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A note on stable Teichmüller quasigeodesics

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Abstract. In this note, we prove that for a cobounded, Lipschitz path $\gamma : I \to \mathcal{T}$ in the Teichmüller space $\mathcal{T}$ of a hyperbolic surface, if the pull back bundle $\mathcal{H}_\gamma \to I$ of the cannonical $\mathbb{H}^2$-bundle $\mathcal{H} \to \mathcal{T}$ is a strongly relatively hyperbolic metric space then there exists a geodesic ξ of $\mathcal{T}$ such that $\gamma(I)$ and ξ are close to each other.

Keywords. Teichmüller space; relatively hyperbolic space; mapping class groups; quasigeodesics.

1. Introduction

Suppose $S_{g,n}$ is a surface of genus g with n punctures such that its Euler characteristic $\chi(S_{g,n}) < 0$. Consider the Teichmüller space $\mathcal{T} = \text{Teich}(S_{g,n})$ of $S_{g,n}$. There is a smooth fiber bundle $\mathcal{S} \to \mathcal{T}$ over $\mathcal{T}$, whose fiber $\mathcal{S}_\sigma$ over $\sigma \in \mathcal{T}$ is $S_{g,n}$ with metric σ. Let $\mathcal{H}$ be the universal cover of $\mathcal{S}$, then the universal covering $\mathcal{H} \to \mathcal{S}$ defines a smooth fiber bundle $\mathcal{H} \to \mathcal{T}$ whose fiber $\mathcal{H}_\sigma$ over $\sigma \in \mathcal{T}$ is isometric to the hyperbolic plane $\mathbb{H}^2$ (refer to §2.2, pp. 49–50 of [13]). Let I be a closed and connected interval of $\mathbb{R}$. The purpose of this note is to prove that for a $\mathcal{B}$-cobounded, Lipschitz path $\gamma : I \to \mathcal{T}$, where $\mathcal{B}$ is a compact subset of $\mathcal{T}$, if the pull back bundle $\mathcal{H}_\gamma$ over I is a strongly relatively hyperbolic metric space then there exists a geodesic ξ in $\mathcal{T}$ such that the Hausdorff distance between $\gamma(I)$ and ξ is bounded. This is a straightforward generalization of a result due to Mosher (Theorem 1.1 of [13]), where the statement was proven for closed surfaces admitting hyperbolic metrics with the assumption that $\mathcal{H}_\gamma$ is a hyperbolic metric space. We refer to [1] for basic facts about hyperbolic metric spaces.

2. Relative hyperbolicity

Let (X, d_X) be a path metric space. A collection of closed subsets $\mathcal{D} = \{D_\alpha\}_{\alpha\in\Lambda}$ of X will be said to be *uniformly separated if* there exists $\mu > 0$ such that $d_X(D_\alpha, D_\beta) \geq \mu$ for all distinct $D_\alpha, D_\beta \in \mathcal{D}$.

Let $Z = X \bigsqcup (\sqcup_\alpha (D_\alpha \times [0, \frac{1}{2}]))$ and we define a distance function on Z as follows:

$$\begin{aligned} d_Z(x, y) &= d_X(x, y), \text{ if } x, y \in X, \\ &= d_{D_\alpha \times [0, \frac{1}{2}]}(x, y), \text{ if } x, y \in H_\alpha \text{ for some } \alpha \in \Lambda, \\ &= \infty, \text{ if } x, y \text{ does not lie on a same set of the disjoint union.} \end{aligned}$$

The metric $d_{D_\alpha \times [0,\frac{1}{2}]}$ above is the product metric on $D_\alpha \times [0, \frac{1}{2}]$. Let $\{v_\alpha\}_{\alpha\in\Lambda}$ be a collection of distinct points indexed by Λ. Let $\sim$ be a relation on Z defined as follows: $(h_\alpha, \frac{1}{2}) \sim v_\alpha$ and $(h_\alpha, 0) \sim h_\alpha$ for all $h_\alpha \in D_\alpha$ and for all $\alpha \in \Lambda$, where $\sim$ is an equivalence relation.

DEFINITION 2.1 [6]

The *electric space* (or coned-off space) $\mathcal{E}(X, \mathcal{D})$ corresponding to the pair $(X, \mathcal{D})$ is the quotient space $Z/\sim$ of Z and the points v_α are said to be cone points. We define a metric $d_{\mathcal{E}(X,\mathcal{D})}$ on $\mathcal{E}(X, \mathcal{H})$ as follows:

$$d_{\mathcal{E}(X,\mathcal{D})}([x], [y]) = \inf \sum_{1\le i\le n} d_Z(x_i, y_i),$$

where the infimum is taken over all sequences $C = \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ of points of Z such that $x_1 \in [x]$, $y_n \in [y]$ and $y_i \sim x_{i+1}$ for $i = 1, \ldots, n-1$. ($\sim$ is the equivalence relation on Z).

Note that $\mathcal{E}(X, \mathcal{D})$ is a metric space obtained from X by coning each D_α to the cone point v_α.

DEFINITION 2.2

X is said to be *weakly hyperbolic* relative to the collection $\mathcal{D}$ if $\mathcal{E}(X, \mathcal{D})$ is a hyperbolic metric space.

For a path $\gamma : [0, 1] \to X$, there is an induced path $\hat{\gamma}$ in $\mathcal{E}(X, \mathcal{D})$ obtained as follows: for each component $\gamma|_{[s,t]}$ of γ lying in a set $D_\alpha \in \mathcal{D}$, replace the component $\gamma|_{[s,t]}$ by the path $[\gamma(s), v_\alpha] \cup [v_\alpha, \gamma(t)]$ of length one passing through cone point v_α, where $[\gamma(s, v_\alpha)]$, $[v_\alpha, \gamma(t)]$ are geodesics in the cone $\mathcal{E}(D_\alpha, \{D_\alpha\})$. If $\hat{\gamma}$ is a geodesic (resp. P-quasigeodesic) in $\mathcal{E}(X, \mathcal{D})$, γ is called a *relative geodesic* (resp. *relative P-quasigeodesic*). γ is said to be *without backtracking* if γ does not return to an D_α after leaving it.

DEFINITION 2.3 [4]

Relative geodesics (resp. P-quasigeodesics) in $(X, \mathcal{D})$ are said to satisfy *bounded region penetration properties* if there exists $K = K(P) > 0$ such that for any two relative geodesics (resp. P-quasigeodesics without backtracking) β, γ joining $x, y \in X$, the following two properties are satisfied:

(1) if precisely one of $\{\beta, \gamma\}$ meets a set $D_\alpha \in \mathcal{D}$, then the distance (measured in the intrinsic path-metric on D_α) from the first (entry) point to the last (exit) point (of the relevant path) is at most K,
(2) if both $\{\beta, \gamma\}$ meet some $D_\alpha \in \mathcal{D}$ then the distance (measured in the intrinsic path-metric on D_α) from the entry point of β to that of γ is at most K; similarly for exit points.

DEFINITION 2.4 [4]

X is said to be strongly hyperbolic relative to the uniformly separated collection $\mathcal{D}$ if X is weakly hyperbolic relative to $\mathcal{D}$ and relative P quasigeodesics without backtracking satisfy the bounded region penetration properties.

Gromov's definition of relative hyperbolicity:

DEFINITION 2.5 [10]

For any geodesic metric space (D, d), the *hyperbolic cone* (analog of a horoball) D^h is the metric space $D \times [0, \infty) = D^h$ equipped with the path metric d_h obtained from two pieces of data

(1) $d_{h,t}((x,t),(y,t)) = 2^{-t} d_D(x,y)$, where $d_{h,t}$ is the induced path metric on $D_t = D \times \{t\}$. Paths joining $(x,t), (y,t)$ and lying on $D_t = D \times \{t\}$ are called *horizontal paths*;
(2) $d_h((x,t),(x,s)) = |t - s|$ for all $x \in D$ and for all $t, s \in [0, \infty)$, and the corresponding paths are called *vertical paths*;
(3) for all $x, y \in D^h$, $d_h(x, y)$ is the path metric induced by the collection of horizontal and vertical paths.

DEFINITION 2.6 [10]

Let $\delta \geq 0$. Let X be a geodesic metric space and $\mathcal{D}$ be a collection of mutually disjoint uniformly separated subsets of X. Let $\mathcal{G}(X, \mathcal{D})$ be the quotient space, obtained by attaching the hyperbolic cones D^h to $D \in \mathcal{D}$ via the identification $(x, 0) \sim x$ for all $x \in D$. The metric on $\mathcal{G}(X, \mathcal{D})$ is the quotient metric. X is said to be strongly δ-hyperbolic relative to $\mathcal{D}$ in the sense of Gromov, if $\mathcal{G}(X, \mathcal{D})$ is a δ-hyperbolic metric space. X is said to be strongly hyperbolic relative to $\mathcal{D}$ in the sense of Gromov if $\mathcal{G}(X, \mathcal{D})$ is a δ-hyperbolic metric space for some $\delta \geq 0$.

Theorem 2.7 [3]. *Let X be a geodesic metric space and $\mathcal{D}$ be a collection of mutually disjoint uniformly separated subsets of X. X is strongly hyperbolic relative to the collection $\mathcal{D}$ of uniformly separated subsets of X in the sense of Farb if and only if X is strongly hyperbolic relative to the collection $\mathcal{D}$ of uniformly separated subsets of X in the sense of Gromov.*

DEFINITION 2.8 (Definition 2.9 of [11])

Let $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ be an ordered quadruple such that the following holds:

(1) X is a geodesic metric space and $\mathcal{H} = \{H_\alpha : \alpha \in \Lambda\}$ is a collection of uniformly ϵ-separated closed subsets of X. X is strongly hyperbolic relative to $\mathcal{H}$,
(2) $\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}$ is a collection of δ-hyperbolic geodesic metric spaces and $\mathcal{G}$ is a collection of uniform large scale retraction $g_\alpha : H_\alpha \to L_\alpha$, i.e., there exists $K, \epsilon > 0$ such that the following holds: $d_{L_\alpha}(g_\alpha(x), g_\alpha(y)) \leq K d_{H_\alpha}(x, y) + \epsilon$ for all $x, y \in H_\alpha$.

The partially electrocuted space or partially coned-off space, denoted by X_{pel}, corresponding to the quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{L})$ is obtained from X by gluing in the (metric) mapping cylinders for the maps $g_\alpha : H_\alpha \to L_\alpha$.

The metric in the partially electrocuted space X_{pel} will be denoted as d_{pel}.

Lemma 2.9 *(Lemma* 2.10 *of* [11]*).* $(X_{\mathrm{pel}}, d_{\mathrm{pel}})$ *is a hyperbolic metric space and the sets L_α are uniformly quasiconvex sets.*

3. Main theorem

Suppose $p_1, \ldots, p_n$ are the punctures of $S_{g,n}$ and σ a Teichmüller metric on $S_{g,n}$. For each p_i, σ corresponds to an infinite collection $\mathcal{D}_\sigma(p_i) = \{D_\sigma(p_i, \alpha) : \alpha \in \Lambda\}$ of horodisks in the fiber $\mathcal{H}_\sigma$ of the bundle $\mathcal{H} \to \mathcal{T}$ such that they satisfy the following properties:

(1) each $\mathcal{D}_\sigma(p_i)$ is invariant under the action of $\pi_1(S_{g,n})$,
(2) for each $\alpha \in \Lambda$, there exists a sub-bundle $\mathcal{D}(p_i, \alpha) \to \mathcal{T}$ such that the fiber over $\sigma \in \mathcal{T}$ is $D_\sigma(p_i, \alpha)$,
(3) elements of $\mathcal{D}_\sigma(p_1) \cup \cdots \cup \mathcal{D}_\sigma(p_n)$ are disjoint from each other.

For each path $\gamma : I \to \mathcal{T}$, $1 \le i \le n$ and $\alpha \in \Lambda$, there exists a pull back bundle $\mathcal{D}_\gamma(p_i, \alpha) \to I$ such that the fiber over $t \in I$ is $D_{\gamma(t)}(p_i, \alpha)$. Let $\mathcal{D}_\gamma$ denote the collection $\{\mathcal{D}_\gamma(p_i, \alpha) : 1 \le i \le n, \alpha \in \Lambda\}$.

DEFINITION 3.1

Consider a subset $\mathcal{B}$ of the modulli space $\mathcal{M} = \mathcal{T}/MCG(S_{g,n})$, a path $\gamma : I \to \mathcal{T}$ is said to be $\mathcal{B}$-*cobounded*, if the image of γ under the projection $\mathcal{T} \to \mathcal{M}$ lies in $\mathcal{B}$.

We prove the following theorem:

Theorem 3.2. *Let I be a closed, connected interval of $\mathbb{R}$. For a compact subset $\mathcal{B}$ of the moduli space $\mathcal{M} = \mathcal{T}/MCG(S_{g,n})$ and for every $\rho \ge 1$, $\delta \ge 0$ there exists $P \ge 0$ such that the following holds*:

If $\gamma : I \to \mathcal{T}$ is $\mathcal{B}$-cobounded and ρ-Lipschitz path, and if $\mathcal{H}_\gamma$ is strongly δ-hyperbolic relative to the collection $\mathcal{D}_\gamma$, then there exists a geodesic $\xi : I \to \mathcal{T}$ joining end points of γ such that the Hausdorff distance between $\gamma(I)$ and $\xi(I)$ is at most P.

Note that the fibers $\mathcal{H}_\sigma = \mathbb{H}^2 \times \sigma$ of $\mathcal{H} \to \mathcal{T}$ are (uniformly) strongly hyperbolic relative to the collections $\mathcal{D}_\sigma = \{D_\sigma(p_i, \alpha) : 1 \le i \le n, \alpha \in \Lambda\}$ of horodisks. Hence the coned-off spaces $\mathcal{E}(\mathcal{H}_\sigma, \mathcal{D}_\sigma)$, $\sigma \in \mathcal{T}$ are (uniformly) hyperbolic metric spaces. Thus for a path $\gamma : I \to \mathcal{T}$ there exists a bundle $\mathcal{PH}_\gamma \to I$ of coned-off hyperbolic metric spaces with fiber $\mathcal{E}(\mathcal{H}_{\gamma(t)}, \mathcal{D}_{\gamma(t)})$. $\mathcal{PH}_\gamma$ is also obtained by partially electrocuting each element $\mathcal{D}_\gamma(p_i, \alpha)$ of $\mathcal{D}_\gamma$ to a hyperbolic space $\mathcal{L}_\gamma(p_i, \alpha)$, where $\mathcal{L}_\gamma(p_i, \alpha)$ is the locus of cone points obtained by coning $D_{\gamma(t)}(p_i, \alpha)$ for all $t \in I$. By Lemma 2.9, if $\mathcal{H}_\gamma$ is strongly hyperbolic relative to the collection $\mathcal{D}_\gamma$ then $\mathcal{PH}_\gamma$ is a hyperbolic metric space.

DEFINITION 3.3

Suppose $\kappa > 1$, n a natural number and $A \ge 0$. Let I be an interval of $\mathbb{R}$ and $\Theta = I \cap \mathbb{Z}$. A sequence of positive numbers $\{r_j : j \in \Theta\}$, is said to satisfy (κ, n, A)-flaring property if $j - n, j + n \in \Theta$ and if $r_j > A$ then $\max\{r_{j-n}, r_{j+n}\} \ge \kappa r_j$.

DEFINITION 3.4

A path $\alpha : J \to \mathcal{PH}_\gamma$, where $J \subset I$, is said to be λ-*quasivertical* if it is λ-Lipschitz and also a section.

Let $\mathcal{B}$ bounded subset of the moduli space $\mathcal{M}$ and let $\rho \ge 1$. By Lemma 4.1 and Proposition 4.3 of [6], it follows that there exists $K \ge 1$ such that if $\gamma : I \to \mathcal{T}$ is ρ-Lipschitz,

$\mathcal{B}$-cobounded path then there exists a connection map $h_{st} : \mathcal{H}_{\gamma(s)} \to \mathcal{H}_{\gamma(t)}$ which is $K^{|s-t|}$-bilipschitz (see also Remark below Proposition 2.3 of [13]). Let $\mathcal{H}^b_{\gamma(t)}$ be obtained from $\mathcal{H}_{\gamma(t)}$ by deleting the interior of horodisks. The connection map h_{st} induces a biLipschitz connection map $h^b_{st} : \mathcal{H}^b_{\gamma(s)} \to \mathcal{H}^b_{\gamma(t)}$. Now, if $\mathcal{H}_\gamma$ is strongly hyperbolic relative to the collection $\mathcal{D}_\gamma$, then the following result is an immediate consequence of converse to strong combination theorem (Theorem 4.7 of [11]). The following flaring properties is also obtained from the fact that $\mathcal{PH}_\gamma$ is a hyperbolic metric space and then applying converse of the Bestvina–Feighn combination theorem [2]. The converse of Bestvina–Feighn combination theorem is the Gersten's theorem (Corollary 6.7 of [7]). For an elaborate discussion, we refer to subsection 3.2, p. 55 and Proposition 3.1 of [13]. Let $d_{\widehat{\sigma}}$ denote the metric of the coned-off space $\mathcal{E}(\mathcal{H}_\sigma, \mathcal{D}_\sigma)$.

PROPOSITION 3.5

Let $\mathcal{B}$ be a compact subset of the moduli space $\mathcal{M} = \mathcal{T}/MCG(S_{g,n})$ and $\rho \geq 1$. If $\gamma : I \to \mathcal{T}$ is $\mathcal{B}$-cobounded and ρ-Lipschitz path such that $\mathcal{H}_\gamma$ is strongly hyperbolic relative to the collection $\mathcal{D}_\gamma$, then for a given $\lambda \geq 1$ there exist $\kappa > 1$, an integer $n \geq 1$ and a number $A > 0$ such that the following holds:

Let $\alpha, \beta : J \to \mathcal{PH}_\gamma$ be two λ-quasivertical paths, then the sequence $s_j = d_{\widehat{(\gamma(j))}}(\alpha(j), \beta(j))$, where $j \in J \cap \mathbb{Z}$, satisfies (κ, n, A)-flaring property.

We refer to [5] for the definitions of measured lamination $\mathcal{ML}$ and measured geodesic lamination $\mathcal{MGL}$ of general hyperbolic surfaces. For each $\mu \in \mathcal{ML}$, let μ_t denote the corresponding measured geodesic lamination on the hyperbolic surface $\mathcal{S}_{\gamma(t)} = \mathcal{H}_{\gamma(t)}/\pi_1(S_{g,n})$. Let $\mathcal{S}^b_{\gamma(t)}$ denote the 'thick part' of $\mathcal{S}_{\gamma(t)}$ i.e. $\mathcal{S}^b_{\gamma(t)}$ is obtained from $\mathcal{S}_{\gamma(t)}$ by deleting the images of interior of horodisks under the projection $\mathcal{H}_{\gamma(t)} \to \mathcal{S}_{\gamma(t)}$. Now each $\mu \in \mathcal{ML}$ induce a geodesic lamination $\mu^b_t (\subset \mu_t)$ on $\mathcal{S}^b_{\gamma(t)}$. A connection path of the sub-bundle $\mathcal{S}^b_\gamma \to I$ is a piecewise smooth section of the projection map which is everywhere tangent to the connection on the bundle $\mathcal{S}^b_\gamma \to I$. The connection map $H^b_{st} : \mathcal{S}^b_{\gamma(s)} \to \mathcal{S}^b_{\gamma(t)}$ $(s \leq t)$ is defined by moving points of $\mathcal{S}_{\gamma(s)}$ to $\mathcal{S}_{\gamma(t)}$ along connection paths. From Lemma 4.1 of [6], it follows that the connection maps H^b_{st} are biLipschitz maps.

Given a hyperbolic structure $\sigma \in \mathcal{T}(S_{g,n})$ and a measured lamination $\mu \in \mathcal{ML}$ on σ, let $\mathrm{d}\mu^T$ denote the transverse measure on μ and let $\mathrm{d}\mu^L$ denote the leafwise Lebesgue measure on leaves. Let $\mathrm{d}\mu = \mathrm{d}\mu^T \times \mathrm{d}\mu^L$ denote the measure on $S_{g,n}$ obtained locally as the Fubini product of $\mathrm{d}\mu^T$ with $\mathrm{d}\mu^L$. The support of $\mathrm{d}\mu$ is μ. The length of μ with respect to σ is defined by $\mathrm{len}_\sigma(\mu) = \int \mathrm{d}\mu$. For a leaf segment l_t of a geodesic lamination μ_t on $\mathcal{S}_\gamma(t)$, the length of l_t is the total measure of l_t under the measure $\mathrm{d}\mu^L$ and is denoted by $\mathrm{len}_t(l_t)$. From Proposition 3.5, it follows that for any leaf segment l_s of μ_s and $l_s \subset \mathcal{S}^b_{\gamma(s)}$, the sequence of lengths $\{\mathrm{len}_{s+i}(H_{s,s+i}(l_s))\}_i$ satisfies flaring property. As a consequence, we have the following theorem:

Theorem 3.6 (Lemma 3.6 of [13]). *For a compact subset $\mathcal{B}$ of the moduli space $\mathcal{M}$ and for every $\rho \geq 1$, there exist constants $L \geq 1$, $\kappa > 1$, $n \in \mathbb{Z}_+$ such that the following holds*:

Let $\gamma : I \to \mathcal{T}$ be a $\mathcal{B}$-cobounded and ρ-Lipschitz path, for any $\mu \in \mathcal{M}$, the sequence $i \to \mathrm{len}_{\gamma(i)}(\mu^b)$, $(i \in I \cap \mathbb{Z})$, satisfies the L-Lipschitz, $(\kappa, n, 0)$-flaring property.

For $\mu \in \mathcal{ML}$, we say μ is *realized* at p, where p is a finite number or $p \in \{-\infty, +\infty\}$, if $\mathrm{len}_{\gamma(i)}(\mu)$ achieves minimum at p.

PROPOSITION 3.7 (Proposition 3.12 of [13])

For each $k \in I \cap \mathbb{Z}$, there exists $\mu \in \mathcal{ML}$ which is finitely realized at k. If I is infinite, for each infinite end $\pm\infty$ of I there exists $\mu_{\pm} \in \mathcal{ML}$ which is realized at $\pm\infty$ respectively.

Now for a compact subset $\mathcal{B} \subset \mathcal{M}$ and numbers $\rho \geq 1$, $\delta \geq 0$, $\eta > 0$, consider $\Gamma_{\beta,\rho,\delta,\eta}$ to be the set of all triples (γ, μ_-, μ_+) with the following properties (see [13]):

(1) $\gamma : I \to \mathcal{T}$ is $\mathcal{B}$-cobounded, ρ-Lipschitz path, such that $\mathcal{H}_\gamma$ is δ-hyperbolic relative to $\mathcal{D}_\gamma$,
(2) $0 \in I$, and each $\mu_{\pm} \in \mathcal{ML}$ is normalized to have length 1 in the hyperbolic structure $\gamma(0)$,
(3) the lamination μ_+ is realized in $\mathcal{S}_\gamma$ near the right end in the following way:

   (a) If I is right infinite, then μ_+ is realized at $+\infty$,
   (b) If I is right finite, with right end point M, then there exists a minimum of length sequence $\text{len}_{\gamma(i)}(\mu_+)$ lying in the interval $[M - \eta, M]$. The lamination μ_- is realized similarly in $\mathcal{S}_\gamma$ near the left end.

The following proposition is a modification of Proposition 3.18 of [13].

PROPOSITION 3.8

The action of $MCG(S_{g,n})$ on $\Gamma_{\beta,\rho,\delta,\eta}$ is cocompact.

Proof. Let $\mathcal{A} \subset \mathcal{T}$ be a compact set such that each $(\gamma, \mu_-, \mu_+) \in \Gamma_{\beta,\rho,\delta,\eta}$, may be translated by the action of $MCG(S_{g,n})$ so that $\gamma(0) \in \mathcal{A}$. Consider the set $\mathcal{A}_{\beta,\rho,\delta,\eta} = \{(\gamma, \mu_-, \mu_+) \in \Gamma_{\beta,\rho,\delta,\eta} : \gamma(0) \in \mathcal{A}\}$. Translates of $\mathcal{A}_{\beta,\rho,\delta,\eta}$ by the action of $MCG(S_{g,n})$ covers $\Gamma_{\beta,\rho,\delta,\eta}$, therefore it suffices to show that the set $\mathcal{A}_{\beta,\rho,\delta,\eta}$ is compact. Let $(\gamma_i, \mu_-^i, \mu_+^i)$ converge to (γ, μ_-, μ_+). γ_i converges to γ implies that in the Gromov–Hausdorff topology, $\mathcal{H}_{\gamma_i}$ converges to $\mathcal{H}_\gamma$ and $\mathcal{D}_{\gamma_i}$ converges to $\mathcal{D}_\gamma$. Hence, $\mathcal{G}(\mathcal{H}_{\gamma_i}, \mathcal{D}_{\gamma_i})$ converges to $\mathcal{G}(\mathcal{H}_\gamma, \mathcal{D}_\gamma)$ in the Gromov–Hausdorff topology. The Gromov–Hausdorff limit of a sequence of δ-hyperbolic spaces is δ-hyperbolic [8]. Therefore, if $\mathcal{H}_{\gamma_i}$ are δ-hyperbolic relative to $\mathcal{D}_{\gamma_i}$ for all i, then $\mathcal{H}_\gamma$ is also δ-hyperbolic relative to $\mathcal{D}_\gamma$. Suppose for all sufficiently large i, μ_+^i (resp. μ_-^i) is realized near the right end (resp. left end), then it is shown in Proposition 3.18 of [13] that μ_+ (resp. μ_-) is realized at the near right end (resp. left end). This justifies the set $\mathcal{A}_{\beta,\rho,\delta,\eta}$ is compact. □

Proof of Theorem 3.2. Let $(\gamma, \mu_-, \mu_+) \in \Gamma_{\beta,\rho,\delta,\eta}$. From Corollary 3.17 and Proposition 3.19 of [9], it follows that there exists a constant $K > 0$ such that if length of the interval I is at least K then μ_-, μ_+ fills $S_{g,n}$.

Case (*i*). If length of I is less than K, then length of the ρ-Lipschitz path γ in $\mathcal{T}$ is at most ρK. Hence, length of a geodesic ξ joining end points of γ is at most ρK. Thus, the Hausdorff distance between γ and ξ is at most ρK.
Case (*ii*). Let length of the interval I be at least K. Any two measured lamination jointly filling a hyperbolic surface defines a conformal structure on the surface (see Example 2.7 of [9]). Therefore, μ_-, μ_+ defines a conformal structure $\sigma(\mu_-, \mu_+)$ on $S_{g,n}$ Let $a_-(t) = \frac{1}{\text{len}_{\gamma(t)}(\mu_-)}$ and $a_+(t) = \frac{1}{\text{len}_{\gamma(t)}(\mu_+)}$. Consider the map $\Sigma(t) = \sigma(a_-(t)\mu_-, a_+(t)\mu_+)$, $t \in I$. It is proved in Theorem 1.1, pp. 76–77 of [13] that the image of the map $\Sigma : I \to \mathcal{T}$ is

a geodesic ξ of $\mathcal{T}$ joining μ_- and μ_+, and there exists $L > 0, \epsilon > 0$ such that $\Sigma : I \to \mathcal{T}$ is a (L, ϵ)-quasigeodesic.

We will show that the Hausdorff distance between Σ and γ is bounded. Consider the map $\Phi : \Gamma_{\beta,\rho,\delta,\eta} \to \mathcal{T} \times \mathcal{T}$ defined by $\Phi(\alpha, \lambda_-, \lambda_+) = (\alpha(0), \sigma(\lambda_-, \lambda_+))$. Then, Φ is $MCG(S_{g,n})$-equivariant, continuous and hence has $MCG(S_{g,n})$ cocompact image.

For $i \in I \cap \mathbb{Z}$, define $\gamma'(s) = \gamma(s + i)$. By Proposition 3.8, there exists an element $f \in MCG(S_{g,n})$ such that the triple $(\gamma', a_-(i)\mu_-, a_+(i)\mu_+)$ lies in a f-translate of the compact set $\mathcal{A}_{\beta,\rho,\delta,\eta}$. Now, $\Phi(f(\mathcal{A}_{\beta,\rho,\delta,\eta}))$ is compact in $\mathcal{T} \times \mathcal{T}$. Therefore, the Teichmüller distance $d_{\mathcal{T}}$ between $\gamma(i) = \gamma'(0)$ and $\sigma(a_-(i)\mu_-, a_+(i)\mu_+) = \Sigma(i)$ is bounded by some constant, say $P_1 > 0$. Now for $t \in I$, there exists $i \in I \cap \mathbb{Z}$ such that $|t - i| \leq 1$. As γ is ρ-Lipschitz, therefore $d_{\mathcal{T}}(\gamma(t), \gamma(i)) \leq \rho$. $\Sigma : I \to \mathcal{T}$ is an (L, ϵ)-quasigeodesic which implies that $d_{\mathcal{T}}(\Sigma(t), \Sigma(i)) \leq L|t - i| + \epsilon$. Thus, by triangle inequality $d_{\mathcal{T}}(\Sigma(t), \gamma(t)) \leq \rho + P_1 + L + \epsilon$. Thus, if $P = \rho + P_1 + L + \epsilon$, then the Hausdorff distance between γ and Σ is at most P. □

4. Application

By a short exact sequence of pair of groups

$$1 \to (G_1, H_1) \to (G_2, H_2) \to (G_3, H_3) \to 1$$

we mean two short exact sequences $1 \to G_1 \to G_2 \to G_3 \to 1$ and $1 \to H_1 \to H_2 \to H_3 \to 1$, where for each $i = 1, 2, 3$, H_i is a subgroup of G_i. For a subgroup K of a group G, $N_G(K)$ will denote the normalizer of K in G.

Consider the following short exact sequence of pair of finitely generated groups:

$$1 \to (\pi_1(S_{g,1}), K_1) \to (G, N_G(K_1)) \to (Q, Q) \to 1,$$

where the K_1 is the peripheral subgroup of $\pi_1(S_{g,1})$, G is strongly hyperbolic relative to $N_G(K_1)$ and Q is a subgroup of $MCG(S_{g,1})$. Let $\Phi : Q \to \mathcal{T}$ denote the orbit map, then for any geodesic $\gamma' : I \to Q$, $\gamma = \Phi \circ \gamma' : I \to \mathcal{T}$ is a cobounded and Lipschitz path. G is strongly hyperbolic relative to $N_G(K_1)$ which implies that the bundle $\mathcal{E}(G, K_1)$ over Q is hyperbolic and hence the bundle $\mathcal{PH}_\gamma \to I$ satisfies the flaring property. Now, by the converse of strong combination theorem in [11], it follows that $\mathcal{H}_\gamma$ is strongly hyperbolic relative to $\mathcal{D}_\gamma$. Hence, as an application of Theorem 3.2, Q is a convex cocompact subgroup of $MCG(S_{g,1})$. The converse of this result is also true (see Proposition 5.27 of [12]). So, we have the following theorem:

Theorem 4.9 (Proposition 5.27 of [12]). *Consider the following short exact sequence of pair of finitely generated groups*

$$1 \to (\pi_1(S_{g,1}), K_1) \to (G, N_G(K_1)) \to (Q, Q) \to 1,$$

where $\pi_1(S_{g,1})$ is strongly hyperbolic relative to K_1. G is strongly hyperbolic relative to $N_G(K_1)$ if and only if Q is a convex cocompact subgroup of $MCG(S_{g,1})$.

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