k,p} - V_{m_{k-1},p})\right\}\right]\\
&\quad = E^{m_k - m_{k-1}}\left[\exp\left\{iu_k\frac{X}{n^{p/\alpha}} + is\left(\frac{|X|}{n^{1/\alpha}}\right)^p\right\}\right].
\end{aligned}
$$

Now,

$$
\begin{aligned}
&E^{m_1}\left[\exp\left\{iu\frac{X}{m_1^{1/\alpha}}\left(\frac{m_1}{n}\right)^{1/\alpha} + iw\left(\frac{|X|}{m_1^{1/\alpha}}\right)^p\left(\frac{m_1}{n}\right)^{p/\alpha}\right\}\right]\\
&\quad = \left[\int\exp\left\{iu\frac{x}{m_1^{1/\alpha}}\left(\frac{m_1}{n}\right)^{1/\alpha} + iw\left(\frac{|x|}{m_1^{1/\alpha}}\right)^p\left(\frac{m_1}{n}\right)^{p/\alpha}\right\}g(x)\mathrm{d}x\right]^{m_1}\\
&\qquad (g \text{ is the density of } X)\\
&\quad = \left[1 + \int\left(\exp\left\{iu\frac{x}{m_1^{1/\alpha}}\left(\frac{m_1}{n}\right)^{1/\alpha} + iw\left(\frac{|x|}{m_1^{1/\alpha}}\right)^p\left(\frac{m_1}{n}\right)^{p/\alpha}\right\} - 1\right)g(x)\mathrm{d}x\right]^{m_1}\\
&\quad = \left[1 + \frac{1}{m_1}\int\left(\exp\left\{iuy\left(\frac{m_1}{n}\right)^{1/\alpha} + iw|y|^p\left(\frac{m_1}{n}\right)^{p/\alpha}\right\} - 1\right)g\left(m_1^{1/\alpha}y\right)\right.\\
&\qquad \left.\times\left(m_1^{1/\alpha}y\right)^{\alpha+1}\frac{\mathrm{d}y}{y^{\alpha+1}}\right]^{m_1} \quad (\text{writing } x/m_1^{1/\alpha} = y).
\end{aligned}
$$

Since $(\exp\{iuy(\frac{m_1}{n})^{1/\alpha} + iw|y|^p(\frac{m_1}{n})^{p/\alpha}\} - 1)$ is bounded by 2 and $m_1^{\frac{1}{\alpha}+1}g(m_1^{1/\alpha}y)$ is integrable we apply bounded convergence theorem to get

$$
\lim_{m_1,n} c_{m_1,n}(u,w) = \int\left[\exp\left\{iuy(t_1)^{1/\alpha} + iw|y|^p(t_1)^{p/\alpha})\right\} - 1\right]\frac{K(y)}{y^{\alpha+1}}\mathrm{d}y,
$$

where $K(y) = \lim_{m\to\infty}(m^{1/\alpha}y)^{\alpha+1}g(m^{1/\alpha}y)$ is

$$K(y) = \begin{cases} r, & \text{if } y > 0, \\ l, & \text{if } y < 0 \end{cases}$$

by the assumption on the tail of X. Therefore

$$\begin{aligned} &\lim_{m_1,n\to\infty, m_1/n\to t_1} E^{m_1}\left[\exp\left\{iu\frac{X}{m_1^{1/\alpha}}\left(\frac{m_1}{n}\right)^{1/\alpha} + iw\left(\frac{|X|}{m_1^{1/\alpha}}\right)^p\left(\frac{m_1}{n}\right)^{p/\alpha}\right\}\right] \\ &= \lim_{m_1,n\to\infty, m_1/n\to t_1}\left[1 + \frac{c_{m_1,n}}{m_1}(u,w)\right]^{m_1} \\ &= \exp\left\{\lim_{m_1,n\to\infty, m_1/n\to t_1} c_{m_1,n}(u,w)\right\}, \end{aligned}$$

where

$$\begin{aligned} c_{m_1,n}(u,w) \;=\; &\int\left(\exp\left\{iuy\left(\frac{m_1}{n}\right)^{1/\alpha} + iw|y|^p\left(\frac{m_1}{n}\right)^{p/\alpha}\right\} - 1\right) g \\ &\times\left(m^{1/\alpha}y\right) m^{1/\alpha+1}\mathrm{d}y. \end{aligned}$$

The same thing can be done for $E^{m_k - m_{k-1}}[\exp\{iu_k\frac{X}{n^{1/\alpha}} + is\frac{|X|^p}{n^{p/\alpha}}\}]$ and let us call it $c_{m_{k-1},m_k,n}(u_k, s)$. Therefore

$$\begin{aligned} &\lim_{m_1,m_2,\ldots,m_k,n\to\infty} \phi_{\mathbf{S}_1}(u_1,u_2,\ldots,u_k,s) \\ &= \exp\Big(\sum_{i=1}^{k}\int[\exp\{iu_i y(t_i)^{1/\alpha} + is|y|^p(t_i)^{p/\alpha})\} - 1]\frac{K(y)}{y^{\alpha+1}}\mathrm{d}y\Big) \\ &\times \lim_{m_k,n\to\infty} E\left(\mathrm{e}^{il|X|^p/n^{p/\alpha}}\right)^{n-m_k}. \end{aligned}$$

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