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Multiplier convergent series and uniform convergence of mapping series

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Abstract. In this paper, we introduce the frame property of complex sequence sets and study the uniform convergence of nonlinear mapping series in β-dual of spaces consisting of multiplier convergent series.

Keywords. Multiplier convergent series; mapping series.

1. Introduction

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\lambda \subseteq \mathbb{C}^{\mathbb{N}}$ and $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$. A series $\sum_{j=1}^{\infty} x_j$ in X is said to be λ-multiplier convergent if the series $\sum_{j=1}^{\infty} t_j x_j$ converges for each $(t_j) \in \lambda$. For example, $\{0, 1\}^{\mathbb{N}}$-multiplier convergent is just the subseries convergent: $\sum_{k=1}^{\infty} x_{j_k}$ converges for each $j_1 < j_2 < \cdots$ and l^{∞}-multiplier convergent is just the bounded multiplier convergent: $\sum_{j=1}^{\infty} t_j x_j$ converges for each bounded complex sequences (t_j), where $l^{\infty} = \{(t_j) \in \mathbb{C}^{\mathbb{N}} : \sup_{j\in\mathbb{N}} |t_j| < +\infty\}$.

There are many results about multiplier convergent series, see, for example [1, 4, 6–8]. Now, we only list a famous one which is known as Orlicz–Pettis theorem [7]: a series $\sum_{j=1}^{\infty} x_j$ which is subseries convergent in the weak topology is actually subseries convergent in the norm topology.

We denote the vector-valued sequence set consisting of λ-multiplier convergent series by

$$MC_{\lambda}(X) = \left\{ (x_j) \in X^{\mathbb{N}} : \sum_{j=1}^{\infty} t_j x_j \text{ converges for each } (t_j) \in \lambda \right\}.$$

As we know, the study of β-dual of sequence spaces is an interesting topic in analysis [2, 3, 6]. For topological vector space E, the β-dual of $MC_{\lambda}(X)$, which drop the linearity restriction on mappings [2], is denoted by

$$MC_{\lambda}(X)^{\beta E} = \left\{ (A_j) \subseteq E^X : \sum_{j=1}^{\infty} A_j(x_j) \text{ converges for each } (x_j) \in MC_{\lambda}(X) \right\}.$$

In this paper, we study an important problem on β-dual of spaces consisting of multiplier convergent series, that is, for mapping series (A_j) in β-dual of $MC_\lambda(X)$, we determine the largest $\mathscr{M} \subseteq 2^{MC_\lambda(X)}$ for which $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathscr{M}$. Moreover, in the last section we give some applications for mapping series.

2. The space of multiplier convergent series

First, we define the frame property of complex sequence set λ, which is important in studying multiplier convergent series.

DEFINITION 2.1

The sequence set $\lambda \in \mathbb{C}^{\mathbb{N}}$ is said to have the frame property, if there is a nonempty subset $\lambda_0 \subseteq \lambda$ such that the following hold. Moreover, λ_0 is said to be a frame subset of λ.

(1) For every integer sequences $m_1 < n_1 < m_2 < n_2 < \ldots$ and $(t_{kj}) \in \lambda_0$, $k \in \mathbb{N}$, there exists a $t_0 \in \mathbb{C}$, define $t_j = t_{kj}$ when $m_k \leq j \leq n_k, k = 1, 2, \ldots$, and otherwise $t_j = t_0$. Then $(t_j) \in \lambda$.
(2) For every $(t_j) \in \lambda$, there exist finitely many $a_1, a_2, \ldots, a_n \in \mathbb{K}$ and $(s_{1j}), (s_{2j}), \ldots, (s_{nj}) \in \lambda_0$, such that $(t_j) = \sum_{i=1}^{n} a_i (s_{ij})$.
(3) For every $i \in \mathbb{N}$, there exists $(t_{ij}) \in \lambda_0$ such that $t_{ii} \neq 0$.
(4) For every $i \in \mathbb{N}$, there exists $b_i > 0$ such that $|t_i| \leq b_i$ for all $(t_j) \in \lambda_0$.

The following examples, which are related to the subseries convergent series $MC_{\{0,1\}^{\mathbb{N}}}(X)$ and bounded multiplier convergent series $MC_{l^\infty}(X)$, indicate that $\{0, 1\}^{\mathbb{N}}$ and l^∞ have the frame property:

Example 2.1. $\{0, 1\}^{\mathbb{N}} \subseteq \mathbb{C}^{\mathbb{N}}$ is a frame subset of itself.

Example 2.2. $B_{l^\infty} = \{(t_j) \in \mathbb{C}^{\mathbb{N}} : \sup_{j \in \mathbb{N}} |t_j| \leq 1\}$ is a frame subset of l^∞.

If λ has a frame subset λ_0, for each $(x_j) \in MC_\lambda(X)$, denote

$$\|(x_j)\|_{\lambda_0} = \sup_{(t_j) \in \lambda_0, n \in \mathbb{N}} \left\| \sum_{j=1}^{n} t_j x_j \right\|.$$

Before the study of $\| \cdot \|_{\lambda_0}$, we need a proposition of frame subset.

PROPOSITION 2.1

Let $(x_j) \in X^{\mathbb{N}}$. If λ has a frame subset λ_0, and $(x_j) \in MC_\lambda(X)$. Then $\sum_{j=1}^{\infty} t_j x_j$ converges uniformly for all $(t_j) \in \lambda_0$.

Proof. Suppose that the convergence of $\sum_{j=1}^{\infty} t_j x_j$ is not uniform for $(t_j) \in \lambda_0 \subseteq \lambda$, that is, there is an $\varepsilon > 0$ such that for every $m_0 \in \mathbb{N}$ we have $m > m_0$ and $(s_j) \in \lambda_0$

for which $\|\sum_{j=m}^{\infty} s_j x_j\| \geq \varepsilon$. Hence, there exist $m_1 > 1$ and $(t_{1j}) \in \lambda_0$ such that $\|\sum_{j=m_1}^{\infty} t_{1j} x_j\| \geq \varepsilon$. Since there is an $n_1 > m_1$ such that $\|\sum_{j=n_1+1}^{\infty} t_{1j} x_j\| < \varepsilon/2$, we have that $\|\sum_{j=m_1}^{n_1} t_{1j} x_j\| > \varepsilon/2$. By induction we get an integer sequence $m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{(t_{kj}) : k \in \mathbb{N}\} \subseteq \lambda_0$ such that $\|\sum_{j=m_k}^{n_k} t_{kj} x_j\| > \varepsilon/2$ for all $k \in \mathbb{N}$. By Definition 2.1(1), there is a $t_0 \in \mathbb{C}$. Let

$$t_j = \begin{cases} t_{kj}, & m_k \leq j \leq n_k, k = 1, 2, \ldots, \\ t_0, & \text{otherwise.} \end{cases}$$

Then $(t_j) \in \lambda$. However, $\sum_{j=1}^{\infty} t_j x_j$ diverges. □

Now, if λ has a frame subset λ_0, we will prove that $\|\cdot\|_{\lambda_0}$ is a norm on $MC_\lambda(X)$, moreover, $(MC_\lambda(X), \|\cdot\|_{\lambda_0})$ is complete.

Theorem 2.1. *$(MC_\lambda(X), \|\cdot\|_{\lambda_0})$ is a Banach space for each frame subset λ_0 of λ.*

Proof. Let $\varepsilon > 0$ and $(x_j) \in MC_\lambda(X)$. By Proposition 2.1, there is an $n_0 \in \mathbb{N}$ such that $\|\sum_{j=n}^{m} t_j x_j\| < \varepsilon$ for all $n > m > n_0$ and $(t_j) \in \lambda_0$. It follows from Definition 2.1(4), for $i = 1, 2, \ldots, n_0$, there exists $b_i > 0$ such that $|t_i| \leq b_i$ for all $(t_j) \in \lambda_0$. Hence, $\|\sum_{j=1}^{n} t_j x_j\| < \sum_{j=1}^{n_0} b_j \|x_j\| + \varepsilon$ for all $n \in \mathbb{N}$ and $(t_j) \in \lambda_0$, that is, $\|\cdot\|_{\lambda_0} : MC_\lambda(X) \to [0, +\infty)$.

It is easy to verify that $\|(x_j) + (y_j)\|_{\lambda_0} \leq \|(x_j)\|_{\lambda_0} + \|(y_j)\|_{\lambda_0}$ and $\|t(x_j)\|_{\lambda_0} = |t|\|(x_j)\|_{\lambda_0}$. Next, if $\|(x_j)\|_{\lambda_0} = 0$, then $\sum_{j=1}^{n} t_j x_j = 0$ for all $n \in \mathbb{N}$ and $(t_j) \in \lambda_0$. By Definition 2.1(3), for $i \in \mathbb{N}$, there exists $(t_{ij}) \in \lambda_0$ such that $t_{ii} \neq 0$. Pick $n = 1$, $t_{11}x_1 = 0$ implies that $x_1 = 0$. Moreover, pick $n = 2$, $t_{21}x_1 + t_{22}x_2 = 0 + t_{22}x_2 = 0$, then $x_2 = 0$. By induction we have that $(x_j) = 0$. It was proved that $\|\cdot\|_{\lambda_0}$ is a norm on $MC_\lambda(X)$.

Let (x_{nj}), $n \in \mathbb{N}$ be Cauchy in $(MC_\lambda(X), \|\cdot\|_{\lambda_0})$. Hence, there exists an $m_0 \in \mathbb{N}$ such that $\|\sum_{j=1}^{k} t_j x_{nj} - \sum_{j=1}^{k} t_j x_{mj}\| < \varepsilon/3$ for all $n > m > m_0$, $k \in \mathbb{N}$ and $(t_j) \in \lambda_0$. Since X is complete, there exist $y_{k,(t_j)} \in X$ and $n_1 \in \mathbb{N}$ such that

$$\left\|\sum_{j=1}^{k} t_j x_{nj} - y_{k,(t_j)}\right\| < \varepsilon/3, \forall n > n_1, k \in \mathbb{N}, (t_j) \in \lambda_0. \tag{1}$$

By Proposition 2.1, for every $n \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $\|\sum_{j=1}^{k} t_j x_{nj} - \sum_{j=1}^{p} t_j x_{nj}\| < \varepsilon/3$ for all $k > p > k_0$ and $(t_j) \in \lambda_0$. Pick $n > n_1$, $\|y_{k,(t_j)} - y_{p,(t_j)}\| < \varepsilon$ for all $k > p > k_0$ and $(t_j) \in \lambda_0$. Since X is complete, $y_{k,(t_j)}$ converges uniformly for $(t_j) \in \lambda_0$, when $k \to +\infty$.

By Definition 2.1(3), for $i \in \mathbb{N}$, there exists $(t_{ij}) \in \lambda_0$ such that $t_{ii} \neq 0$. Hence, $|t_{ii}|\|x_{ni} - x_{mi}\| \leq \|\sum_{j=1}^{i} t_{ij}(x_{nj} - x_{mj})\| + \|\sum_{j=1}^{i-1} t_{ij}(x_{nj} - x_{mj})\| < 2\varepsilon/3$ for all $n > m > m_0$. Since X is complete, there exists an $(z_j) \in X^{\mathbb{N}}$ such that $\lim_n \|x_{n_j} - z_j\| = 0$ for all $j \in \mathbb{N}$.

Let $(t_j) \in \lambda_0$ and $k \in \mathbb{N}$ be arbitrary. There is a $n_2 > n_1$ such that $\|x_{n_j} - z_j\| < \varepsilon$ for all $n > n_2$ and $j = 1, 2, \ldots, k$. Hence, $\|\sum_{j=1}^{k} t_j z_j - y_{k,(t_j)}\| \leq \|\sum_{j=1}^{k} t_j(z_j - x_{nj})\| + \|\sum_{j=1}^{k} t_j x_{nj} - y_{k,(t_j)}\| < (\sum_{j=1}^{k} |t_j|)\varepsilon + \varepsilon$. This implies that $\sum_{j=1}^{k} t_j z_j = y_{k,(t_j)}$ for all $(t_j) \in \lambda_0$ and $k \in \mathbb{N}$. By (1), $\lim_n \|(x_{nj}) - (z_j)\|_{\lambda_0} = 0$.

Finally, let $(t_j) \in \lambda$. By Definition 2.1(2), there exist $a_1, a_2, \ldots, a_n \in \mathbb{K}$ and $(s_{1j}), (s_{2j}), \ldots, (s_{nj}) \in \lambda_0$, such that $(t_j) = \sum_{i=1}^{n} a_i (s_{ij})$. Hence, $\sum_{j=1}^{k} t_j z_j = \sum_{i=1}^{n} a_i y_{k,(s_{ij})}$. Since $y_{k,(s_{ij})}$ converges when $k \to +\infty$, we have that $(z_j) \in MC_\lambda(X)$. Now, we prove that $MC_\lambda(X)$ is complete. □

3. Main theorem

In the following sections, we only care about the λ which has at least one frame subset λ_0, for example, $\lambda = \{0, 1\}^{\mathbb{N}}$ or l^∞, etc. First, we discuss the totally bounded subsets of $(MC_\lambda(X), \|\cdot\|_{\lambda_0})$, where λ_0 is any frame subset of λ. Recall that a subset B of a topological vector space E is totally bounded or precompact if for every neighborhood U of $0 \in E$ there is a finite subset $F \subseteq E$ such that $B \subseteq F + U$ (p. 83 of [9]).

PROPOSITION 3.1

Let M be a totally bounded subset of $(MC_\lambda(X), \|\cdot\|_{\lambda_0})$. Then $\lim_n \|\sum_{j=n}^{\infty} t_j x_j\| = 0$ *uniformly for $(x_j) \in M$ and $(t_j) \in \lambda_0$.*

Proof. Let $\varepsilon > 0$ be arbitrary and let $U = \{(u_j) \in MC_\lambda(X) : \|(u_j)\|_{\lambda_0} < \varepsilon/3\}$. Since M is totally bounded, there is a finite subset $F = \{(z_{ij}) : i = 1, 2, \ldots, n\} \subseteq MC_\lambda(X)$ such that $M \subseteq F + U$. By Proposition 2.1, there exists an $n_0 \in \mathbb{N}$ such that $\|\sum_{j=m}^{n} t_j z_{ij}\| < \varepsilon/3$ for all $n, m > n_0$, $i = 1, 2, \ldots, n$ and $(t_j) \in \lambda_0$. Moreover, $\|\sum_{j=m}^{n} t_j u_j\| \le \|\sum_{j=1}^{n} t_j u_j\| + \|\sum_{j=1}^{m-1} t_j u_j\| < 2\varepsilon/3$ for all $n, m > n_0$, $(u_j) \in U$ and $(t_j) \in \lambda_0$. Hence, $\|\sum_{j=m}^{n} t_j x_j\| \le \|\sum_{j=m}^{n} t_j z_{i_0 j}\| + \|\sum_{j=m}^{n} t_j u_j\| < \varepsilon$ for all $n, m > n_0$, $(x_j) \in M$ and $(t_j) \in \lambda_0$.

However, the converse is not always true.

Example 3.1. Let $M = \{(kx, 0, 0, \ldots) : k \in \mathbb{N}\}$ where $0 \neq x \in X$. In fact, $M \subseteq MC_\lambda(X)$ and $\lim_n \|\sum_{j=n}^{\infty} t_j x_j\| = 0$ uniformly for $(x_j) \in M$ and $(t_j) \in \lambda_0$, but there is a $(t_{1j}) \in \lambda_0$ such that $t_{11} \neq 0$. Pick $(x_j) = (kx, 0, 0, \ldots) \in M$, we have $\|(x_j)\|_{\lambda_0} = k\|t_{11}x\|$. Hence, M is not totally bounded.

Now, based on the proposition of totally bounded sets, we characterize the uniform convergence of mapping series in β-dual of $MC_\lambda(X)$.

Theorem 3.1. *Let $M \subseteq MC_\lambda(X)$ and λ_0 be a frame subset of λ. Then the following are equivalent*:

(I) $\lim_n \|\sum_{j=n}^{\infty} t_j x_j\| = 0$ *uniformly for $(x_j) \in M$ and $(t_j) \in \lambda_0$.*

(II) *For every Fréchet space E and $(A_j) \in MC_\lambda(X)^{\beta E}$, $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly for $(x_j) \in M$.*

Proof.

(I) $\Longrightarrow$ (II). If (II) fails, there is a Fréchet space $(E, p(\cdot))$ and $(A_j) \in MC_\lambda(X)^{\beta E}$ such that the convergence of $\sum_{j=1}^{\infty} A_j(x_j)$ is not uniform for $(x_j) \in M$. Hence, there is an

$\varepsilon > 0$ such that for every $m_0 \in \mathbb{N}$ we have $n > m > m_0$ and $(x_j) \in M$ for which $p(\sum_{j=m}^{n} A_j(x_j)) > \varepsilon$.

By (I), there is a $j_1 \in \mathbb{N}$ such that $\| \sum_{j=n}^{\infty} t_j z_j \| < 1/2$ for all $(z_j) \in M$, $n > j_1$ and $(t_j) \in \lambda_0$. Then, there exist $n_1 > m_1 > j_1$ and $(x_{1j}) \in M$ such that $p(\sum_{j=m_1}^{n_1} A_j(x_{1j})) > \varepsilon$ and $\| \sum_{j=m_1}^{n_1} t_j x_{1j} \| < 1/2$ for all $(t_j) \in \lambda_0$. Pick $j_2 > n_1$ for which $\| \sum_{j=n}^{\infty} t_j z_j \| < 1/2^2$ for all $(z_j) \in M$, $n > j_2$ and $(t_j) \in \lambda_0$. Then, there exist $n_2 > m_2 > j_2$ and $(x_{2j}) \in M$ such that $p(\sum_{j=m_2}^{n_2} A_j(x_{2j})) > \varepsilon$ and $\| \sum_{j=m_2}^{n_2} t_j x_{2j} \| < 1/2^2$ for all $(t_j) \in \lambda_0$. Continuing this construction produces an integer sequence $m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{(x_{kj}) : k \in \mathbb{N}\} \subseteq M$ such that

$$p\left(\sum_{j=m_k}^{n_k} A_j(x_{kj})\right) > \varepsilon \quad \text{and} \quad \left\| \sum_{j=m_k}^{n_k} t_j x_{kj} \right\| < 1/2^k, \forall (t_j) \in \lambda_0, k \in \mathbb{N}.$$

Let

$$x_j = \begin{cases} x_{kj}, & m_k \leq j \leq n_k, k = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

For every $(t_j) \in \lambda$, it follows from Definition 2.1(2) that there exist $a_1, a_2, \ldots, a_n \in \mathbb{K}$ and $(s_{1j}), (s_{2j}), \ldots, (s_{nj}) \in \lambda_0$ such that $(t_j) = \sum_{i=1}^{n} a_i (s_{ij})$. Hence, $\sum_{j=1}^{\infty} t_j x_j = \sum_{i=1}^{n} a_i \sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k} s_{ij} x_{kj}$. Since $\sum_{k=1}^{\infty} 1/2^k = 1$ and X is complete, $\sum_{k=1}^{\infty} \sum_{j=m_k}^{n_k} s_{ij} x_{kj}$ converges for each $i = 1, 2, \ldots, n$. Then, $(x_j) \in MC_\lambda(X)$. However, $\sum_{j=1}^{\infty} A_j(x_j)$ diverges which contradicts $(A_j) \in MC_\lambda(X)^{\beta E}$.

(II) $\Longrightarrow$ (I). If (I) fails, there exist $\varepsilon > 0$, $m_1 < n_1 < m_2 < n_2 < \cdots$, $\{(x_{kj}) : k \in \mathbb{N}\} \subseteq M$ and $\{(t_{kj}) : k \in \mathbb{N}\} \subseteq \lambda_0$ such that $\| \sum_{j=m_k}^{n_k} t_{kj} x_{kj} \| > \varepsilon$ for all $k \in \mathbb{N}$.

For each $j \in \mathbb{N}$ define $A_j : X \to MC_\lambda(X)$ by $A_j(x) = (0, \ldots, 0, \overset{(j)}{x}, 0, \ldots)$ for all $x \in X$. For every $(x_j) \in MC_\lambda(X)$, it follows from Proposition 2.1 that

$$\begin{aligned} \lim_n \left\| \sum_{j=1}^{n} A_j(x_j) - (x_j) \right\|_{\lambda_0} &= \lim_n \| (0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots) \|_{\lambda_0} \\ &= \lim_n \sup_{(t_j) \in \lambda_0, k \in \mathbb{N}} \left\| \sum_{j=n+1}^{n+k} t_j x_j \right\| = 0. \end{aligned}$$

Then, $(A_j) \in MC_\lambda(X)^{\beta E}$, where $E = MC_\lambda(X)$ is a Banach space. However,

$$\begin{aligned} \left\| \sum_{j=m_k}^{n_k} A_j(x_{kj}) \right\|_{\lambda_0} &= \| \big(0, \ldots, 0, x_{km_k}, x_{kn_k}, \ldots\big) \|_{\lambda_0} \\ &= \sup_{(t_j) \in \lambda_0, n \in \mathbb{N}} \left\| \sum_{j=m_k}^{m_k+n} t_j x_{kj} \right\| \geq \left\| \sum_{j=m_k}^{n_k} t_j x_{kj} \right\| > \varepsilon. \end{aligned}$$

This contradicts (II). □

4. Applications

Let X, Y be Banach spaces, $\lambda \subseteq \mathbb{C}^{\mathbb{N}}$ which has a frame subset λ_0, and

$$\mathscr{M}_{\lambda,\lambda_0} = \left\{ M \subseteq MC_\lambda(X) : \lim_n \| \sum_{j=n}^{\infty} t_j x_j \| = 0 \text{ uniformly for } (x_j) \in M \text{ and } (t_j) \in \lambda_0 \right\}.$$

By Proposition 3.1, any totally bounded subset of $MC_\lambda(X)$ belongs to $\mathscr{M}_{\lambda,\lambda_0}$.

The Banach–Steinhaus theorem says that if the linear operator $T_n : X \to Y$ is continuous and $\lim_n T_n(x) = T(x)$ at each $x \in X$, then $T : X \to Y$ is also linear and continuous. Moreover, $\lim_n T_n(x) = T(x)$ uniformly for x in any totally bounded subset of X (pp. 299–300 of [5]).

In general, the Banach–Steinhaus theorem fails to hold for nonlinear mappings. However, from Theorem 3.1, we directly have the following.

Theorem 4.1. *If* $(A_j) \in MC_\lambda(X)^{\beta Y}$ *and* $f_n[(x_j)] = \sum_{j=1}^{n} A_j(x_j)$, $f[(x_j)] = \sum_{j=1}^{\infty} A_j(x_j)$ *for* $(x_j) \in MC_\lambda(X)$. *Then* $\lim_n f_n[(x_j)] = f[(x_j)]$ *uniformly for* (x_j) *in any totally bounded subset of* $MC_\lambda(X)$.

COROLLARY 4.1

If $(A_j) \in MC_\lambda(X)^{\beta Y}$ *and* A_j *is continuous, then* $\langle (A_j), (x_j) \rangle = \sum_{j=1}^{\infty} A_j(x_j)$ *defines a continuous mapping* $\langle (A_j), \cdot \rangle : MC_\lambda(X) \to Y$.

Proof. Suppose that $(x_j^{(n)}) \to (x_j)$ in $MC_\lambda(X)$ when $n \to +\infty$. By Definition 2.1(3), for every $k \in \mathbb{N}$, there exist $(t_{kj}) \in \lambda_0$ such that $t_{kk} \neq 0$. Hence, $\|t_{kk}(x_k^{(n)} - x_k)\| \leq \| \sum_{j=1}^{k} t_{kj}(x_j^{(n)} - x_j)\| + \| \sum_{j=1}^{k-1} t_{kj}(x_j^{(n)} - x_j)\| \leq 2\|(x_j^{(n)}) - (x_j)\|_{\lambda_0} \to 0$, that is, $\lim_n x_k^{(n)} = x_k$ for all $k \in \mathbb{N}$. So $\lim_n \sum_{j=1}^{m} A_j(x_j^{(n)}) = \sum_{j=1}^{m} A_j(x_j)$ for all $m \in \mathbb{N}$. Since $\{(x_j^{(n)}) : n \in \mathbb{N}\}$ is totally bounded, it follows from Theorem 4.1 that $\lim_n \sum_{j=1}^{m} A_j(x_j^{(n)}) = \sum_{j=1}^{\infty} A_j(x_j^{(n)})$ uniformly with respect to $n \in \mathbb{N}$. Then, $\lim_n \sum_{j=1}^{\infty} A_j(x_j^{(n)}) = \lim_n \lim_m \sum_{j=1}^{m} A_j(x_j^{(n)}) = \lim_m \lim_n \sum_{j=1}^{m} A_j(x_j^{(n)}) = \lim_m \sum_{j=1}^{m} A_j(x_j) = \sum_{j=1}^{\infty} A_j(x_j)$. □

Finally, we suppose that λ satisfies the following condition: for any $(t_j) \in \lambda$ and $j_1 < j_2 < \cdots$, let

$$t'_j = \begin{cases} t_j, & j = j_k, k = 1, 2, \ldots, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Then $(t'_j) \in \lambda$. For example, $\lambda = \{0, 1\}^{\mathbb{N}}$ or l^∞, etc. Then by the Orlicz–Pettis theorem and Theorem 3.1, we can get the following.

Theorem 4.2. *If* $(A_j) \subseteq Y^X$ *such that* $A_j(0) = 0$ *for all* $j \in \mathbb{N}$ *and* $\sum_{j=1}^{\infty} A_j(x_j)$ *converges weakly at each* $(x_j) \in MC_\lambda(X)$. *Then* $\sum_{j=1}^{\infty} A_j(x_j)$ *converges uniformly for* (x_j) *in any totally bounded subset of* $MC_\lambda(X)$.

Proof. For any $(x_j) \in MC_\lambda(X)$, $(t_j) \in \lambda$ and $j_1 < j_2 < \cdots$, let (t'_j) by (2) and

$$u_j = \begin{cases} x_j, & j = j_k, k = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\sum_{j=1}^n t_j u_j = \sum_{k=1}^n t_{j_k} x_{j_k} = \sum_{j=1}^n t'_j x_j$ converges when $n \to +\infty$. Then $(u_j) \in MC_\lambda(X)$ so $\sum_{j=1}^\infty A_j(u_j)$ is weakly convergent. Since $A_j(0) = 0$ for all $j \in \mathbb{N}$, it follows from $\sum_{k=1}^n A_{j_k}(x_{j_k}) = \sum_{j=1}^{j_n} A_j(u_j)$ that $\sum_{k=1}^\infty A_{j_k}(x_{j_k})$ is weakly convergent. By the Orlicz–Pettis theorem, $\sum_{j=1}^\infty A_j(x_j)$ converges in Y. Hence, $(A_j) \in MC_\lambda(X)^{\beta Y}$. □

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