

Two remarks on normality preserving Borel automorphisms of \mathbb{R}^n

K R PARTHASARATHY

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute,
7, S. J. S. Sansanwal Marg, New Delhi 110 016, India
E-mail: krp@isid.ac.in

MS received 18 November 2011; revised 7 May 2012

Abstract. Let T be a bijective map on \mathbb{R}^n such that both T and T^{-1} are Borel measurable. For any $\theta \in \mathbb{R}^n$ and any real $n \times n$ positive definite matrix Σ , let $N(\theta, \Sigma)$ denote the n -variate normal (Gaussian) probability measure on \mathbb{R}^n with mean vector θ and covariance matrix Σ . Here we prove the following two results: (1) Suppose $N(\theta_j, I)T^{-1}$ is gaussian for $0 \leq j \leq n$, where I is the identity matrix and $\{\theta_j - \theta_0, 1 \leq j \leq n\}$ is a basis for \mathbb{R}^n . Then T is an affine linear transformation; (2) Let $\Sigma_j = I + \varepsilon_j \mathbf{u}_j \mathbf{u}'_j$, $1 \leq j \leq n$ where $\varepsilon_j > -1$ for every j and $\{\mathbf{u}_j, 1 \leq j \leq n\}$ is a basis of unit vectors in \mathbb{R}^n with \mathbf{u}'_j denoting the transpose of the column vector \mathbf{u}_j . Suppose $N(\mathbf{0}, I)T^{-1}$ and $N(\mathbf{0}, \Sigma_j)T^{-1}$, $1 \leq j \leq n$ are gaussian. Then $T(\mathbf{x}) = \sum_{\mathbf{s}} 1_{E_{\mathbf{s}}}(\mathbf{x}) V \mathbf{s} U \mathbf{x}$ a.e. \mathbf{x} , where \mathbf{s} runs over the set of 2^n diagonal matrices of order n with diagonal entries ± 1 , U, V are $n \times n$ orthogonal matrices and $\{E_{\mathbf{s}}\}$ is a collection of 2^n Borel subsets of \mathbb{R}^n such that $\{E_{\mathbf{s}}\}$ and $\{V \mathbf{s} U(E_{\mathbf{s}})\}$ are partitions of \mathbb{R}^n modulo Lebesgue-null sets and for every j , $V \mathbf{s} U \Sigma_j (V \mathbf{s} U)^{-1}$ is independent of all \mathbf{s} for which the Lebesgue measure of $E_{\mathbf{s}}$ is positive. The converse of this result also holds.

Our results constitute a sharpening of the results of Nabeya and Kariya (*J. Multivariate Anal.* **20** (1986) 251–264) and part of Khatri (*Sankhyā Ser. A* **49** (1987) 395–404).

Keywords. Borel automorphism; normality preserving Borel automorphism; Lebesgue partition.

1. Introduction and preliminaries

Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$ be measurable spaces and let \mathcal{P}_i be a set of probability measures on \mathcal{F}_i for each i . In the context of statistical theory, Basu and Khatri [1] posed the important problem of characterizing the set of all measurable maps $T : \Omega_1 \rightarrow \Omega_2$ satisfying the property that $\mu T^{-1} \in \mathcal{P}_2$ whenever $\mu \in \mathcal{P}_1$. They present many examples and solve the problem completely when $\Omega_1 = \mathbb{R}^n$, $\Omega_2 = \mathbb{R}^m$, \mathcal{F}_1 and \mathcal{F}_2 are Borel σ -algebras and \mathcal{P}_1 and \mathcal{P}_2 are sets of all gaussian laws. When $\Omega_1 = \Omega_2 = \mathbb{R}^m$,

$$\mathcal{P}_1 = \mathcal{P}_2 = \{\mu A^{-1}, A \text{ an affine linear transformation of } \mathbb{R}^m\},$$

where μ is a fixed probability measure, the problem looks particularly interesting. When $n = 1$ and μ is the symmetric Cauchy law this problem has been given a complete solution

by Letac [4]. In the gaussian case on the real line there are many variations due to Ghosh [2] and Mase [6].

Nabeya and Kariya [7] have studied the problem when T is a *Borel automorphism* of \mathbb{R}^n , i.e., a bijective map on \mathbb{R}^n such that T and T^{-1} are Borel measurable, \mathcal{P}_2 is the class of all nonsingular gaussian measures on \mathbb{R}^n and \mathcal{P}_1 is a restricted class of such gaussian measures. Here we follow their approach and obtain a sharpened version of their results after a more detailed analysis in two cases when \mathcal{P}_2 is the set of all nonsingular gaussian measures in \mathbb{R}^n but \mathcal{P}_1 is (1) a set of $n + 1$ gaussian measures with a fixed nonsingular covariance matrix and mean vectors θ_j , $0 \leq j \leq n$ where $\{\theta_j - \theta_0, 0 \leq j \leq n\}$ is a basis for \mathbb{R}^n ; (2) a set of $(n + 1)$ gaussian measures with a fixed mean vector θ and nonsingular covariance matrices Σ_j , $0 \leq j \leq n$ such that $\text{Rank}(\Sigma_j - \Sigma_0) = 1$ and there exists a basis $\{\mathbf{u}_j, 1 \leq j \leq n\}$ of unit vectors in \mathbb{R}^n satisfying the condition $\mathbf{u}_j \in \text{Range}(\Sigma_j - \Sigma_0)$ for each $j = 1, 2, \dots, n$. The main result of Nabeya and Kariya in the gaussian case follows as a corollary.

Theorem 2 of Khatri's paper [3] has a more general version of our first characterization result of Corollary 3.2 when there are two covariance matrices Σ_1, Σ_2 instead of a single Σ . For the case of several covariance matrices his conjecture is still open. However, in the case of a fixed mean vector our second characterization result in Theorem 4.4 seems to be new.

We conclude this section with some notations and definitions that will be used in the following sections. Let $GL(n)$, $\mathcal{S}_+(n)$ and D_n denote respectively the set of all real $n \times n$ nonsingular matrices, positive definite matrices and diagonal matrices with entries ± 1 . For any matrix A , denote by A' its transpose. We express any element of \mathbb{R}^n as a $n \times 1$ matrix and for any $\theta \in \mathbb{R}^n$, $\Sigma \in \mathcal{S}_+(n)$, denote by $N(\theta, \Sigma)$ the gaussian probability measure in \mathbb{R}^n with density function $(2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\{-1/2(\mathbf{x} - \theta)' \Sigma^{-1} (\mathbf{x} - \theta)\}$, $|\Sigma|$ being the determinant of Σ . A Borel map T on \mathbb{R}^n is called an *affine automorphism* if $T(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$ a.e. \mathbf{x} with respect to the Lebesgue measure where $A \in GL(n)$ and $\mathbf{a} \in \mathbb{R}^n$. All almost everywhere statements in \mathbb{R}^n will be with respect to Lebesgue measure. By a *Lebesgue partition* over D_n we mean a collection $\{E_s, \mathbf{s} \in D_n\}$ of Borel sets with the property that all the sets $\mathbb{R}^n \setminus \cup_{\mathbf{s} \in D_n} E_s$, $E_s \cap E_t$, $\mathbf{s} \neq \mathbf{t}$ have zero Lebesgue measure.

If T is an affine automorphism of \mathbb{R}^n such that $T(\mathbf{x}) = A\mathbf{x} + \mathbf{a}$ a.e. \mathbf{x} for some $A \in GL(n)$, $\mathbf{a} \in \mathbb{R}^n$ and $N(\theta, \Sigma)$ is a gaussian probability measure with mean vector θ and covariance matrix Σ then $N(\theta, \Sigma)T^{-1} = N(A\theta + \mathbf{a}, A\Sigma A')$. We say that two Borel automorphisms T_1 and T_2 are *affine equivalent* if $T_2 = RT_1S$ a.e. for some affine automorphisms R and S . If T is a Borel automorphism of \mathbb{R}^n and $N(\theta, \Sigma)T^{-1} = N(\varphi, \Psi)$ for some $\theta, \varphi \in \mathbb{R}^n$, $\Sigma, \Psi \in GL(n)$ then there exists a Borel automorphism \tilde{T} affine equivalent to T such that $N(\mathbf{0}, I)\tilde{T}^{-1} = N(\mathbf{0}, I)$, i.e., \tilde{T} preserves the standard gaussian probability measure with mean vector $\mathbf{0}$ and covariance matrix I . Thus problems of the type we have described in the context of gaussian measures can be translated to the case when the Borel automorphism preserves the standard gaussian measure in \mathbb{R}^n .

2. Borel automorphisms preserving the normality of a pair of normal distributions

Let T be a Borel automorphism of \mathbb{R}^n such that

$$N(\theta_i, \Sigma_i)T^{-1} = N(\varphi_i, \Psi_i), \quad i = 1, 2 \quad (2.1)$$

where $\Sigma_i, \Psi_i \in \mathcal{S}_+(n)$, $\theta_i, \varphi_i \in \mathbb{R}^n$ for each i . Our aim is to establish the following proposition which, together with its proof, is a slight variation of Lemmas 2.1 and 2.2 together in the paper of Nabeya and Kariya [7].

PROPOSITION 2.1

Under condition (2.1) the following hold:

- (1) $\Psi_1^{1/2} \Psi_2^{-1} \Psi_1^{1/2}$ and $\Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2}$ have the same characteristic polynomial.
- (2) $(\varphi_1 - \varphi_2)' \Psi_2^{-1} (\Psi_1^{-1} - z \Psi_2^{-1})^{-1} \Psi_2^{-1} (\varphi_1 - \varphi_2) = (\theta_1 - \theta_2)' \Sigma_2^{-1} (\Sigma_1^{-1} - z \Sigma_2^{-1})^{-1} \Sigma_2^{-1} (\theta_1 - \theta_2)$ as analytic functions of z .
- (3) $(\varphi_1 - \varphi_2)' \Psi_2^{-1} (\varphi_1 - \varphi_2) = (\theta_1 - \theta_2)' \Sigma_2^{-1} (\theta_1 - \theta_2)$.
- (4) $(T(\mathbf{x}) - \varphi_2)' \Psi_2^{-1} (T(\mathbf{x}) - \varphi_2) - (T(\mathbf{x}) - \varphi_1)' \Psi_1^{-1} (T(\mathbf{x}) - \varphi_1) = (\mathbf{x} - \theta_2)' \Sigma_2^{-1} (\mathbf{x} - \theta_2) - (\mathbf{x} - \theta_1)' \Sigma_1^{-1} (\mathbf{x} - \theta_1)$, a.e. \mathbf{x} .

Proof. Let f_i, \tilde{f}_i denote respectively the density functions of $N(\theta_i, \Sigma_i)$, $N(\varphi_i, \Psi_i)$, $i = 1, 2$. Equation (2.1) implies, in particular, that the Lebesgue measure L in \mathbb{R}^n is equivalent to the measure LT^{-1} and for any real t ,

$$\left[\frac{\tilde{f}_1}{\tilde{f}_2}(T(\mathbf{x})) \right]^{2t} = \left[\frac{f_1}{f_2}(\mathbf{x}) \right]^{2t} \quad \text{a.e. } \mathbf{x}. \quad (2.2)$$

Integrating both sides with respect to the probability measure $N(\theta_1, \Sigma_1)$ and using (2.1) for $i = 1$ we get for all t in a neighbourhood of 0,

$$\begin{aligned} & |\Psi_2|^t |\Psi_1|^{-t(t+1/2)} \left| \left(t + \frac{1}{2} \right) \Psi_1^{-1} - t \Psi_2^{-1} \right|^{-\frac{1}{2}} \\ & \quad \times \exp t (\varphi_1 - \varphi_2)' \Psi_2^{-1} (\varphi_1 - \varphi_2) \\ & \quad + t^2 \left\{ (\varphi_1 - \varphi_2)' \Psi_2^{-1} \left(\left(t + \frac{1}{2} \right) \Psi_1^{-1} - t \Psi_2^{-1} \right)^{-1} \Psi_2^{-1} (\varphi_1 - \varphi_2) \right\} \\ & = |\Sigma_2|^t |\Sigma_1|^{-t(t+\frac{1}{2})} \left| \left(t + \frac{1}{2} \right) \Sigma_1^{-1} - t \Sigma_2^{-1} \right|^{-\frac{1}{2}} \\ & \quad \times \exp t (\theta_1 - \theta_2)' \Sigma_2^{-1} (\theta_1 - \theta_2) \\ & \quad + t^2 \left\{ (\theta_1 - \theta_2)' \Sigma_2^{-1} \left(\left(t + \frac{1}{2} \right) \Sigma_1^{-1} - t \Sigma_2^{-1} \right)^{-1} \Sigma_2^{-1} (\theta_1 - \theta_2) \right\}. \end{aligned}$$

Squaring both sides and rearranging the terms this can be expressed as

$$\frac{|\Psi_1|}{|\Sigma_1|} \frac{\left| \left(t + \frac{1}{2} \right) \Psi_1^{-1} - t \Psi_2^{-1} \right|}{\left| \left(t + \frac{1}{2} \right) \Sigma_1^{-1} - t \Sigma_2^{-1} \right|} = e^{\chi(t)},$$

where the left-hand side and the function $\chi(t)$ in the exponent of the right-hand side are rational functions of t in a neighbourhood of 0 in \mathbb{R} . By exactly the same arguments and using the theory of analytic functions as in the proof of Lemma 2.2 in [7] we now conclude that

$$\begin{aligned} |\Sigma_1| \left| \left(t + \frac{1}{2} \right) \Sigma_1^{-1} - t \Sigma_2^{-1} \right| &= |\Psi_1| \left| \left(t + \frac{1}{2} \right) \Psi_1^{-1} - t \Psi_2^{-1} \right|, \\ \chi(t) &= 0 \end{aligned} \quad (2.3)$$

for all t in a neighbourhood of 0. The first part of equation (2.3) implies property (1) of the proposition and, in particular,

$$|\Sigma_1| |\Sigma_2|^{-1} = |\Psi_1| |\Psi_2|^{-1}. \quad (2.4)$$

Now the second part of equation (2.3) can be written as

$$\begin{aligned} & t\{(\theta_1 - \theta_2)' \Sigma_2^{-1} (\theta_1 - \theta_2) - (\varphi_1 - \varphi_2)' \Psi_2^{-1} (\varphi_1 - \varphi_2)\} \\ & + t^2 \left\{ (\theta_1 - \theta_2)' \Sigma_2^{-1} \left(\left(t + \frac{1}{2} \right) \Sigma_1^{-1} - t \Sigma_2^{-1} \right)^{-1} \Sigma_2^{-1} (\theta_1 - \theta_2) \right. \\ & \left. - (\varphi_1 - \varphi_2)' \Psi_2^{-1} \left(\left(t + \frac{1}{2} \right) \Psi_1^{-1} - t \Psi_2^{-1} \right)^{-1} \Psi_2^{-1} (\varphi_1 - \varphi_2) \right\} = 0 \end{aligned}$$

in a neighbourhood of 0. Dividing by t and letting $t \rightarrow 0$ we get property (3). After deleting the first term, dividing by t^2 and replacing the parameter $(t + \frac{1}{2})^{-1}t$ by z we get property (2). Going back to equation (2.2), putting $t = 1$, using (2.4) and taking logarithms we get property (4). This completes the proof. \square

COROLLARY 2.2

Under condition (2.1) the following hold:

- (i) *If $\Sigma_1 = \Sigma_2$, then $\Psi_1 = \Psi_2$;*
- (ii) *If $\theta_1 = \theta_2$, then $\varphi_1 = \varphi_2$.*

Proof. Property (i) follows from property (1) in Proposition 2.1 whereas property (ii) is a consequence of property (3) of the same proposition. \square

3. The first characterization theorem

Here we consider a Borel automorphism T of \mathbb{R}^n satisfying the property that $N(\theta, \Sigma)T^{-1}$ is a gaussian probability measure for a fixed Σ in $\mathcal{S}_+(n)$ and the mean vector θ varying in a natural set of $(n + 1)$ points in \mathbb{R}^n . We shall make use of the remark at the end of §1.

Theorem 3.1 *Let T be a Borel automorphism of \mathbb{R}^n and let $\{\theta_j, 0 \leq j \leq n\} \subset \mathbb{R}^n$ be a set of $(n + 1)$ points with the property that $\theta_j - \theta_0, j = 1, 2, \dots, n$ is a basis for \mathbb{R}^n . Suppose*

$$N(\theta_j, I)T^{-1} = N(\varphi_j, \Psi_j), \quad 0 \leq j \leq n.$$

Then T is an affine automorphism.

Proof. By part (i) of Corollary 2.2 there exists $\Psi \in \mathcal{S}_+(n)$ such that $\Psi_j = \Psi$ for every j . From property (3) of Proposition 2.1 we have

$$(\varphi_i - \varphi_j)' \Psi^{-1} (\varphi_i - \varphi_j) = (\theta_i - \theta_j)' (\theta_i - \theta_j) \quad (3.1)$$

for all $i, j \in \{0, 1, 2, \dots, n\}$. Hence the correspondence

$$(\theta_i - \theta_0) \rightarrow \Psi^{-1/2}(\varphi_i - \varphi_0), \quad 1 \leq i \leq n$$

is distance preserving in \mathbb{R}^n with the Euclidean metric. Since $\{\theta_i - \theta_0, 1 \leq i \leq n\}$ is a basis for \mathbb{R}^n it follows that there exists an orthogonal matrix U such that

$$U(\theta_i - \theta_0) = \Psi^{-1/2}(\varphi_i - \varphi_0), \quad 1 \leq i \leq n. \quad (3.2)$$

Using property (4) of Proposition 2.1 we have

$$\begin{aligned} & (T(\mathbf{x}) - \varphi_i)' \Psi^{-1} (T(\mathbf{x}) - \varphi_i) - (T(\mathbf{x}) - \varphi_j)' \Psi^{-1} (T(\mathbf{x}) - \varphi_j) \\ &= (\mathbf{x} - \theta_i)' (\mathbf{x} - \theta_i) - (\mathbf{x} - \theta_j)' (\mathbf{x} - \theta_j), \quad i, j \in \{0, 1, 2, \dots, n\} \\ & \text{a.e. } \mathbf{x}. \end{aligned}$$

Expressing $T(\mathbf{x}) - \varphi_i = T(\mathbf{x}) - \varphi_0 + \varphi_0 - \varphi_i$, $\mathbf{x} - \theta_i = \mathbf{x} - \theta_0 + \theta_0 - \theta_i$ for each i and expanding both sides of the equation above we obtain by using (3.1) the equation

$$(\varphi_i - \varphi_j)' \Psi^{-1} (T(\mathbf{x}) - \varphi_0) = (\theta_i - \theta_j)' (\mathbf{x} - \theta_0) \text{ a.e. } \mathbf{x}.$$

Writing

$$A = \begin{bmatrix} (\theta_1 - \theta_0)' \\ \vdots \\ (\theta_n - \theta_0)' \end{bmatrix}, \quad B = \begin{bmatrix} (\varphi_1 - \varphi_0)' \Psi^{-1} \\ \vdots \\ (\varphi_n - \varphi_0)' \Psi^{-1} \end{bmatrix}$$

and using (3.2) we conclude that A and B are nonsingular matrices and

$$T(\mathbf{x}) = \varphi_0 + B^{-1}A(\mathbf{x} - \theta_0) \text{ a.e.}$$

In other words, T is an affine automorphism. \square

COROLLARY 3.2

Let T be a Borel automorphism of \mathbb{R}^n and let $\{\theta_j, 0 \leq j \leq n\} \subset \mathbb{R}^n$ be a set of $n + 1$ points such that $\{\theta_j - \theta_0, 1 \leq j \leq n\} \subset \mathbb{R}^n$ is a basis for \mathbb{R}^n . Suppose there exists an $n \times n$ nonsingular covariance matrix Σ such that

$$N(\theta_j, \Sigma)T^{-1} = N(\varphi_j, \Psi_j), \quad 0 \leq j \leq n.$$

Then T is an affine automorphism.

Proof. Define $T'(\mathbf{x}) = T(\Sigma^{1/2}\mathbf{x})$. Then T' is a Borel automorphism for which

$$N(\Sigma^{-1/2}\theta_j, I)T'^{-1} = N(\varphi_j, \Psi_j), \quad 0 \leq j \leq n.$$

Since $\{\Sigma^{-1/2}(\theta_j - \theta_0), 1 \leq j \leq n\}$ is a basis for \mathbb{R}^n it follows that T' and therefore T is an affine automorphism. \square

Remark 3.3. Corollary 3.2 is false if T is just a Borel measurable map. Using a result of Linnik (Theorem 4.3.1 of [5]), Mase [6] has shown that given any finite set of normal

distributions $\{N_j, 1 \leq j \leq k\}$ on \mathbb{R} there exists a Borel map $T : \mathbb{R} \rightarrow \mathbb{R}$ such that $N_j T^{-1}$ is the probability measure of the standard normal distribution on \mathbb{R} .

4. The second characterization theorem

Here we consider the case of a Borel automorphism T of \mathbb{R}^n such that $N(\boldsymbol{\theta}, \Sigma)T^{-1}$ is a gaussian probability measure for a fixed $\boldsymbol{\theta}$ in \mathbb{R}^n but Σ varying in a set of $n+1$ covariance matrices with n of them being rank one perturbations of the remaining one. Without loss of generality one may assume the fixed mean vector $\boldsymbol{\theta}$ to be $\mathbf{0}$. We begin with a lemma.

Lemma 4.1. Let T be a Borel automorphism of \mathbb{R}^n such that

$$\begin{aligned} N(\mathbf{0}, I)T^{-1} &= N(\mathbf{0}, I), \\ N(\mathbf{0}, \Sigma)T^{-1} &= N(\boldsymbol{\eta}, \Psi), \end{aligned}$$

where $\Sigma = I + \varepsilon \mathbf{u}\mathbf{u}'$, \mathbf{u} is a unit vector in \mathbb{R}^n and ε is a real nonzero scalar such that $\varepsilon > -1$. Then $\boldsymbol{\eta} = \mathbf{0}$ and $\Psi = I + \varepsilon \mathbf{v}\mathbf{v}'$ for some unit vector \mathbf{v} . Furthermore,

$$(\mathbf{v}'T(\mathbf{x}))^2 = (\mathbf{u}'\mathbf{x})^2 \quad \text{a.e. } \mathbf{x}. \quad (4.1)$$

Proof. By property (ii) of Corollary 2.2, $\boldsymbol{\eta} = \mathbf{0}$. By property (1) of Proposition 2.1, Σ and Ψ have the same characteristic polynomial and therefore $\Psi = I + \varepsilon \mathbf{v}\mathbf{v}'$ for some unit vector \mathbf{v} . By property (4) of Proposition 2.1 we have

$$\begin{aligned} T(\mathbf{x})'(I + \varepsilon \mathbf{v}\mathbf{v}')^{-1}T(\mathbf{x}) - T(\mathbf{x})'T(\mathbf{x}) \\ = \mathbf{x}'(I + \varepsilon \mathbf{u}\mathbf{u}')^{-1}\mathbf{x} - \mathbf{x}'\mathbf{x} \quad \text{a.e. } \mathbf{x}. \end{aligned} \quad (4.2)$$

Since, for any unit vector \mathbf{w} , $(I + \varepsilon \mathbf{w}\mathbf{w}')^{-1} = I - \frac{\varepsilon}{1+\varepsilon} \mathbf{w}\mathbf{w}'$, (4.1) follows from (4.2). \square

Lemma 4.2. Let T be a Borel automorphism of \mathbb{R}^n such that

- (i) $N(\mathbf{0}, I)T^{-1} = N(\mathbf{0}, I)$,
- (ii) $N(\mathbf{0}, \Sigma_j)T^{-1} = N(\boldsymbol{\varphi}_j, \Psi_j)$, $1 \leq j \leq n$,

where $\Sigma_j = I + \varepsilon_j \mathbf{u}_j \mathbf{u}_j'$, $1 \leq j \leq n$, $0 \neq \varepsilon_j > -1 \forall j$ and $\{\mathbf{u}_j, 1 \leq j \leq n\}$ is a basis of unit vectors in \mathbb{R}^n . Then there exist A, B in $GL(n)$ and a Lebesgue partition $\{E_s, \mathbf{s} \in D_n\}$ of \mathbb{R}^n over D_n such that

- (i) $\{B_s A E_s, \mathbf{s} \in D_n\}$ is a Lebesgue partition over D_n ,
- (ii) $T(\mathbf{x}) = \sum_{\mathbf{s} \in D_n} 1_{E_s}(\mathbf{x}) B \mathbf{s} A \mathbf{x}$ a.e. \mathbf{x} ,
- (iii) $T^{-1}(\mathbf{x}) = \sum_{\mathbf{s} \in D_n} 1_{B_s A(E_s)}(\mathbf{x}) A^{-1} \mathbf{s} B^{-1} \mathbf{x}$ a.e. \mathbf{x} .

Proof. From Lemma 4.1 we have $\boldsymbol{\varphi}_j = \mathbf{0} \forall j$ and

$$\Psi_j = I + \varepsilon_j \mathbf{v}_j \mathbf{v}_j', \quad 1 \leq j \leq n$$

where each \mathbf{v}_j is a unit vector. Furthermore,

$$(\mathbf{v}_j' T(\mathbf{x}))^2 = (\mathbf{u}_j' \mathbf{x})^2 \quad \text{a.e. } \mathbf{x}, \quad 1 \leq j \leq n.$$

Denoting $\mathbf{s} = \text{diag}(s_1, s_2, \dots, s_n)$ with $s_j = \pm 1$ and ‘diag’ indicating diagonal matrix define the Borel sets

$$E_{\mathbf{s}} = \{\mathbf{x} | \mathbf{v}'_j T(\mathbf{x}) = s_j \mathbf{u}'_j \mathbf{x} \quad \forall j = 1, 2, \dots, n\}, \quad \mathbf{s} \in D_n.$$

Then it follows that for the Lebesgue measure L ,

$$L\left(\mathbb{R}^n \setminus \bigcup_{\mathbf{s} \in D_n} E_{\mathbf{s}}\right) = 0.$$

Write

$$A = \begin{bmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \\ \vdots \\ \mathbf{u}'_n \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{v}'_1 \\ \mathbf{v}'_2 \\ \vdots \\ \mathbf{v}'_n \end{bmatrix}.$$

Then

$$CT(\mathbf{x}) = \mathbf{s}A\mathbf{x} \quad \forall \mathbf{x} \in E_{\mathbf{s}}.$$

Choose $\mathbf{s} \in D_n$ such that $L(E_{\mathbf{s}}) > 0$. Then we know that $\{\mathbf{s}A\mathbf{x}, \mathbf{x} \in E_{\mathbf{s}}\}$ spans \mathbb{R}^n and therefore $\{CT(\mathbf{x}), \mathbf{x} \in E_{\mathbf{s}}\}$ spans \mathbb{R}^n . Thus $C \in GL(n)$. Writing $B = C^{-1}$ we have

$$T(\mathbf{x}) = B\mathbf{s}A\mathbf{x} \quad \forall \mathbf{x} \in E_{\mathbf{s}}.$$

Now let $\mathbf{s}, \mathbf{t} \in D_n$, $\mathbf{s} \neq \mathbf{t}$. If $\mathbf{x} \in E_{\mathbf{s}} \cap E_{\mathbf{t}}$ we have $\mathbf{s}A\mathbf{x} = \mathbf{t}A\mathbf{x}$. In other words, $A\mathbf{x}$ is an eigenvector of $\mathbf{s}\mathbf{t}$ and $\mathbf{s}\mathbf{t} \neq I$. Hence $L(E_{\mathbf{s}} \cap E_{\mathbf{t}}) = 0$. Thus $\{E_{\mathbf{s}}, \mathbf{s} \in D_n\}$ is a Lebesgue partition over D_n and

$$T(\mathbf{x}) = \sum_{\mathbf{s}} 1_{E_{\mathbf{s}}}(\mathbf{x}) B\mathbf{s}A\mathbf{x} \quad \text{a.e. } \mathbf{x}.$$

Putting $T(\mathbf{x}) = \mathbf{y}$ and solving for \mathbf{x} from this equation we get

$$T^{-1}(\mathbf{y}) = \sum_{\mathbf{s}} 1_{B\mathbf{s}A(E_{\mathbf{s}})}(\mathbf{y}) A^{-1} \mathbf{s} B^{-1} \mathbf{y} \quad \text{a.e. } \mathbf{y},$$

where $\{B\mathbf{s}A(E_{\mathbf{s}}), \mathbf{s} \in D_n\}$ is also a Lebesgue partition. This proves all the required properties (i)–(iii) of the lemma. \square

Lemma 4.3. In Lemma 4.2, the matrices A and B can be chosen to be orthogonal.

Proof. Following the notations of Lemma 4.2 and its proof denote $F_{\mathbf{s}} = B\mathbf{s}A(E_{\mathbf{s}})$ and by (Σ, Ψ) any of the pairs (I, I) , (Σ_j, Ψ_j) , $1 \leq j \leq n$. Then we have $|\Sigma| = |\Psi|$ and $N(\mathbf{0}, \Sigma)T^{-1} = N(\mathbf{0}, \Psi)$. Thus, for any $\lambda \in \mathbb{R}^n$ we have

$$\begin{aligned} & \int e^{\lambda' \mathbf{x} - \frac{1}{2} \mathbf{x}' \Psi' \mathbf{x}} d\mathbf{x} \\ &= \int e^{\lambda' T(\mathbf{x}) - \frac{1}{2} \mathbf{x}' \Sigma^{-1} \mathbf{x}} d\mathbf{x} \\ &= \sum_{\mathbf{s}} \int_{E_{\mathbf{s}}} e^{\lambda' B\mathbf{s}A\mathbf{x} - \frac{1}{2} \mathbf{x}' \Sigma^{-1} \mathbf{x}} \\ &= ||AB|^{-1}| \sum_{\mathbf{s}} \int_{F_{\mathbf{s}}} e^{\lambda' \mathbf{x} - \frac{1}{2} \mathbf{x}' (A^{-1} \mathbf{s} B^{-1})' \Sigma^{-1} A^{-1} \mathbf{s} B^{-1} \mathbf{x}} d\mathbf{x}. \end{aligned}$$

By the uniqueness of Laplace transforms we conclude that

$$e^{-\frac{1}{2}\mathbf{x}'\Psi^{-1}\mathbf{x}} = ||AB|^{-1}|_{\Sigma_{\mathbf{s} \in D_n}} 1_{F_{\mathbf{s}}}(\mathbf{x}) e^{-\frac{1}{2}\mathbf{x}'(A^{-1}\mathbf{s}B^{-1})'\Sigma^{-1}A^{-1}\mathbf{s}B^{-1}\mathbf{x}} \quad \text{a.e. } \mathbf{x}.$$

Suppose $L(E_{\mathbf{s}}) > 0$. Then $L(F_{\mathbf{s}}) > 0$. Since $\{F_t\}$ is a Lebesgue partition we have

$$\log ||AB|| = \frac{1}{2}\mathbf{x}'(\Psi^{-1} - (A^{-1}\mathbf{s}B^{-1})'\Sigma^{-1}A^{-1}\mathbf{s}B^{-1})\mathbf{x} \quad \text{a.e. } \mathbf{x} \in F_{\mathbf{s}}.$$

In other words, the quadratic form on the right-hand side is a constant on a set of positive Lebesgue measure. A simple argument based on Fubini's theorem implies that $||AB|| = 1$ and

$$B\mathbf{s}A\Sigma(B\mathbf{s}A)' = \Psi, \quad \text{whenever } L(E_{\mathbf{s}}) > 0.$$

Choosing $(\Sigma, \Psi) = (I, I)$ we conclude that $B\mathbf{s}A$ is orthogonal. Let $B = VK$, $A = HU$ where U, V are orthogonal and K, H are positive definite. Then $K\mathbf{s}H^2\mathbf{s}K = I$ and therefore $K = \mathbf{s}H^{-1}\mathbf{s}$ and $B\mathbf{s}A = V\mathbf{s}U$. This shows that

$$V\mathbf{s}U\Sigma(V\mathbf{s}U)' = \Psi, \quad \text{whenever } L(E_{\mathbf{s}}) > 0$$

and Lemma 4.2 holds with A, B replaced by U, V respectively. \square

Theorem 4.4 *Let T be a Borel automorphism of \mathbb{R}^n and let $\Sigma_j = I + \varepsilon_j \mathbf{u}_j \mathbf{u}_j'$, $1 \leq j \leq n$ where $\varepsilon_j > -1$ for every j and $\{\mathbf{u}_j, 1 \leq j \leq n\}$ is a basis of unit vectors in \mathbb{R}^n . Then $N(\mathbf{0}, I)T^{-1} = N(\mathbf{0}, I)$ and $N(\mathbf{0}, \Sigma_j)T^{-1}$ is a gaussian probability measure for every j if and only if there exist $n \times n$ orthogonal matrices U, V and a Lebesgue partition $\{E_{\mathbf{s}}, \mathbf{s} \in D_n\}$ of \mathbb{R}^n such that the following hold:*

- (i) $T(\mathbf{x}) = \Sigma_{\mathbf{s} \in D_n} 1_{E_{\mathbf{s}}}(\mathbf{x}) V\mathbf{s}U\mathbf{x}$ a.e. \mathbf{x} ;
- (ii) $\{V\mathbf{s}U(E_{\mathbf{s}}), \mathbf{s} \in D_n\}$ is also a Lebesgue partition. In such a case there exists a basis $\{\mathbf{v}_j, 1 \leq j \leq n\}$ of unit vectors in \mathbb{R}^n such that
- (iii) $V\mathbf{s}U\mathbf{u}_j = \mathbf{v}_j$ if $L(E_{\mathbf{s}}) > 0, \forall 1 \leq j \leq n$;
- (iv) $N(\mathbf{0}, \Sigma_j)T^{-1} = N(\mathbf{0}, \Psi_j), \forall j$ where $\Psi_j = I + \varepsilon_j \mathbf{v}_j \mathbf{v}_j', 1 \leq j \leq n$.

Proof. The 'only if' part is contained in Lemmas 4.2 and 4.3 and their proofs. Conversely, suppose there exist orthogonal matrices U, V and a Lebesgue partition $\{E_{\mathbf{s}}, \mathbf{s} \in D_n\}$ such that (i), (ii) and (iii) hold. Then for every pair $(\Sigma, \Psi) = (I, I), (\Sigma_j, \Psi_j), 1 \leq j \leq n$ we have

$$\begin{aligned} \int e^{\lambda'T(\mathbf{x}) - \frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}} d\mathbf{x} &= \Sigma_{\mathbf{s}} \int_{E_{\mathbf{s}}} e^{\lambda'V\mathbf{s}U\mathbf{x} - \frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}} d\mathbf{x} \\ &= \Sigma_{\mathbf{s}} \int_{V\mathbf{s}U(E_{\mathbf{s}})} e^{\lambda'\mathbf{x} - \frac{1}{2}\mathbf{x}'\Psi^{-1}\mathbf{x}} d\mathbf{x} \\ &= \int e^{\lambda'\mathbf{x} - \frac{1}{2}\mathbf{x}'\Psi^{-1}\mathbf{x}} d\mathbf{x}. \end{aligned}$$

Since $|\Sigma| = |\Psi|$ it follows that $N(\mathbf{0}, \Sigma)T^{-1} = N(\mathbf{0}, \Psi)$. \square

COROLLARY 4.5

Let T be a Borel automorphism of \mathbb{R}^n and let Σ_j , $0 \leq j \leq n$ be elements in $\mathcal{S}_+(n)$ satisfying the following conditions:

- (1) $\text{rank}(\Sigma_j - \Sigma_0) = 1$,
- (2) $\text{Range}(\Sigma_j - \Sigma_0) = \mathbb{R}\mathbf{u}_j$ with \mathbf{u}_j as a unit vector, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is a basis for \mathbb{R}^n .

Then $N(\mathbf{0}, \Sigma_j)T^{-1} = N(\mathbf{0}, \Psi_j)$ for every j , for some covariance matrices $\Psi_0, \Psi_1, \dots, \Psi_n$ if and only if there exist orthogonal matrices U, V of order n and a Lebesgue partition $\{E_s, \mathbf{s} \in D_n\}$ such that $\{VsU(E_s), \mathbf{s} \in D_n\}$ is also a Lebesgue partition and

$$T(\mathbf{x}) = \sum_{\mathbf{s} \in D_n} 1_{E_s}(\Sigma_0^{-1/2}\mathbf{x})\Psi_0^{1/2}VsU\Sigma_0^{-1/2}\mathbf{x} \quad \text{a.e. } \mathbf{x}.$$

Proof. This is immediate from Theorem 4.4 and the fact that the Borel automorphism T' defined by

$$T'(\mathbf{x}) = \Psi_0^{-1/2}T(\Sigma_0^{-1/2}\mathbf{x})$$

preserves $N(\mathbf{0}, I)$ and transforms $N(\mathbf{0}, \Sigma_0^{-1/2}\Sigma_j\Sigma_0^{-1/2})$ to a mean zero gaussian probability measure for each $j = 1, 2, \dots, n$. \square

Remark 4.6. Let $\mathbb{R} = F_{j+} \cup F_{j-}$ be a partition of \mathbb{R} into two disjoint symmetric Borel subsets, i.e. $F_{j+} = -F_{j+}$ and $F_{j-} = -F_{j-}$ for each $j = 1, 2, \dots, n$ and for $\mathbf{s} = \text{diag}(s_1, s_2, \dots, s_n)$ in D_n let

$$F_s = F_{s_1} \times F_{s_2} \times \dots \times F_{s_n}$$

where

$$F_{s_j} = \begin{cases} F_{j+}, & \text{if } s_j = +1, \\ F_{j-}, & \text{if } s_j = -1. \end{cases}$$

If U, V are orthogonal matrices of order n , putting $E_s = U^{-1}F_s$ for every \mathbf{s} we note that $\{E_s, \mathbf{s} \in D_n\}$ and $\{VsUE_s, \mathbf{s} \in D_n\}$ are Lebesgue partitions of \mathbb{R}^n . Indeed, $VsUE_s = VF_s$ for every \mathbf{s} .

Remark 4.7. If $\{E_s, \mathbf{s} \in D_n\}$ happens to be a trivial Lebesgue partition in the sense that for some \mathbf{s}_0 , $L(\mathbb{R}^n \setminus E_{\mathbf{s}_0}) = 0$ then the Borel automorphism T in Theorem 4.4 is an orthogonal transformation a.e. whereas in Corollary 4.5 it is a nonsingular linear transformation a.e. in \mathbb{R}^n .

Remark 4.8. In Theorem 4.4, let

$$D = \{\mathbf{s} \mid L(E_s) > 0\} \subset D_n$$

and let T satisfy (i)–(iv). Suppose

$$\mathcal{C} = \{A \mid A \in \mathcal{S}_+(n), \mathbf{s}UAU^{-1}\mathbf{s} \text{ is independent of } \mathbf{s} \text{ varying in } D\}.$$

Then

$$N(\mathbf{0}, A)T^{-1} = N(\mathbf{0}, VsUAU^{-1}sV^{-1}) \quad \forall A \in \mathcal{C}, \mathbf{s} \in D.$$

Acknowledgements

The author is grateful to B G Manjunath for bringing his attention to all the references in this paper and particularly to the results of Nabeya and Kariya [7]. He wishes to thank Manjunath and the referee for citing the importance of Khatri's paper [3] in the course of revision of this paper.

References

- [1] Basu D and Khatri C G, On some characterizations of statistics, *Sankhyā Ser. A* **31** (1969) 199–208
- [2] Ghosh J K, Only linear transformations preserve normality, *Sankhyā Ser. A* **31** (1969) 309–312
- [3] Khatri C G, Characterization of bijective and bimeasurable transformations of normal variates, *Sankhyā Ser. A* **49** (1987) 395–404
- [4] Letac G, Which functions preserve Cauchy laws? *Proc. Amer. Math. Soc.* **67** (1977) 277–286
- [5] Linnik Yu V, Statistical Problems with Nuisance Parameters (1968) (American Mathematical Society)
- [6] Mase S, Some theorems on normality preserving transformations, *Sankhyā Ser. A* **39** (1977) 186–190
- [7] Nabeya S and Kariya T, Transformations preserving normality and Wishartness, *J. Multivariate Anal.* **20** (1986) 251–264