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On commuting operator exponentials, II

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Abstract. We prove that, under sufficient conditions on the spectra,

$$e^M e^N \subseteq e^N e^M \Rightarrow MN \subseteq NM,$$

where N is an unbounded normal operator and M is a bounded normal operator in the Hilbert space.

Keywords. Self-adjoint and normal operator; commuting normal operator exponentials; Borel functional calculus.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For $A, B \in B(\mathcal{H})$, the implication

$$AB = BA \Rightarrow e^A e^B = e^{A+B} = e^B e^A$$

is a standard result in operator theory (in fact, this is true in any Banach algebra (p. 268 of [8]). In the present paper we are concerned with the reverse implication, when A and B are normal operators. Commuting exponentials of bounded normal operators have been studied in [4], [5] and [9]. Here, we extend these results to the case where one of the operators is unbounded.

2. Commuting bounded normal exponentials

Theorem 1. *Let $A, B \in B(\mathcal{H})$ be self-adjoint operators. Then*

$$e^A e^B = e^B e^A \Rightarrow AB = BA. \tag{1}$$

Proof. Since $\sigma(A)$ and $\sigma(B)$ are contained in $\mathbb{R}$, the exponential function is one-to-one on the spectra of A and B. Now, the result follows from the spectral theorem for self-adjoint operators [2] (cf. Theorem 1 of [5]). □

Let $A \in B(\mathcal{H})$ be self-adjoint, and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ its spectral family. The spectrum, $\sigma(A)$ of A is called *generalized* 2π*-congruence free* if there exist a Borel subset Δ of $\sigma(A)$ such that

$$E(\Delta \cap (\Delta + 2k\pi)) = 0 \qquad \text{for } k = \pm 1, \pm 2, \ldots$$

and

$$E(\sigma(A)\backslash\Delta) = 0,$$

where E is the spectral measure of A, $E((-\infty, \lambda]) = E_\lambda$.

Theorem 2. *Let $N \in B(\mathcal{H})$ be a normal operator and $S \in B(\mathcal{H})$ a self-adjoint operator. Let $N = A + iB$ be the Cartesian decomposition of N, and E the spectral measure of B. If $\sigma(B)$ is generalized 2π-congruence free, then*

$$e^N e^S = e^S e^N \Rightarrow NS = SN. \tag{2}$$

Proof. The real and imaginary parts of N are the self-adjoint operators $A = \frac{1}{2}(N + N^*)$ and $B = \frac{1}{2i}(N - N^*)$, respectively. By the Borel functional calculus, e^N is also normal, and $(e^N)^* = e^{N^*}$. Taking adjoints in the first side of (2) we get $e^S e^{N^*} = e^{N^*} e^S$. Moreover, the normality of N implies

$$e^S e^A = e^S e^{\frac{1}{2}(N+N^*)} = e^S e^{\frac{1}{2}N} e^{\frac{1}{2}N^*} = e^{\frac{1}{2}N} e^{\frac{1}{2}N^*} e^S = e^A e^S.$$

From Theorem 1, we get $SA = AS$. Similarly,

$$e^S e^{iB} = e^S e^{\frac{1}{2}(N-N^*)} = e^S e^{\frac{1}{2}N} e^{-\frac{1}{2}N^*} = e^{\frac{1}{2}N} e^{-\frac{1}{2}N^*} e^S = e^{iB} e^S.$$

Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of B. Since $\sigma(B)$ is generalized 2π-congruence free, there is a Borel subset Δ of $\sigma(B)$ such that $E(\Delta \cap (\Delta + 2k\pi)) = 0$ for $k = \pm 1, \pm 2, \ldots$ and $E(\sigma(B)\backslash\Delta) = 0$. Then the bounded Borel function $f(t) = e^{it}$ is one-to-one on $\Omega = \Delta \backslash \bigcup_{k\neq 0}(\Delta \cap (\Delta + 2k\pi))$ and $E(\Omega) = I$. Using the Borel functional calculus as in Theorem 6 of [5], we get $SB = BS$. Thus,

$$SN = S(A + iB) = SA + iSB = AS + iBS = (A + iB)S = NS.$$

□

Theorem 3. *Let $N, M \in B(\mathcal{H})$ be normal operators with Cartesian decompositions $N = A + iB$ and $M = C + iD$. If $\sigma(B)$ and $\sigma(D)$ are generalized 2π-congruence free, then*

$$e^M e^N = e^N e^M \Rightarrow MN = NM. \tag{3}$$

Proof. Since $e^M e^N = e^N e^M$, Fuglede's theorem tells us that $e^M e^{N^*} = e^{N^*} e^M$. The same argument as above, gives $e^M e^A = e^A e^M$. From Theorem 2, it follows that $MA = AM$. Similarly, $e^M e^{iB} = e^{iB} e^M$. Since e^M is normal, Fuglede's theorem implies that $e^{M^*} e^{iB} = e^{iB} e^{M^*}$. So, as before, $e^C e^{iB} = e^{iB} e^C$ and $e^{iD} e^{iB} = e^{iB} e^{iD}$. The hypothesis on the spectra yields $CB = BC$ and $DB = BD$. Therefore $MB = BM$. Thus $MN = NM$. □

Let $A \in B(\mathcal{H})$ be a normal operator, and let E be its spectral measure. The spectrum, $\sigma(A)$ of A is called *generalized* $2\pi i$*-congruence free* if there exist a Borel subset Δ of $\sigma(A)$ such that

$$E(\Delta \cap (\Delta + 2k\pi i)) = 0 \qquad \text{for } k = \pm 1, \pm 2, \ldots$$

and

$$E(\sigma(A) \backslash \Delta) = 0.$$

Note that the generalized 2π-congruence free on the spectra of B and D in Theorem 3, is equivalent to the generalized $2\pi i$-congruence free hypothesis on $\sigma(N)$ and $\sigma(M)$. In addition, our hypothesis on the spectra of the operators B and D is weaker than the hypothesis $\sigma(B), \sigma(D) \subseteq (0, \pi)$ given in [4]. Indeed, if $\sigma(B) \subseteq (0, \pi)$, then $\sigma(B) \cap \sigma(B + 2k\pi) = \emptyset$, that is, $\sigma(B)$ is 2π*-congruence free*. But then $E(\sigma(B) \cap \sigma(B + 2k\pi)) = E(\emptyset) = 0$, which implies that $\sigma(B)$ is generalized 2π-congruence free by Remark 5 of [5]. Moreover note that the proof of Theorems 2 and 3 bypasses Berberian's result [1] on cramped operators, on which the proof of these results in [4] is based.

3. Commuting unbounded normal exponentials

We denote by $\text{Op}(\mathcal{H})$ the set of unbounded densely defined linear operators on $\mathcal{H}$. For $A \in \text{Op}(\mathcal{H})$ we denote the domain of A by $D(A)$. An operator $A \in \text{Op}(\mathcal{H})$ is called *closed* if whenever $x_n \in D(A)$ with $x_n \to x$ and with $Ax_n \to y$ it implies $x \in D(A)$ and $Ax = y$. It is easy to see that if $A \in \text{Op}(\mathcal{H})$ is closed and $T \in B(\mathcal{H})$, then AT is also closed. Let $A, B \in \text{Op}(\mathcal{H})$, we recall that B is called an *extension* of A, denoted by $A \subseteq B$, if $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$. A closed operator $A \in \text{Op}(\mathcal{H})$ is said to *commute* with $T \in B(\mathcal{H})$, if $TA \subseteq AT$, that is, for $x \in D(A)$, we have $Tx \in D(A)$ and $TAx = ATx$.

Let $\{A\}' = \{T \in B(\mathcal{H}) : TA \subseteq AT\}$. In case $A \in B(\mathcal{H})$ this notion agrees with the usual notion of *commutant*. It is easily seen that $\{A\}'$ is a strongly closed subalgebra of $B(\mathcal{H})$. Furthermore, $T \in \{A\}'$ if and only if $T^* \in \{A^*\}'$. Thus, $\{A\}' \cap \{A^*\}'$ is a von Neumann algebra.

DEFINITION 1

Let $A \in \text{Op}(\mathcal{H})$ be closed and $\mathcal{A}$ a von Neumann algebra. The operator A is said to be *affiliated* with $\mathcal{A}$, denoted by $A\eta\mathcal{A}$, if $\mathcal{A}' \subseteq \{A\}'$.

Note that, if $A \in B(\mathcal{H})$, the double commutant theorem tells us that $A\eta\mathcal{A}$ if and only if $A \in \mathcal{A}$. Note also that $A\eta\mathcal{A}$ if and only if $A^*\eta\mathcal{A}$. Equivalently, $A\eta\mathcal{A}$ if and only if $\mathcal{A}' \subseteq \{A\}' \cap \{A^*\}'$ or $\left\{\{A\}' \cap \{A^*\}'\right\}' \subseteq \mathcal{A}$. We denote by $W^*(A) = \left\{\{A\}' \cap \{A^*\}'\right\}'$. Clearly, $W^*(A)$ is the smallest von Neumann algebra with which A is affiliated and is referred to as *von Neumann algebra generated* by A.

DEFINITION 2

Let $A \in \text{Op}(\mathcal{H})$. A *bounding sequence* for A is a non-decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of projections on $\mathcal{H}$ such that $\bigvee_{n=1}^{\infty} F_n = I$, $F_nA \subseteq AF_n$ and $AF_n \in B(\mathcal{H})$ for all $n \in \mathbb{N}$, where $\bigvee_{n=1}^{\infty} F_n$ denotes the least upper bound of $\{F_n\}_{n\in\mathbb{N}}$.

We prove that for a closed operator $A \in \mathrm{Op}(\mathcal{H})$ affiliated with an abelian von Neumann algebra $\mathcal{A}$, we can always construct a bounding sequence $\{F_n\}_{n\in\mathbb{N}}$ for A belonging in $\mathcal{A}$. Before doing so, we require the following lemma.

Lemma 1. *Let $A \in \mathrm{Op}(\mathcal{H})$ be closed and $\mathcal{A}$ a von Neumann algebra. If $A\eta\mathcal{A}$, then $A^*A\eta\mathcal{A}$.*

Proof. Since A is closed, A^*A is self-adjoint (Theorem 2.7.8 of [3]) and hence closed. Let $T \in \mathcal{A}'$. Since $\mathcal{A}' \subseteq \{A\}' \cap \{A^*\}'$, we have $TA^*A \subseteq A^*TA \subseteq A^*AT$. Hence $\mathcal{A}' \subseteq \{A^*A\}'$, and so $A^*A\eta\mathcal{A}$. □

The following lemma is part of the statement of the spectral theorem for unbounded self-adjoint operators given in Theorem 4.3 of [6].

Lemma 2. *Let $A \in \mathrm{Op}(\mathcal{H})$ be self-adjoint. Then there is a (unique) spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ such that for each interval $(a, b]$ of $\mathbb{R}$, if $F_J = E_b - E_a$, then we have $F_J \in \{A\}''$, $\bigvee_{J\in J} F_J = I$ (where J denotes the set of half open intervals of $\mathbb{R}$ ordered by inclusion), and AF_J is a bounded self-adjoint operator.*

Theorem 4. *If $\mathcal{A}$ is an abelian von Neumann algebra and $A\eta\mathcal{A}$, then there is a bounding sequence $\{F_n\}$ for A such that $F_n \in \mathcal{A}$ and $AF_n \in \mathcal{A}$ for all $n \in \mathbb{N}$.*

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the self-adjoint operator A^*A and let $F_n = E_n - E_{-n}$ for each $n \in \mathbb{N}$. Then from Lemma 2, we have $F_n \in \{A^*A\}''$, $\bigvee_{n=1}^{\infty} F_n = I$, and $(A^*A)F_n \in B(\mathcal{H})$. Since $D(AF_n) \supseteq D(A^*AF_n) = \mathcal{H}$, the closed operator AF_n is defined everywhere, and the closed graph theorem implies that $AF_n \in B(\mathcal{H})$. Since $A^*A\eta\mathcal{A}$, it follows that $\{A^*A\}'' \subseteq \mathcal{A}'' = \mathcal{A}$. Hence, $F_n \in \mathcal{A}$. As $\mathcal{A}$ is abelian, we also have $F_nA \subseteq AF_n$ for all $n \in \mathbb{N}$. Finally, since $A\eta\mathcal{A}$ and $F_n \in \mathcal{A}$, if $T \in \mathcal{A}'$ it follows that $T \in \{A\}'$ and $TF_n = F_nT$. Therefore, $TAF_n \subseteq ATF_n = AF_nT$. That is, $TAF_n = AF_nT$, and so $AF_n \in \mathcal{A}'' = \mathcal{A}$. □

A closed operator $N \in \mathrm{Op}(\mathcal{H})$ is called *normal* if $N^*N = NN^*$. It turns out that the von Neumann algebra $W^*(N)$ generated by N is abelian, and $W^*(N) = \{N\}''$ [7]. Thus, given an unbounded normal operator N, there is a bounding sequence $\{F_n\}$ for N, with $F_n \in W^*(N)$.

Now, we are ready to state and prove the main theorem in this paper.

Theorem 5. *Let $N \in \mathrm{Op}(\mathcal{H})$ be normal and $M \in B(\mathcal{H})$ be normal. If $\sigma(M)$ and $\sigma(N)$ are generalized $2\pi i$-congruence free, then*

$$e^M e^N \subseteq e^N e^M \Rightarrow MN \subseteq NM. \tag{4}$$

Proof. Let $\{F_n\}$ be a bounding sequence for N. Since $\sigma(N)$ is generalized $2\pi i$-congruence free, the exponential function $f(\lambda) = e^\lambda$ is one-to-one on $\sigma(N)$. By the spectral mapping theorem $\sigma(e^N) = \overline{e^{\sigma(N)}}$. Hence f has a (Borel) inverse f^{-1} on $\sigma(e^N)$

such that $f^{-1} \circ f = id$, where $id(\lambda) = \lambda$ for all $\lambda \in \sigma(N)$. The composition property of the Borel functional calculus (Corollary 5.6.29 of [3]), yields

$$N = (f^{-1} \circ f)(N) = f^{-1}(f(N)) = f^{-1}(e^N).$$

Set $T = e^M$. Then $Te^N \subseteq e^N T$. Since e^N is normal and f^{-1} is a Borel function on $\sigma(e^N)$, the spectral theorem for unbounded normal operators (Theorem 4.11 of [2]) implies that $Tf^{-1}(e^N) \subseteq f^{-1}(e^N)T$, that is, $TN \subseteq NT$. Note that the latter also implies $TF_n = F_n T$ for all $n \in \mathbb{N}$.

Thus $e^M N \subseteq N e^M$. Multiplying by F_n, we have $e^M N F_n \subseteq N e^M F_n = N F_n e^M$. As $e^M N F_n \in B(\mathcal{H})$, it follows that $e^M N F_n = N F_n e^M$. Since NF_n is a bounded normal operator, the bounded case implies that $MNF_n = NF_n M$ for all $n \in \mathbb{N}$.

Now, for $x \in D(N)$ we have $F_n x \to x$, and since M is bounded, $MF_n x \to Mx$. Furthermore, $NF_n Mx = MNF_n x = MF_n Nx \to MNx$. Since N is closed, it follows that $Mx \in D(N)$ and $NMx = MNx$. Thus $MN \subseteq NM$. □

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