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The multiplication operators on some analytic function spaces of the unit ball

BENOÎT F SEHBA

African Institute of Mathematical Sciences, 6 Melrose Road, Muizenberg,
Cape Town 7945, South Africa
Email: bsehba@gmail.com

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Abstract. We give equivalent definitions of the multipliers of the space of functions of bounded mean oscillation, the Bloch space and their logarithmic counterparts.

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1. Introduction

Our setting is the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ $(n \geq 1)$, that is $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$. We denote by $\mathrm{d}V$ the normalized Lebesgue measure on $\mathbb{B}^n$ and σ the normalized measure on $\mathbb{S}^n = \partial\mathbb{B}^n$ the boundary of $\mathbb{B}^n$. By $\mathcal{H}(\mathbb{B}^n)$, we denote the space of holomorphic functions on $\mathbb{B}^n$.

As usual, for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we let $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ so that $|z|^2 = \langle z, z\rangle = |z_1|^2 + \cdots + |z_n|^2$.

For $a \in \mathbb{B}^n$, $a \neq 0$, let φ_a denote the automorphism of $\mathbb{B}^n$ taking 0 to a defined by

$$\varphi_a(z) = \frac{a - P_a(z) - \left(1 - |z|^2\right)^{\frac{1}{2}} Q_a(z)}{1 - \langle z, a\rangle},$$

where P_a is the projection of $\mathbb{C}^n$ onto the one-dimensional subspace span of a, and $Q_a = I - P_a$ where I is the identity. It is easy to see that

$$\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \varphi_a \circ \varphi_a(z) = z.$$

These automorphisms are quite useful in the study of analytic function spaces and will be used all over the text.

For $0 < p < \infty$, let $\mathcal{H}^p(\mathbb{B}^n)$ denote the Hardy space which is the space of all $f \in \mathcal{H}(\mathbb{B}^n)$ such that

$$||f||_p^p := \sup_{0 \leq r < 1} \int_{\mathbb{S}^n} |f(r\xi)|^p \mathrm{d}\sigma(\xi) < \infty.$$

We denote by $\mathcal{H}^\infty(\mathbb{B}^n)$, the space of bounded analytic functions in $\mathbb{B}^n$.

The weighted Bergman space $A^p_\alpha(\mathbb{B}^n)$ ($\alpha > -1$) is the space of analytic function f on $\mathbb{B}^n$ satisfying the following integrability condition

$$||f||^p_{p,\alpha} := \int_{\mathbb{B}^n} |f(z)|^p \mathrm{d}V_\alpha(z) < \infty,$$

where $\mathrm{d}V_\alpha(z) = (1-|z|^2)^\alpha \mathrm{d}V(z)$.

We next introduce the usual definition of the space of analytic function of bounded mean oscillation BMOA. For this, we first give some more definitions. For any $\xi \in \mathbb{S}^n$ and $\delta > 0$, let

$$B_\delta(\xi) = \{w \in \mathbb{S}^n : |1 - \langle w, \xi\rangle| < \delta\}$$

and

$$Q_\delta(\xi) = \{z \in \mathbb{B}^n : |1 - \langle z, \xi\rangle| < \delta\}.$$

These are the high dimension analogues of Carleson regions.

Given $f \in \mathcal{H}^1(\mathbb{B}^n)$ we still denote by $f(\xi)$ ($\xi \in \mathbb{S}^n$) its admissible limit at the boundary which exists a.e. A function $f \in \mathcal{H}^2(\mathbb{B}^n)$ belongs to BMOA if

$$||f||^2_* := \sup_B \frac{1}{\sigma(B)} \int_B |f - f_B|^2 \mathrm{d}\sigma < \infty, \tag{1}$$

where $B = B_\delta(\xi)$, $\xi \in \mathbb{S}^n$, $0 < \delta < 1$, and f_B is the mean of f over B, that is $f_B = \frac{1}{\sigma(B)}\int_B f \mathrm{d}\sigma$. BMOA is a Banach space under the following norm:

$$||f||^2_{\mathrm{BMOA}} := |f(0)|^2 + ||f||^2_* < \infty.$$

There are at least three equivalent definitions of the class BMOA.

Theorem 1.1. *The following assertions are equivalent*:

(i) $f \in$ BMOA *that is* (1) *holds*.
(ii) $\sup_{a\in\mathbb{B}^n} ||f \circ \varphi_a - f(a)||_2 < \infty$.
(iii) $|\nabla f(z)|^2(1-|z|^2)\mathrm{d}V(z)$ *is a Carleson measure.*

If the definitions (i) and (iii) are quite often used, this is not always the case for (ii). We will be making use of the two last ones in this text.

In the spirit of the equivalent definition (ii), we define LMOA to be the space of all $f \in \mathcal{H}(\mathbb{B}^n)$ such that

$$||f||_{\mathrm{LMOA}} = |f(0)| + \sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right) ||f \circ \varphi_a - f(a)||_2 < \infty.$$

For $f \in \mathcal{H}(\mathbb{B}^n)$, the radial derivative Rf of f is given by

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

We recall that the Bloch space $\mathcal{B}$ consists of all $f \in \mathcal{H}(\mathbb{B}^n)$ such that

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in B^n} |Rf(z)|(1-|z|^2) < \infty.$$

We also recall the following definition of the logarithmic Bloch space $L\mathcal{B}$. An analytic function f belongs to $L\mathcal{B}$ if

$$\sup_{z \in B^n} |Rf(z)|(1-|z|^2) \log \frac{4}{1-|z|^2} < \infty.$$

The natural norm on $L\mathcal{B}(\mathbb{B}^n)$ is given by

$$||f||_{L\mathcal{B}} = |f(0)| + \sup_{z \in B^n} |Rf(z)|(1-|z|^2) \log \frac{4}{1-|z|^2} < \infty.$$

Given $f \in \mathcal{H}(\mathbb{B}^n)$, the multiplication operator by f *denoted* M_f is the operator defined on $\mathcal{H}(\mathbb{B}^n)$ by

$$M_f(g)(z) = f(z)g(z), \quad \text{for every } \; g \in \mathcal{H}(\mathbb{B}^n) \; \text{ and } \; z \in \mathbb{B}^n.$$

The boundedness of this operator between $\mathcal{H}^p$, A^p_α, BMOA, LMOA, $\mathcal{B}$, $L\mathcal{B}$ has been considered by several authors in various domains, we refer for example to [1–6, 8, 9]. We come back to the boundedness of this operator on BMOA, $\mathcal{B}$, and their logarithmic counterparts LMOA and $L\mathcal{B}$. As the boundedness of the multiplication operator on these spaces is well understood in [6] and [8], our aim here is just to revisit the proof, emphasizing on the use of the equivalent definition (ii) above of BMOA and the corresponding one for the other spaces as this is not the case in [6]. Our presentation is even less demanding as we do not require all the developments in [6]. We also insist on the equivalent definitions of these multipliers. Even if the equivalent definitions we provide here may not appear surprising at all for specialists, the fact is that the question is not presented in this way nowhere else as far as we know. In the sequel, we will need some simplified equivalent characterizations of some weighted Carleson measures already presented in [6] where they were the key argument in the characterization of the multipliers of the above spaces. Here, our presentation, contrary to [6], allows the parameter to be taken in the whole range $(0, \infty)$ even if we will only be interested in the interval $[1, \infty)$.

As usual, given two positive quantities A and B, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there is an absolute positive constant C such that $A \leq CB$ (resp. $A \geq CB$). When $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say A and B are equivalent. Finally, throughout, C will be a positive constant not necessarily the same at different occurrences.

2. Some useful tools

Here we provide some useful notions. In particular, we present a simplified and somewhat extended version of Theorems 1.3 and 1.4 of [6].

Let μ be a positive Borel measure on $\mathbb{B}^n$ and $0 < s < \infty$. For ρ a positive function defined on $(0, 1)$, we say μ is a (ρ, s)-Carleson measure if there is a constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \le C \frac{(\sigma(B_\delta(\xi)))^s}{\rho(\delta)}.$$

If $s = 1$, μ is called a ρ-Carleson measure. When ρ is a constant function, the measures μ are the so-called s-Carleson measures and the case $s = 1$ corresponds to the usual Carleson measures. As a particular case of such measures, we have the so-called logarithmic s-Carleson measures corresponding to the case $\rho(t) = (\log \frac{4}{t})^2$.

We recall in [7] that an analytic function f belongs to LMOA if and only if the measure $|\nabla f(z)|^2(1-|z|^2)\mathrm{d}V(z)$ is a logarithmic Carleson measure. The following result is well known (see for example [10]).

Theorem 2.1. *For a positive Borel measure μ on $\mathbb{B}^n$, and $s > 0$, the following are equivalent*:

(i) *The measure μ is a s-Carleson measure*,
(ii) *There is a constant $C > 0$ such that, for all $a \in \mathbb{B}^n$*,

$$\int_{\mathbb{B}^n} \frac{(1-|a|^2)^{ns}}{|1-\langle a, w\rangle|^{2ns}} \mathrm{d}\mu(w) \le C.$$

We will need the following lemma.

Lemma 2.2. *The following assertions are satisfied*:

(i) *There exists a constant $C > 0$ such that for any $f \in \mathcal{B}$*,

$$|f(z)| \le C \left(\log \frac{4}{1-|z|}\right) ||f||_{\mathcal{B}}, \quad z \in \mathbb{B}^n.$$

(ii) *Given $a \in \mathbb{B}^n$, the functions $f_a(z) = \log \frac{4}{1-\langle z,a\rangle}$ and*

$$g_a(z) = \left(\log\log \frac{e^4}{1-|a|}\right)^{-1} \left(\log \frac{4}{1-\langle z, a\rangle}\right) \left(\log\log \frac{e^4}{1-\langle z, a\rangle}\right)$$

are uniformly in $BMOA$.

Proof. The assertion (i) is well known (see for example [11]). That the function $f_a(z) = \log \frac{4}{1-\langle z,a\rangle}$ is uniformly in BMOA is also a well known result (see for example [6]). The proof in [6] consists in proving that the measure $\mathrm{d}\mu_a(z) = |Rf_a(z)|^2(1-|z|^2)\mathrm{d}V(z)$ is a Carleson measure with uniform bound. Thus, that the function g_a defined for all $z \in \mathbb{B}^n$ by

$$g_a(z) = \left(\log\log \frac{e^4}{1-|a|}\right)^{-1} \left(\log \frac{4}{1-\langle z, a\rangle}\right) \left(\log\log \frac{e^4}{1-\langle z, a\rangle}\right)$$

is uniform in BMOA follows easily by observing

$$|Rg_a(z)| \leq 2|Rf_a(z)| \text{ for all } z \in \mathbb{B}^n.$$

□

The following can be proved in the same way (see [6]).

Lemma 2.3. *The following assertions are satisfied.*

(i) *There exists a constant* $C > 0$ *such that for any* $f \in L\mathcal{B}$,

$$|f(z)| \leq C\left(\log\log\frac{e^4}{1-|z|}\right)||f||_{L\mathcal{B}}, \quad z \in \mathbb{B}^n.$$

(ii) *Given* $a \in \mathbb{B}^n$, *the function* $f_a(z) = \log\log\frac{e^4}{1-\langle z,a\rangle}$ *is uniform in LMOA.*

We remark that BMOA embeds continuously in $\mathcal{B}$ while LMOA continuously embeds in $L\mathcal{B}$.

Let us set

$$K_a(z) = \frac{(1-|a|^2)^n}{|1-\langle z,a\rangle|^{2n}}.$$

We have the following equivalent definitions.

Theorem 2.4. *Let* μ *be a positive Borel measure on* $\mathbb{B}^n$ *and* $0 < s < \infty$. *Then the following conditions are equivalent.*

(i) *There is* $C_1 > 0$ *such that for any* $\xi \in \mathbb{S}^n$ *and* $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C_1\frac{(\sigma(B_\delta(\xi)))^s}{(\log\frac{4}{\delta})^2}.$$

(ii) *There is* $C_2 > 0$ *such that*

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^2\int_{\mathbb{B}^n}K_a^s(z)\mathrm{d}\mu(z) \leq C_2 < \infty.$$

(iii) *There is* $C_3 > 0$ *such that*

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}\left|\log\frac{4}{1-\langle a,z\rangle}\right|^2 K_a^s(z)\mathrm{d}\mu(z) \leq C_3 < \infty.$$

(iv) *There is* $C_4 > 0$ *such that*

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}|g_a(z)|^2K_a^s(z)\mathrm{d}\mu(z) \leq C_4 < \infty,$$

$$g_a(z) = \left(\log\log\frac{e^4}{1-|a|}\right)^{-1}\left(\log\frac{4}{1-\langle a,z\rangle}\right)\left(\log\log\frac{e^4}{1-\langle a,z\rangle}\right).$$

Proof. The implication (i) $\Rightarrow$ (ii) is in Theorem 2.2 of [6]. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) follow by recalling that the functions $f_a(z) = \log \frac{4}{1-\langle a,z\rangle}$ and $h_a(z) = \log\log \frac{e^4}{1-\langle a,z\rangle}$ are uniform in BMOA and LMOA respectively, and by using the pointwise estimates for the function in these spaces provided by Lemmas 2.2 and 2.3. To complete the proof, let us show that (iv) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (i): For $0 < \delta < 1$ and $\xi \in \mathbb{S}^n$, we consider $a = (1-\delta)\xi$. We remark that for any $z \in Q_\delta(\xi)$, $K_a(z) \geq \frac{C}{\sigma(B_\delta(\xi))}$,

$$\log \frac{4}{1-|a|} \leq \left|\log \frac{4}{1-\langle a,z\rangle}\right| \quad \text{and} \quad \log\log \frac{e^4}{1-|a|} \leq \left|\log\log \frac{e^4}{1-\langle a,z\rangle}\right|.$$

It follows easily from (iv) that

$$\frac{1}{(\sigma(B_\delta(\xi)))^s}\left(\log \frac{4}{1-|a|}\right)^2 \int_{Q_\delta} \mathrm{d}\mu(z) \leq C$$

or equivalently

$$\mu(Q_\delta(\xi)) \leq C_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log \frac{4}{\delta})^2}.$$

The proof is complete. □

The following is also a consequence of Theorem 2.2 of [6] and the arguments used in the above proof.

Theorem 2.5. *Let μ be a positive Borel measure on $\mathbb{B}^n$ and $0 < s < \infty$. Then the following conditions are equivalent*:

(i) *There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,*

$$\mu(Q_\delta(\xi)) \leq C_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log \frac{4}{\delta})^2 (\log\log \frac{e^4}{\delta})^2}.$$

(ii) *There is $C_2 > 0$ such that*

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)^2 \left(\log\log \frac{e^4}{1-|a|}\right)^2 \int_{\mathbb{B}^n} K_a(z)^s \mathrm{d}\mu(z) \leq C_2 < \infty.$$

(iii) *There is $C_3 > 0$ such that*

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)^2 \int_{\mathbb{B}^n} \left|\log\log \frac{e^4}{1-\langle a,z\rangle}\right|^2 K_a^s(z)\mathrm{d}\mu(z) \leq C_3 < \infty.$$

The above theorems will be used when dealing with the spaces $\mathcal{B}$, BMOA and their logarithmic counterparts $L\mathcal{B}$ and LMOA.

3. Multipliers of BMOA and LMOA

This section is devoted to the equivalent characterizations of multipliers of BMOA and LMOA. As a motivation to our approach, we start with the (well-known) case of Hardy spaces.

PROPOSITION 3.1

Let $0 < p < \infty$ *and for* $a \in \mathbb{B}^n$, *let* $f_a(z) = \left[\frac{(1-|a|^2)^n}{(1-\langle a,z\rangle)^{2n}}\right]^{1/p}$. *For* f *an analytic function in* $\mathbb{B}^n$, *the following assertions are equivalent*:

(i) M_f *is bounded on* $\mathcal{H}^p(\mathbb{B}^n)$.
(ii) $\sup_{a\in\mathbb{B}^n} ||f_a f||_p < \infty$.
(iii) $f \in \mathcal{H}^\infty(\mathbb{B}^n)$.

Moreover,

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_p \approx ||M_f|| \approx ||f||_\infty.$$

Proof. We first observe that the functions $f_a(z) = \left[\frac{(1-|a|^2)^n}{(1-\langle a,z\rangle)^{2n}}\right]^{1/p}$ are uniform in $\mathcal{H}^p(\mathbb{B}^n)$. Now, (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. To conclude, we prove that (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iii). We recall the following pointwise estimate in $\mathcal{H}^p(\mathbb{B}^n)$.

$$|f(z)| \le C(1-|z|^2)^{-n/p}||f||_p.$$

It follows that if (ii) holds, then for any $a \in \mathbb{B}^n$,

$$|f_a(z)f(z)| \le C(1-|z|^2)^{-n/p}||f_a f||_p.$$

Taking $z = a$, we obtain that

$$|f(z)| \le C \sup_{a\in\mathbb{B}^n} ||f_a f||_p.$$

That means $f \in \mathcal{H}^\infty(\mathbb{B}^n)$.

We have already obtained that there exists a positive constant C such that for any $z \in \mathbb{B}^n$,

$$|f(z)| \le C \sup_{a\in\mathbb{B}^n} ||f_a f||_p.$$

It is also clear that

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_p \lesssim \sup_{||g||_p\le 1} ||gf||_p = ||M_f||.$$

Since the functions f_a are uniform in $\mathcal{H}^p(\mathbb{B}^n)$, one easily has that for any $a \in \mathbb{B}^n$,

$$||f_a f||_p \le C||f||_\infty.$$

We conclude that

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_p \approx ||f||_\infty \lesssim \sup_{||g||_p\le 1} ||gf||_p.$$

The proof is complete. □

Remark 3.2. The family $f_a(z) = \left[\frac{(1-|a|^2)^n}{(1-\langle a,z\rangle)^{2n}}\right]^{1/p}$, $a \in \mathbb{B}^n$ can be replaced by any of the following families:

$$F_{a,k,l,s}(z) = g_{a,k}(z)h_{a,l}(z)\frac{(1-|a|^2)^{\frac{n}{p}+s}}{(1-\langle z,a\rangle)^{\frac{2n}{p}+s}},$$

where

$$g_{a,k}(z) = \left(\log\frac{4}{1-|a|}\right)^{-k}\left(\log\frac{4}{1-\langle z,a\rangle}\right)^{k},$$

$$h_{a,l}(z) = \left(\log\log\frac{e^4}{1-|a|}\right)^{-l}\left(\log\log\frac{e^4}{1-\langle z,a\rangle}\right)^{l}$$

and $0\le k,l,s<\infty$.

We next prove the following.

Theorem 3.3. *Let f be an analytic function on $\mathbb{B}^n$. The following assertions are equivalent*:

(i) *M_f is bounded on $BMOA$.*

(ii)

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_{\mathrm{BMOA}} < \infty,$$

where $f_a(z) = \log\frac{4}{1-\langle z,a\rangle}$, $z\in\mathbb{B}^n$.

(iii)

$$\sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}} < \infty,$$

where

$$g_a(z) = \left(\log\log\frac{e^4}{1-|a|}\right)^{-1}\left(\log\frac{4}{1-\langle z,a\rangle}\right)\left(\log\log\frac{e^4}{1-\langle z,a\rangle}\right),\quad z\in\mathbb{B}^n.$$

(iv) *$f\in\mathcal{H}^\infty(\mathbb{B}^n)$ and*

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)||f\circ\varphi_a - f(a)||_2 < \infty.$$

Moreover,

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_{\mathrm{BMOA}} \approx ||M_f|| \approx \sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}} \approx ||f||_{\mathrm{LMOA}} + ||f||_\infty.$$

Proof. (i) $\Rightarrow$ (ii) is obvious. Let us show that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).
(ii) $\Rightarrow$ (iii): We first show that if $f \in H(\mathbb{B}^n)$ is such that (ii) holds, then $f \in \mathcal{H}^\infty(\mathbb{B}^n)$. We recall that there exists a positive constant C such that for any $f \in$ BMOA, the following estimate holds:

$$|f(z)| \leq C \log \frac{4}{1-|z|} ||f||_{\text{BMOA}}.$$

It follows that if f is such that (ii) holds, then for any $a \in \mathbb{B}^n$,

$$|f_a(z) f(z)| \leq C \log \frac{4}{1-|z|} ||f_a f||_{\text{BMOA}}, \quad \text{for all } z \in \mathbb{B}^n.$$

Taking $z = a$, we obtain that

$$|f(z)| \leq C \sup_{a \in \mathbb{B}^n} ||f_a f||_{\text{BMOA}}, \quad \text{for all } z \in \mathbb{B}^n,$$

that is $f \in \mathcal{H}^\infty(\mathbb{B}^n)$.

Now, given $a \in \mathbb{B}^n$, recalling that $f_a(z) = \log \frac{4}{1-\langle z,a\rangle}$ and

$$g_a(z) = \left(\log\log \frac{e^4}{1-|a|}\right)^{-1} \left(\log \frac{4}{1-\langle z,a\rangle}\right)\left(\log\log \frac{e^4}{1-\langle z,a\rangle}\right), \quad z \in \mathbb{B}^n,$$

we set

$$h_a(z) = \left(\log\log \frac{e^4}{1-|a|}\right)^{-1} \left(\log\log \frac{e^4}{1-\langle z,a\rangle}\right)$$

so that $g_a = h_a f_a$.

Next we recall that an analytic function ψ is in BMOA if and only if

$$\sup_{a \in \mathbb{B}^n} ||\psi \circ \varphi_a - \psi(a)||_2 < \infty.$$

Now for any $b \in \mathbb{B}^n$, let $J_b := ||(g_a \circ \varphi_b)(f \circ \varphi_b) - g_a(b) f(b)||_2^2$. Using the above remarks and the pointwise estimate in BMOA, we obtain

$$\begin{aligned} J_b &= ||(f_a \circ \varphi_b)(h_a \circ \varphi_b)(f \circ \varphi_b) - (h_a \circ \varphi_b) f_a(b) f(b) \\ &\quad + (h_a \circ \varphi_b) f_a(b) f(b) - g_a(b) f(b)||_2^2 \\ &= ||(f_a \circ \varphi_b)(h_a \circ \varphi_b)(f \circ \varphi_b) - (h_a \circ \varphi_b) f_a(b) f(b) \\ &\quad + (h_a \circ \varphi_b - h_a)\, f_a(b) f(b)||_2^2 \\ &\lesssim ||h_a o \varphi_b||_\infty^2 ||(f_a o \varphi_b)(f o \varphi_b) - f_a(b) f(b)||_2^2 \\ &\quad + ||f||_\infty^2 \left(\log \frac{4}{1-|b|}\right)^2 ||h_a o \varphi_b - h_a(b)||_2^2 \\ &\lesssim ||f_a f||_{\text{BMOA}}^2 + ||f||_\infty^2 ||h_a||_{\text{LMOA}}^2. \end{aligned}$$

Recalling that the functions h_a $(a \in \mathbb{B}^n)$ are uniformly in LMOA, we easily conclude that

$$\sup_{a \in \mathbb{B}^n} ||g_a f||_{\text{BMOA}} \lesssim \sup_{a \in \mathbb{B}^n} ||f_a f||_{\text{BMOA}} + ||f||_\infty < \infty.$$

(iii) $\Rightarrow$ (iv): It follows as above that if a holomorphic function f is such that (iii) holds, then f belongs to $\mathcal{H}^\infty(\mathbb{B}^n)$. Now, let us show that if (iii) holds for some $f \in \mathcal{H}(\mathbb{B}^n)$, then

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)||fo\varphi_a - f(a)||_2 < \infty$$

or equivalently

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^2\int_{\mathbb{B}^n}|Rf(z)|^2K_a(z)(1-|z|^2)\mathrm{d}V(z) \le C < \infty.$$

By Theorem 2.4, the latter is equivalent to

$$I := \sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}|g_a(z)|^2|Rf(z)|^2K_a(z)(1-|z|^2)\mathrm{d}V(z) \le C < \infty.$$

One can easily have $I \lesssim \sup_{a\in\mathbb{B}^n} I_1(a) + \sup_{a\in\mathbb{B}^n} I_2(a)$, where

$$\begin{aligned} I_1(a) &= \int_{\mathbb{B}^n}|f(z)|^2|R(g_a)(z)|^2K_a(z)(1-|z|^2)\mathrm{d}V(z)\\ &\le ||f||_\infty^2\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}|R(g_a)(z)|^2K_a(z)(1-|z|^2)\mathrm{d}V(z)\\ &\lesssim ||f||_\infty^2, \end{aligned}$$

$$\begin{aligned} I_2(a) &= \int_{\mathbb{B}^n}|R(g_af)(z)|^2K_a(z)(1-|z|^2)\mathrm{d}V(z)\\ &\le \sup_{b\in\mathbb{B}^n}||g_bf||_{\mathrm{BMOA}}^2. \end{aligned}$$

Thus, $I \le ||f||_\infty^2 + \sup_{a\in\mathbb{B}^n}||g_af||_{\mathrm{BMOA}}^2 < \infty$.
(iv) $\Rightarrow$ (i): Let f be an analytic function satisfying (iv). For any $g \in \mathrm{BMOA}$, we would like to prove that $||gf||_{\mathrm{BMOA}} < \infty$ or equivalently

$$\sup_{a\in\mathbb{B}^n}||(g\circ\varphi_a)(f\circ\varphi_a) - g(a)f(a)||_2^2 < \infty.$$

Using the pointwise estimate in BMOA and the fact that f is a bounded function, we easily obtain

$$\begin{aligned} I_a &:= ||(g\circ\varphi_a)(f\circ\varphi_a) - g(a)f(a)||_2^2\\ &= ||(g\circ\varphi_a)(f\circ\varphi_a) - (f\circ\varphi_a)g(a) + (f\circ\varphi_a)g(a) - g(a)f(a)||_2^2\\ &\lesssim ||(g\circ\varphi_a)(f\circ\varphi_a) - (f\circ\varphi_a)g(a)||_2^2 + ||(f\circ\varphi_a)g(a) - g(a)f(a)||_2^2\\ &\lesssim ||f||_\infty^2||g\circ\varphi_a - g(a)||_2^2 + |g(a)|^2||f\circ\varphi_a - f(a)||_2^2\\ &\lesssim ||f||_\infty^2||g||_{\mathrm{BMOA}}^2 + ||g||_{\mathrm{BMOA}}^2\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^2||fo\varphi_a - f(a)||_2^2 < \infty. \end{aligned}$$

Let us now prove that

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_{\mathrm{BMOA}} \approx ||M_f|| \approx \sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}} \approx ||f||_{\mathrm{LMOA}} + ||f||_\infty.$$

A careful observation of the proof of the implication (iii) $\Rightarrow$ (iv) shows that $||f||_\infty \lesssim \sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}}$ and

$$||f||_{\mathrm{LMOA}} \lesssim \sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}}.$$

On the other hand, following the proof of the implication (iv) $\Rightarrow$ (i) , we see that

$$||gf||_{\mathrm{BMOA}} \lesssim (||f||_{\mathrm{LMOA}} + ||f||_\infty)||g||_{\mathrm{BMOA}},$$

which shows that

$$\begin{aligned}\sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}} &\lesssim \sup_{||g||_{\mathrm{BMOA}}\leq 1} ||gf||_{\mathrm{BMOA}} \lesssim ||f||_{\mathrm{LMOA}} + ||f||_\infty \\ &\lesssim \sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{BMOA}}.\end{aligned}$$

The same inequalities hold when g_a is replaced by f_a by the same observations. The proof is complete. □

We obtain the following results using Theorem 2.5.

Theorem 3.4. *Let f be an analytic function on $\mathbb{B}^n$. Then the following assertions are equivalent.*

(i) *M_f is bounded on LMOA.*

(ii)

$$\sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{LMOA}} < \infty,$$

where $g_a(z) = \log\log \frac{e^4}{1-\langle z,a\rangle}$, $z \in \mathbb{B}^n$.

(iii) $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ *and*

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)\left(\log\log \frac{e^4}{1-|a|}\right) ||f\circ\varphi_a - f(a)||_2 < \infty.$$

Moreover,

$$A + ||f||_\infty \approx ||M_f|| \approx \sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathrm{LMOA}},$$

$$A = \sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)\left(\log\log \frac{e^4}{1-|a|}\right) ||f\circ\varphi_a - f(a)||_2.$$

Remark 3.5. The functions f_a and g_a appearing in the above theorems are the usual ones used in various questions about BMOA and LMOA, they can be replaced by any appropriate functions. For example by those obtained by multiplying the previous by adapted bounded functions.

4. The case of the Bloch space and the weighted Bloch space

As in the previous section one can easily obtain the following characterization of pointwise multipliers of Bergman spaces which we give here as a motivation.

PROPOSITION 4.1

Let $0 < p < \infty$, $\alpha > -1$ and f be an analytic function on $\mathbb{B}^n$. For $a \in \mathbb{B}^n$, let $f_a(z) = \left[\frac{(1-|a|^2)^{n+1+\alpha}}{(1-\langle z,a\rangle)^{2(n+1+\alpha)}}\right]^{1/p}$. Then the following assertions are equivalent:

(i) *M_f is bounded on $A^p_\alpha(\mathbb{B}^n)$.*
(ii) $\sup_{a\in\mathbb{B}^n} ||f_a f||_{p,\alpha} < \infty$.
(iii) $f \in \mathcal{H}^\infty(\mathbb{B}^n)$.

Moreover,

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_{p,\alpha} \approx ||M_f|| \approx ||f||_\infty.$$

Let us recall the following equivalent norm for the Bloch space. A holomorphic function f is in the Bloch space $\mathcal{B}$ if and only if

$$||f||_{\mathcal{B}} \approx \sup_{a\in\mathbb{B}^n} ||f \circ \varphi_a - f(a)||_{2,\alpha} < \infty,$$

for all $\alpha > -1$. One can easily prove the following:

$$||f||_{L\mathcal{B}} \approx \sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right) ||f \circ \varphi_a - f(a)||_{2,\alpha} < \infty, \quad \text{for all } \alpha > -1.$$

Using the above observations, Theorems 2.4 and 2.5, one obtain as in the previous section the following results.

Theorem 4.2. *Let f be an analytic function on $\mathbb{B}^n$. Then the following assertions are equivalent.*

(i) *M_f is bounded on $\mathcal{B}$.*
(ii)

$$\sup_{a\in\mathbb{B}^n} ||f_a f||_{\mathcal{B}} < \infty,$$

where $f_a(z) = \log \frac{4}{1-\langle z,a\rangle}$, $z \in \mathbb{B}^n$.

(iii)

$$\sup_{a\in\mathbb{B}^n} ||g_a f||_{\mathcal{B}} < \infty,$$

where

$$g_a(z) = \left(\log\log \frac{e^4}{1-|a|}\right)^{-1} \left(\log \frac{4}{1-\langle z,a\rangle}\right)\left(\log\log \frac{e^4}{1-\langle z,a\rangle}\right),$$

$z \in \mathbb{B}^n$.

(iv) $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ *and*

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)||f\circ\varphi_a - f(a)||_{2,\alpha} < \infty \text{ for all } \alpha > -1.$$

Moreover,

$$\sup_{a\in\mathbb{B}^n}||f_a f||_{\mathcal{B}} \approx ||M_f|| \approx \sup_{a\in\mathbb{B}^n}||g_a f||_{\mathcal{B}} \approx ||f||_{L\mathcal{B}} + ||f||_\infty.$$

Theorem 4.3. *Let f be an analytic function on $\mathbb{B}^n$. Then the following assertions are equivalent*:

(i) M_f *is bounded on $L\mathcal{B}$.*

(ii)

$$\sup_{a\in\mathbb{B}^n}||g_a f||_{L\mathcal{B}} < \infty,$$

where $g_a(z) = \log\log\frac{e^4}{1-\langle z,a\rangle}$, $z \in \mathbb{B}^n$.

(iii) $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ *and*

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)\left(\log\log\frac{e^4}{1-|a|}\right)||f\circ\varphi_a - f(a)||_{2,\alpha} < \infty \text{ for all } \alpha > -1.$$

Moreover,

$$\sup_{a\in\mathbb{B}^n}||g_a f||_{L\mathcal{B}} \approx ||M_f|| \approx B + ||f||_\infty,$$

$$B = \sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)\left(\log\log\frac{e^4}{1-|a|}\right)||f\circ\varphi_a - f(a)||_{2,\alpha}.$$

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