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Divisibility of class numbers of imaginary quadratic function fields by a fixed odd number

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Abstract. In this paper we find a new lower bound on the number of imaginary quadratic extensions of the function field $\mathbb{F}_q(x)$ whose class groups have elements of a fixed odd order. More precisely, for q, a power of an odd prime, and g a fixed odd positive integer ≥ 3, we show that for every $\epsilon > 0$, there are $\gg q^{L(\frac{1}{2}+\frac{3}{2(g+1)}-\epsilon)}$ polynomials $f \in \mathbb{F}_q[x]$ with $\deg f = L$, for which the class group of the quadratic extension $\mathbb{F}_q(x, \sqrt{f})$ has an element of order g. This sharpens the previous lower bound $q^{L(\frac{1}{2}+\frac{1}{g})}$ of Ram Murty. Our result is a function field analogue which is similar to a result of Soundararajan for number fields.

Keywords. Divisibility; class numbers; quadratic extensions; function fields.

1. Introduction

For a square-free integer D, let $\mathrm{Cl}(-D)$ denote the ideal class group of $\mathbb{Q}(\sqrt{-D})$, and let $h(-D) = \#\mathrm{Cl}(-D)$ denote the class number. In his 1801 *Disquisitiones Arithmeticae*, Gauss put forward the problem of finding all positive square-free D such that $h(-D)$ is some fixed number C. Heegner [15], Baker [5] and Stark [25] solved Gauss's problem completely for $C = 1$. Subsequently, Baker [6] and Stark [26] provided solutions to the case $C = 2$. Recently, Watkins [27] extended the range of the complete solutions to Gauss's problem for $C \leq 100$.

A related problem of interest is to determine the existence of g-torsion subgroups of $\mathrm{Cl}(-D)$ for positive integers g. Gauss studied the case $g = 2$. Davenport and Heilbronn [10] proved that the proportion of D with $3 \nmid h(-D)$ is at least $1/2$. For any g the infinitude of such fields was established by Nagell [21], Honda [17], Ankeny and Chowla [3], Hartung [16], Yamamoto [30] and Weinberger [28].

For a positive integer g, let $N_g(X)$ denote the number of positive square-free $D \leq X$ such that $g|h(-D)$. Gauss's genus theory (see [7]) demonstrates that $2|h(-D)$ whenever D is a product of at least two odd prime numbers. This, in particular, implies that $N_2(X) \sim$

$6X/\pi^2$. In general, it is believed that $N_g(X) \sim C_g X$ for some positive constant C_g. For odd primes g, Cohen and Lenstra [8] conjectured that

$$C_g = \frac{6}{\pi^2}\left(1 - \prod_{i=1}^{\infty}\left(1 - \frac{1}{g^i}\right)\right).$$

Ankeny and Chowla [3] were among the first to achieve an estimate for $N_g(X)$ for $g \geq 3$. Although they did not explicitly point this out, their method shows that for $g \geq 3$, $N_g(X) \gg X^{1/2}$. Recently, Murty [20] improved this lower bound to $N_g(X) \gg X^{\frac{1}{2}+\frac{1}{g}}$, which was subsequently sharpened by Soundararajan [24] who showed

$$N_g(X) \gg \begin{cases} X^{\frac{1}{2}+\frac{2}{g}-\epsilon}, & \text{if } g \equiv 0 \pmod 4 \\ X^{\frac{1}{2}+\frac{3}{g+2}-\epsilon}, & \text{if } g \equiv 2 \pmod 4. \end{cases}$$

For q, a power of an odd prime, we define $k := \mathbb{F}_q(x)$ to be the function field over the finite field $\mathbb{F}_q$ and $\mathcal{A} := \mathbb{F}_q[x]$, its ring of integers. For a square-free $f \in \mathcal{A}$, we will denote the quadratic field extension $k(\sqrt{f})$ by K, and its ring of integers $\mathcal{A}[\sqrt{f}]$ by $\mathcal{B}$. The function field analogue of the class number divisibility problem was initiated by Artin [4]. Friesen [13] constructed infinitely many polynomials $f \in \mathcal{A}$ of even degree such that the class groups for K have an element of order g where g is not divisible by q. In [19], Murty and Cardon proved that for $q \geq 5$ there are $\gg q^{L(\frac{1}{2}+\frac{1}{g})}$ polynomials $f \in \mathcal{A}$ with $\deg(f) \leq L$ such that the class groups for the quadratic extensions K have an element of order g, which is analogous to the result $N_g(X) \gg X^{\frac{1}{2}+\frac{1}{g}}$ of Murty [20]. Further, the lower bound of Murty and Cardon was extended by Pacelli [22] to $q^{L(\frac{1}{l}+\frac{1}{g})}$ for cyclic extensions $\mathbb{F}_q(x, \sqrt[l]{f})$ of $\mathbb{F}_q(x)$ where l is a prime dividing $q-1$. In [9], Chakraborty and Mukhopadhyay have shown that there are $\gg q^{L/2g}$ monic polynomials $f \in \mathcal{A}$ of even degree with $\deg(f) \leq L$ such that the ideal class group of the (real) quadratic extensions K have an element of order g. This is a function field analogue of Murty's result [20] $N_g(X) \gg X^{1/2g}$ for real quadratic number fields.

The case when $\deg f$ is odd is analogous to the case of an imaginary quadratic number field in which the prime at infinity ramifies and the unit group has rank 0. Recently, Merberg [18] used a function field analogue to the diophantine method of Soundararajan [24] for finding imaginary quadratic function fields whose class groups have elements of a given order. He further proved that if either $c = 4$, or c is any odd prime distinct from the characteristic, then there are infinitely many such fields whose class numbers are not divisible by c. Wong [29] gives a lower bound on the number of such pairwise distinct quadratic extensions whose class numbers are not divisible by c, in the case when c is an odd prime distinct from the characteristic. Precisely, he shows that if $L \geq 5$, then for any odd prime $c \nmid q$, there are at least $(\ln L)/(\ln 5) + 1$ pairwise coprime $D \in \mathbb{F}_q[x]$ which are square-free and of odd degree $\leq L$, such that c does not divide the class number of the imaginary quadratic fields $\mathbb{F}_q(x)(\sqrt{D})/\mathbb{F}_q(x)$.

Friedman and Washington [12] have studied the Cohen–Lenstra conjecture in the function field case, and Yu [31] has established the Cohen–Lenstra conjecture when the characteristic p of $\mathbb{F}_q$ tends to infinity for fixed discriminantal degree. For recent developments in this direction, the reader may refer to [1], [2] and [11]. In the present work, we follow the classical approach and obtain a lower bound on the number of imaginary

quadratic function fields whose class groups have an element of order g for any odd $g \geq 3$. Specifically, we prove the following:

Theorem 1. *Let $g \geq 3$ be a fixed positive odd integer. Let q be a power of an odd prime. For odd L, let $N_g(L)$ denote the number of square-free polynomials $f \in \mathbb{F}_q[x]$ with $\deg f \leq L$ such that the class group of the quadratic extension $\mathbb{F}_q(x, \sqrt{f})$ contain an element of order g. Then, for sufficiently large L we have*

$$N_g(L) \gg q^{L(\frac{1}{2}+\frac{3}{2(g+1)}-\epsilon)}.$$

We will work with polynomials f with $\deg f = L$. This, however does not affect the statement of our result. We will use ideas from [24] to achieve our result. From our construction of the quadratic extensions of $\mathbb{F}_q(x)$ it will become evident that the case when $g \equiv 0 \pmod 4$ cannot be handled by our method. However, we remark that by a straightforward group theoretic argument and Theorem 1, a new lower bound when $g \equiv 2 \pmod 4$ can be achieved if one can first settle the function field analogue of Gauss's genus theory.

For basic function field related concepts, we refer the reader to [23]. We will denote by $\mathbb{F}_q^\times$ the multiplicative group of non-zero elements in $\mathbb{F}_q$. For an integer U, we let $\pi(U)$ count the number of irreducible monic polynomials of degree U. For a $f \in \mathcal{A}$, define the norm $|f|$ of f as $|f| := q^{\deg f}$, and let $\operatorname{sgn}(f)$ denote the leading coefficient of f. Let the Möbius function $\mu(f)$ be 0 if f is not square-free, and $(-1)^t$ if f is constant times a product of t distinct irreducible monic polynomials in $\mathcal{A}$. We will let $d(f)$ denote the number of distinct monic divisors of f (including $f/\operatorname{sgn} f$). We further define the Euler function $\phi(f)$ to be the order of the unit group $(\mathcal{A}/f\mathcal{A})^\times$ of the ring $\mathcal{A}/f\mathcal{A}$. It can be verified that

$$\phi(f) = |f| \prod_{p|f} \left(1 - \frac{1}{|p|}\right),$$

where the product is taken over irreducible monic polynomials. For a, b in $\mathcal{A}$, the symbol (a, b) will denote the greatest common monic divisor of a and b, and $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol whenever relevant. For functions F and G, we will use the notation $F \asymp G$ whenever $F \gg\ll G$. Finally, we would like to point out to the reader that the ϵ's appearing at different places are different.

We prove our result by first giving a criteria for the existence of elements of order g in $\mathrm{Cl}(f)$, the class group of K. This will be achieved in § 2. In order to obtain the lower bound in the theorem, we need to count the number of square-free f meeting the divisibility criteria. We will do this in § 3. Sections 4 and 5 provide the technical details needed in § 3. The last section contains the conclusion of the proof.

2. A divisibility criteria for the class number of $\mathbb{F}_q(x, \sqrt{f})$

For an element $c + d\sqrt{f}$ in $\mathcal{B} = A[\sqrt{f}]$, with c, d in $\mathcal{A}$, define the norm $N(c + d\sqrt{f})$ of $c + d\sqrt{f}$ as

$$N(c + d\sqrt{f}) = (c + d\sqrt{f})(c - d\sqrt{f}) = c^2 - d^2 f \in \mathcal{A}.$$

For an ideal $\mathfrak{v}$ in $\mathcal{B}$, we consider the ideal $\mathfrak{u}$ in $\mathcal{A}$ generated by the set $\{N(a) : a \in \mathfrak{v}\}$. Since $\mathcal{A}$ is a principal ideal domain, the ideal $\mathfrak{u}$ is principal, say $\mathfrak{u} = (a)$, where $a \in \mathcal{A}$. We define the norm $N(\mathfrak{v})$ of the ideal $\mathfrak{v}$ to be $N(\mathfrak{v}) := |a| = q^{\deg a}$, where $|\cdot|$ is the usual norm of an element in $\mathcal{A}$ as defined earlier. We note that for a principal ideal $(c + d\sqrt{f})$ in $\mathcal{B}$, $N((c + d\sqrt{f})) = |N(c + d\sqrt{f})| = |c^2 - d^2 f| = q^{\deg(c^2 - d^2 f)}$.

In the following proposition, we construct quadratic extensions of k whose class groups contain an element of order g.

PROPOSITION 1

Let $g \geq 3$ be an odd integer. Let $f \in \mathcal{A}$ be a square-free polynomial of odd degree. If there exist nonzero $m, n, t \in \mathcal{A}$ such that $t^2 f = n^2 - m^g$ with $(m, n) = 1$ and $\deg m^g > \max\{\deg n^2, \deg t^4\}$, then the class group for K has an element of order g.

Proof. Suppose m, n and t exist as in the lemma. Rewriting $t^2 f = n^2 - m^g$ as $m^g = n^2 - t^2 f$, we see that the ideal $(m)^g$ factors in $\mathcal{B}$ as

$$(m)^g = (n + t\sqrt{f})(n - t\sqrt{f}).$$

We note that any common divisor $\mathfrak{d}$ of the ideals $(n + t\sqrt{f})$ and $(n - t\sqrt{f})$ contains $2n$. As 2 is a unit in $\mathcal{A}$, we deduce that $n \in \mathfrak{d}$. On the other hand, $\mathfrak{d}$ also contains m^g, but $(m^g, n) = 1$. Thus $\mathfrak{d} = \mathcal{B}$, that is the ideals $(n + t\sqrt{f})$ and $(n - t\sqrt{f})$ are co-prime in $\mathcal{B}$. Thus there exist ideals $\mathfrak{a}$ and $\mathfrak{a}'$ in $\mathcal{B}$ such that $(n + t\sqrt{f}) = \mathfrak{a}^g$ and $(n - t\sqrt{f}) = \mathfrak{a}'^g$.

We claim that the ideal class of $\mathfrak{a}$ has order g. Assume otherwise that there is a positive integer $r < g$ such that $\mathfrak{a}^r$ is principal, say $\mathfrak{a}^r = (u + v\sqrt{f})$ for some $u, v \in \mathcal{A}$. It is clear that $r|g$. Taking norms we have $N(\mathfrak{a})^r = q^{\deg(u^2 - v^2 f)}$. We also have $(n + t\sqrt{f}) = (u + v\sqrt{f})^{g/r}$. Since $t \neq 0$, it immediately follows that $v \neq 0$. Thus $v^2 f \neq 0$ has odd degree, and since u^2 has even degree, $\deg(u^2 - v^2 f) \geq \deg f$. Therefore $N(\mathfrak{a})^r = q^{\deg(u^2 - v^2 f)} \geq q^{\deg f}$. On the other hand,

$$N(\mathfrak{a})^g = q^{\deg(n^2 - t^2 f)} = q^{\deg m^g} = q^{g \deg m}.$$

Thus $N(\mathfrak{a}) = q^{\deg m}$.

Now from $q^{r \deg m} = N(\mathfrak{a})^r \geq q^{\deg f}$ we see that

$$r \deg m \geq \deg f = \deg\left(\frac{n^2 - m^g}{t^2}\right) = g \deg m - 2 \deg t. \tag{1}$$

The last equality above follows from our assumption that $\deg m^g > \max\{\deg n^2, \deg t^4\}$. Rearranging terms in inequality (1), we have $\deg m \leq \frac{2 \deg t}{g - r}$. But from our assumption that $\deg m^g > \deg t^4$, it now follows that

$$\frac{4 \deg t}{g} < \deg m \leq \frac{2 \deg t}{g - r},$$

giving rise to $\frac{g}{r} < 2$, thereby contradicting the fact that $r|g$ since $g \geq 3$. This proves our claim and hence the proposition. □

3. Counting square-free f

In this section we shall obtain a lower bound on the number of square-free $f \in \mathcal{A}$ meeting the criteria of Proposition 1. The bound obtained in this section will depend on some parameter T to be determined in § 6 (see eq. (22)).

Thus we will be interested in counting the number of square-free polynomials $f \in \mathcal{A}$ satisfying

$$n^2 - m^g = t^2 f, \quad (m, n) = 1 \quad \text{and} \quad \deg m^g > \max\{n^2, t^4\}. \tag{2}$$

Let $\deg m = M$, $\deg n = N$, $\deg t = T$ and $\deg f = L$. In view of Proposition 1, we assume that

$$T < L/2, \quad Mg = 2T + L \quad \text{and} \quad N = T + \frac{L}{2} - 1. \tag{3}$$

From the above choice of M, N and T it follows that

$$Mg > \max\{2N, 4T\},$$

that is $\deg m^g > \max\{n^2, t^4\}$. Thus if f admits a solution to (2), then by Proposition 1, $\mathrm{Cl}(f)$ has an element of order g.

Let $N_g(L, T)$ count the number of square-free f with $\deg f = L$ satisfying (2). For a square-free polynomial $f \in \mathcal{A}$ of degree L, let $\mathcal{R}(f)$ denote the number of solutions in monic m, n and t to (2). If we define the characteristic function $\chi(f)$ as

$$\chi(f) = \begin{cases} 0, & \text{if } \mathcal{R}(f) = 0, \\ 1, & \text{if } \mathcal{R}(f) \neq 0, \end{cases}$$

then we can write $N_g(L, T)$ as

$$N_g(L, T) = \sum_{\deg f = L} \chi(f).$$

By the Cauchy–Schwarz inequality, we have

$$\Big(\sum_{\deg f = L} \chi(f)^2 \Big) \Big(\sum_{\deg f = L} \mathcal{R}(f)^2 \Big) \geq \Big(\sum_{\deg f = L} \chi(f) \mathcal{R}(f) \Big)^2,$$

which can be rewritten as

$$N_g(L, T) \geq \Big(\sum_{\deg f = L} \mathcal{R}(f) \Big)^2 \Big(\sum_{\deg f = L} \mathcal{R}(f)^2 \Big)^{-1}. \tag{4}$$

Thus, in order to determine a lower bound on $N_g(L, T)$, we need to establish a lower bound on $(\sum_{\deg f = L} \mathcal{R}(f))^2$ and an upper bound on $\sum_{\deg f = L} \mathcal{R}(f)^2$.

In the next section we will obtain the lower bound on $(\sum_{\deg f = L} \mathcal{R}(f))^2$ by establishing the following lemma.

Lemma 1. $\sum_{\deg f=L} \mathcal{R}(f) \asymp q^{M+N-T}$.

By a counting argument, we will show in § 5 the following lemma.

Lemma 2. $\sum_{\deg f=L} \mathcal{R}(f)(\mathcal{R}(f)-1) \ll q^{\epsilon L+2M+2T}$ *for every* $\epsilon > 0$ *and* $L \gg_\epsilon 0$.

Below we demonstrate how Lemmas 1 and 2 give a lower bound on $N_g(L,T)$. Observe that

$$\sum_{\deg f=L} \mathcal{R}(f)^2 = \sum_{\deg f=L} \mathcal{R}(f)(\mathcal{R}(f)-1) + \sum_{\deg f=L} \mathcal{R}(f) \ll q^{M+N-T} + q^{\epsilon L+2M+2T}.$$

Thus

$$\sum_{\deg f=L} \mathcal{R}(f)^2 \ll q^{\epsilon L+2M+2T} \tag{5}$$

provided

$$M+N-T \le \epsilon L+2M+2T. \tag{6}$$

Therefore, from (4), (5) and Lemma 1 we have

$$N_g(L,T) \gg \frac{q^{2(M+N-T)}}{q^{\epsilon L+2M+2T}} = q^{2N-4T-\epsilon L}.$$

Putting the value of N from (3) we get

$$N_g(L,T) \gg q^{L-2T-2-\epsilon L} \gg q^{L-2T-\epsilon L}. \tag{7}$$

The inequality in (6) and the lower bound in Theorem 1 will be achieved by suitably choosing the parameter T in § 6.

4. Proof of Lemma 1

Let $(m,n,t) \in \mathcal{A}^3$ be a tuple of pairwise relatively prime monic polynomials with $\deg m = M$, $\deg n = N$ and $\deg t = T$ satisfying $n^2 \equiv m^g \pmod{t^2}$, and M, N and T are as in (3). We define sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ of such tuples $(m,n,t) \in \mathcal{A}^3$ as follows:

$$\mathcal{S}_1 = \left\{(m,n,t) : p^2 \nmid \frac{n^2-m^g}{t^2} \text{ for all monic primes } p \text{ with } \deg p \le \log L\right\},$$

$$\mathcal{S}_2 = \left\{(m,n,t) : p^2 \mid \frac{n^2-m^g}{t^2} \text{ for some monic primes } p \text{ with } \log L < \deg p \le Q\right\}$$

and

$$\mathcal{S}_3 = \left\{(m,n,t) : p^2 \mid \frac{n^2-m^g}{t^2} \text{ for some monic primes } p \text{ with } Q < \deg p\right\}.$$

Here logarithms are taken to the base q, and Q is some real parameter to be described below.

Let $N_i = |\mathcal{S}_i|$ for $i = 1, 2, 3$. Note that $N_g(L, T) \geq N_1 - N_2 - N_3$. Thus in order to obtain a lower bound on $N_g(L, T)$, we would want N_1 to be large compared to $N_2 + N_3$. In other words, the sum we desire is $N_1 + O(N_2 + N_3)$. We shall show below that by optimally choosing $Q := (L - T + 2\log L)/3$, one obtains

$$N_1 \asymp q^{M+N-T} + o(q^{M+\frac{L}{3}+\frac{2T}{3}}),$$
$$N_2 \ll q^{M+N-T}/L + o(q^{M+\frac{L}{3}+\frac{2T}{3}})$$

and

$$N_3 = o(q^{M+\frac{L}{3}+\frac{2T}{3}}),$$

where q is fixed in the above $o(\cdot)$ notation. Observe that for $L > 4T + 6$, it follows from (3) that $M + N - T \geq M + (L/3) + (2T/3)$, and hence $N_1 \asymp q^{M+N-T}$, and N_2, N_3 are small. The choice of T as in eq. (22), and by taking $L > 2(g + 1)$, it is ensured that $L > 4T + 6$. Thus it follows that

$$\sum_{\deg f = L} R(f) \asymp q^{M+N-T}.$$

Estimation of N_1. For a fixed monic m and t with $\deg m = M$ and $\deg t = T$, we count the number of monic polynomials n with $\deg n = N$ such that $n^2 \equiv m^g (\mathrm{mod}\ t^2)$, and p^2 does not divide $\frac{n^2 - m^g}{t^2}$ for all irreducible monic p with $\deg p \leq \log L$.

Let $\rho_m(l)$ denote the number of solutions (mod l) to the congruence $n^2 \equiv m^g (\mathrm{mod}\ l)$. It is worth noting that $\rho_m(l)$ is a multiplicative function of l, and if $p \nmid m$ is irreducible, then for $\alpha \geq 1$ one has

$$\rho_m(p^\alpha) = \rho_m(p) = 1 + \left(\frac{m^g}{p}\right) = 1 + \left(\frac{m}{p}\right), \tag{8}$$

as g is odd.

Set $P = \prod_{\deg p \leq \log L} p$, where the product is taken over all irreducible monic polynomials p. Thus, the sum $\sum_{l^2 | (f, P^2)} \mu(l) = 1$ or 0 depending on whether $p^2 \nmid f$ for all p with $\deg p \leq \log L$ or not. Here l is assumed to be monic. Thus in order to estimate N_1, the sum over n (with m and t fixed), what we seek is

$$\sum_{\substack{\deg n = N \\ n^2 \equiv m^g (\mathrm{mod}\ t^2) \\ (n,m)=1}} \sum_{l^2 | \left(\frac{n^2 - m^g}{t^2}, P^2\right)} \mu(l) = \sum_{\substack{l | P \\ (l,m)=1}} \mu(l) \sum_{\substack{\deg n = N \\ n^2 \equiv m^g (\mathrm{mod}\ l^2 t^2)}} 1. \tag{9}$$

If $N \geq \deg l^2 t^2$, then for fixed l we have

$$\sum_{\substack{\deg n = N \\ n^2 \equiv m^g (\mathrm{mod}\ l^2 t^2)}} 1 = \frac{q^N}{|l^2 t^2|} \rho_m(l^2 t^2) = \frac{q^{N-2T} \rho_m(l^2 t^2)}{|l^2|},$$

while if $N \leq \deg l^2 t^2$ then

$$\sum_{\substack{\deg n = N \\ n^2 \equiv m^g (\mathrm{mod}\ l^2 t^2)}} 1 \leq \rho_m(l^2 t^2).$$

Thus the sum in (9) is

$$\begin{aligned}
&\sum_{\substack{\deg n=N\\ n^2\equiv m^g \pmod{t^2}\\ (n,m)=1}} \sum_{l^2\mid\left(\frac{n^2-m^g}{t^2},P^2\right)} \mu(l)\\
&\quad= \sum_{\substack{l\mid P\\ (l,m)=1}} \mu(l)\frac{q^N}{|l^2t^2|}\rho_m(l^2t^2)+O\left(\sum_{\substack{l\mid P\\ (l,m)=1}}\rho_m(l^2t^2)\right)\\
&\quad= q^{N-2T}\rho_m(t^2)\sum_{\substack{l\mid P\\ (l,m)=1}}\frac{\mu(l)}{|l|^2}\rho_m(l/(l,t))+O\left(\sum_{\substack{l\mid P\\ (l,m)=1}}\rho_m(l^2t^2)\right),
\end{aligned}$$

which can be written as

$$q^{N-2T}\rho_m(t^2)\prod_{\substack{p\mid P\\ p\text{-monic}\\ (p,m)=1}}\left(1-\frac{\rho_m\big(p/(p,t)\big)}{|p|^2}\right)+O\left(\sum_{\substack{l\mid P\\ (l,m)=1}}\rho_m(l^2t^2)\right), \tag{10}$$

where the product is taken over irreducible monic polynomials p.

We trivially see that

$$\prod_{\substack{p\mid P\\ p\text{-monic}\\ (p,m)=1}}\left(1-\frac{\rho_m\big(p/(p,t)\big)}{|p|^2}\right)<1.$$

Also, it can be seen from $\rho_m(p/(p,t))=1+\left(\frac{m}{p}\right)\le 2$ that

$$\begin{aligned}
\prod_{\substack{p\mid P\\ p\text{-monic}\\ (p,m)=1}}\left(1-\frac{\rho_m(p/(p,t))}{|p|^2}\right) &\ge \prod_{\substack{\text{all } p\\ p\text{-monic}}}\left(1-\frac{2}{|p|^2}\right)\\
&= \prod_{\substack{\text{all } p\\ p\text{-monic}}}\left(1-\frac{1}{|p|^2}\right)^2\left(1+\frac{1}{|p|^2(|p|^2-2)}\right)^{-1}.
\end{aligned}$$

Now, for $x>2$ we have

$$\left(1+\frac{1}{x(x-2)}\right)=\frac{(x-1)^2}{x(x-2)}\le\frac{x^2}{x(x-1)}=\left(1-\frac{1}{x}\right)^{-1}.$$

Since $|p|>2$, we have

$$\prod_{\substack{p\mid P\\ p\text{-monic}\\ (p,m)=1}}\left(1-\frac{\rho_m(p/(p,t))}{|p|^2}\right)\ge\prod_{\substack{\text{all } p\\ \text{p-monic}}}\left(1-\frac{1}{|p|^2}\right)^3=\zeta_{\mathcal{A}}(2)^{-3}=\left(1-\frac{1}{q}\right)^3.$$

We have used $\zeta_{\mathcal{A}}(s) = \frac{1}{1-q^{1-s}}$ above. This may easily be derived by looking at the series expansion of $\zeta_{\mathcal{A}}(s)$ (see [23]). Therefore the main term in (10) is $\asymp q^{N-2T}\rho_m(t^2)$. For the error term in (10), we first note from (8) that

$$\rho_m(l^2t^2) = \rho_m(lt) = \prod_{p|lt}\rho_m(p) = \prod_{p|lt}\left(1+\left(\frac{m}{p}\right)\right) \leq \prod_{p|lt} 2 \leq d(lt).$$

As l^2t^2 divides $n^2 - m^g$, we have from (3) that

$$2\deg l + 2\deg t \leq Mg = L + 2T = L + 2\deg t.$$

Therefore $\deg l \leq L/2$. Also from (3) we have $\deg t = T < L/2$. Hence $\deg lt \leq L$.

For a polynomial $r(x) \in \mathcal{A}$ with $\deg r \leq X$, it is an easy exercise to show that $d(r) = O(q^{\epsilon X})$, where the O-constant depends on ϵ only (see pages 260–262 of [14] for the classical divisor function). Therefore,

$$\rho_m(l^2t^2) \leq d(lt) = O(q^{\epsilon L}). \tag{11}$$

Thus the error term in (10) is $O(d(P)q^{\epsilon L})$. Now,

$$d(P) = 2^{\pi(1)+\pi(2)+\cdots+\pi(\log L)} \leq 2^{q+\frac{q^2}{2}+\cdots+\frac{q^{\log L}}{\log L}} \ll 2^{\frac{L}{\log L}},$$

for all sufficiently large L. Here we have used that $\pi(U) \leq q^U/U$ for all $U \in \mathbb{N}$ (see Proposition 2.1 of [23]). Thus the error term in (10) is $O(q^{\epsilon L})$. Therefore, the sum in (9) is

$$\asymp q^{N-2T}\rho_m(t^2) + O(q^{\epsilon L}).$$

Now, summing over all monic m with $\deg m = M$, and monic t with $\deg t = T$ we have

$$N_1 \asymp q^{N-2T} \sum_{\substack{\deg m = M \\ \deg t = T}} \rho_m(t^2) + O(q^{\epsilon L + M + T}). \tag{12}$$

We now show that the error term in (12) is $o(q^{M+\frac{L}{3}+\frac{2T}{3}})$. We choose $0 < \delta < \frac{1}{2}$ so that $q^{L/2} = o(q^{L(1-\delta)})$. Since we have $T < L/2$ from (3), hence $q^T < q^{L/2} = o(q^{L(1-\delta)})$.

Taking $\epsilon = \frac{\delta}{3}$, we have $q^{T/3} = o(q^{L/3}q^{-\epsilon L})$, that is $q^{\epsilon L} = o(q^{L/3}q^{-T/3})$.

Thus from (12) we have

$$N_1 \asymp q^{N-2T} \sum_{\substack{\deg m = M \\ \deg t = T}} \rho_m(t^2) + o(q^{M+\frac{L}{3}+\frac{2T}{3}}). \tag{13}$$

We next show that

$$\sum_{\substack{\deg m = M \\ \deg t = T}} \rho_m(t^2) \asymp q^{M+T}.$$

In order to prove this result we will need a couple of lemmas. The following lemma is an easy exercise (see Ex. 12, page 20 of [23]).

Lemma 3. *For an integer $U \geq 2$, we have*

$$\sum_{\substack{y\text{-monic} \\ \deg y = U}} \mu(y) = 0.$$

The next lemma is based upon Lemma 17.10, Proposition 17.11 and Proposition 17.12 of [23] which we state without proof as follows.

Lemma 4. *Suppose* $b \notin \mathbb{F}_q^{\times}$ *is not a square in* $\mathcal{A}$, *and let* $\deg b = B$. *Then*

(i) *for* $D \geq B$,

$$\sum_{\substack{a\text{-monic} \\ \deg a = D}} \left(\frac{b}{a}\right) = 0.$$

(ii) *For* $1 \leq D \leq B-1$,

$$\sum_{\substack{b\text{-monic} \\ \deg b = B}} \sum_{\substack{a\text{-monic} \\ \deg a = D}} \left(\frac{b}{a}\right) = (q-1)\Phi(D/2, M),$$

where

$$\Phi(D/2, M) = \begin{cases} \left(1 - \frac{1}{q}\right) q^{M+D/2}, & \text{if } D \equiv 0 \pmod 2, \\ 0, & \text{if } D \equiv 1 \pmod 2. \end{cases}$$

We are now ready to estimate the average value of $\rho_m(t^2)$.

Lemma 5. *Assume that* m *and* $t \in A$ *are monic and relatively prime. Then we have*

$$\sum_{\deg m = M} \sum_{\deg t = T} \rho_m(t^2) = q^{M+T} + O(q^{M/2+T}).$$

Proof. Since $\rho_m(\cdot)$ is multiplicative and $\rho_m(p^{\alpha}) = \rho_m(p)$ for any irreducible $p \in \mathcal{A}$ and $\alpha \geq 1$, we have the following product to sum formula for $\rho_m(t^2)$.

$$\rho_m(t^2) = \rho_m(t) = \prod_{p \mid t} \left(1 + \left(\frac{m}{p}\right)\right) = \sum_{d \mid t} \mu^2(d) \left(\frac{m}{d}\right).$$

We derive our result by showing that the main contribution in the above sum comes from $d = 1$. For $d = 1$, the sum over t, we are interested in

$$\begin{aligned} \sum_{\substack{\deg t = T \\ (t,m)=1}} 1 &= \sum_{\substack{\deg t = T \\ s \mid t}} \sum_{s \mid m} \mu(s) = \sum_{s \mid m} \mu(s) \sum_{\substack{\deg t = T \\ s \mid t}} 1 \\ &= \sum_{s \mid m} \mu(s) \sum_{\substack{l \\ ls = t}} 1 = \sum_{s \mid m} \mu(s) \sum_{\substack{l \\ \deg l = T - \deg s}} 1 \\ &= \sum_{s \mid m} \mu(s) q^{T - \deg s} = q^T \prod_{p \mid m} \left(1 - \frac{1}{q^{\deg p}}\right) \\ &= q^T \frac{\phi(m)}{|m|} = q^{T-M} \phi(m). \end{aligned}$$

Now summing over m, and using Proposition 2.7 of [23] we have

$$q^{T-M} \sum_{\deg m=M} \phi(m) = q^{T-M} \cdot q^{2M}\left(1 - \frac{1}{q}\right).$$

Thus the contribution from $d = 1$ is indeed $\asymp q^{M+T}$.

We next demonstrate that the contribution from $d \neq 1$ is $O(q^{M/2+T})$. The sum we seek to bound is

$$\sum_{\deg m=M} \sum_{\substack{\deg t=T \\ (t,m)=1}} \sum_{\substack{d|t \\ d\neq 1}} \mu^2(d)\left(\frac{m}{d}\right).$$

Let us denote $\deg d$ by Z. We split the above sum into $1 \leq Z \leq M$ and $Z \geq M+1$, where $M = \deg m$. The sum corresponding to $1 \leq Z \leq M$ (after changing the order of summation) is

$$\sum_{\substack{\deg t=T \\ (t,m)=1}} \sum_{\substack{d|t \\ Z\leq M}} \mu^2(d) \sum_{\deg m=M} \left(\frac{m}{d}\right).$$

Observe that if d is a square then $\mu^2(d) = 0$, and if d is not a square, then from quadratic reciprocity law we have

$$\left(\frac{m}{d}\right)\left(\frac{d}{m}\right) = (-1)^{\frac{q-1}{2}(\deg m)(\deg d)} \operatorname{sgn}(m)^{\deg d} = (-1)^{\frac{q-1}{2}MZ}.$$

Since $d \neq 1$, Lemma 4 implies that

$$\sum_{\deg m=M} \left(\frac{m}{d}\right) = (-1)^{\frac{q-1}{2}MZ} \sum_{\deg m=M} \left(\frac{d}{m}\right) = 0$$

for $\deg d = Z \leq M$. So the sum over $1 \leq Z \leq M$ is 0.

Consider the sum over $Z \geq M+1$,

$$\begin{aligned}
&\sum_{\deg m=M} \sum_{\substack{\deg t=T \\ (t,m)=1}} \sum_{\substack{d|t \\ M+1\leq Z\leq T}} \mu^2(d)\left(\frac{m}{d}\right) \\
&\quad = \sum_{\deg m=M} \sum_{M+1\leq Z\leq T} \sum_{\substack{\deg d=Z \\ (d,m)=1}} \mu^2(d)\left(\frac{m}{d}\right) q^{T-Z} \\
&\quad = q^T \sum_{M+1\leq Z\leq T} q^{-Z} \sum_{\deg m=M} \sum_{\substack{\deg d=Z \\ (d,m)=1}} \mu^2(d)\left(\frac{m}{d}\right).
\end{aligned}$$

Since $\left(\frac{m}{d}\right) = 0$ when $(d, m) \neq 1$, we can ignore the condition $(d, m) = 1$ in the above summation. Let us denote the inner sum by

$$S := \sum_{\deg m=M} \sum_{\deg d=Z} \mu^2(d)\left(\frac{m}{d}\right).$$

We write $d = l^2 s$. Further without loss of generality, we assume that l and s are monic. Observe that for monic d and m we have by quadratic reciprocity law that

$$\Big(\frac{m}{d}\Big)\Big(\frac{d}{m}\Big) = (-1)^{\frac{q-1}{2}(\deg m)(\deg d)} = (-1)^{\frac{q-1}{2}MZ}.$$

Noting that $d = l^2 s$ we have from above that

$$\Big(\frac{m}{d}\Big)\Big(\frac{s}{m}\Big) = (-1)^{\frac{q-1}{2}MZ}.$$

Similarly, for monic m and s we have

$$\Big(\frac{m}{s}\Big)\Big(\frac{s}{m}\Big) = (-1)^{\frac{q-1}{2}(\deg m)(\deg s)} = (-1)^{\frac{q-1}{2}M(Z-2\deg l)} = (-1)^{\frac{q-1}{2}MZ},$$

since q is odd. Therefore, $\Big(\frac{m}{d}\Big) = \Big(\frac{m}{s}\Big)$. Now using $\sum_{l^2|d}\mu(d) = \mu^2(d)$, we have

$$\begin{aligned} S &= \sum_{\deg m = M}\ \sum_{\deg d = Z}\ \sum_{l^2|d} \mu(l)\Big(\frac{m}{s}\Big) \\ &= \sum_{\deg m = M}\ \sum_{\deg l \le \frac{Z}{2}} \mu(l) \sum_{\deg s = Z-2\deg l} \Big(\frac{m}{s}\Big) \end{aligned}$$

If $\deg l = Z/2$, then $s = 1$. For such l, the corresponding contribution in S is

$$\sum_{\deg m = M}\ \sum_{\deg l = \frac{Z}{2}} \mu(l).$$

For $Z \ge 2$, the sum $\sum_{\deg l = \frac{Z}{2}} \mu(l)$ is zero by Lemma 3. Since $Z \ge M+1 > 2$, we deduce that the contribution in S corresponding to $s = 1$ is 0. Therefore,

$$\begin{aligned} S &= \sum_{\deg m = M}\ \sum_{\deg l < \frac{Z}{2}} \mu(l) \sum_{\substack{\deg s = Z-2\deg l \\ s \ne 1}} \Big(\frac{m}{s}\Big) \\ &= \sum_{\deg l < \frac{Z}{2}} \mu(l) \sum_{\deg m = M} \sum_{\substack{\deg s = Z-2\deg l \\ s \ne 1}} \Big(\frac{m}{s}\Big), \end{aligned}$$

which is

$$\le \sum_{\deg l < \frac{Z}{2}} \Bigg| \sum_{\deg m = M} \sum_{\substack{\deg s = Z-2\deg l \\ s \ne 1}} \Big(\frac{m}{s}\Big) \Bigg|. \tag{14}$$

Observe that since m satisfies equation (2), and since we have assumed that $\deg f$ and g are odd in (2), m cannot be a square in $\mathcal{A}$. Also $\deg m = M > 1$ implies that $m \notin \mathbb{F}_q^{\times}$. Thus appealing to the first part of Lemma 4 we deduce that if $M \le Z - 2\deg l$, then

$$\sum_{\substack{\deg s = Z-2\deg l \\ s \ne 1}} \Big(\frac{m}{s}\Big) = 0,$$

while if $M \geq Z - 2\deg l$, then from the second part of Lemma 4 we have

$$\sum_{\deg m = M} \sum_{\substack{\deg s = Z - 2\deg l \\ s \neq 1}} \left(\frac{m}{s}\right) \leq \left(1 - \frac{1}{q}\right) q^{\frac{Z}{2} - \deg l + M}.$$

Summing over l in (14) we deduce that $S \leq q^{M + \frac{Z}{2}}$. Thus the contribution from $d \neq 1$ is less than

$$q^{M+T} \sum_{Z \geq M+1} q^{-Z/2} = q^{M+T} q^{-\frac{M+1}{2}} \left(1 - \frac{1}{\sqrt{q}}\right)^{-1} = O(q^{M/2+T}).$$

This completes the proof of the lemma. □

As an immediate consequence of Lemma 5, from (13) we have

$$N_1 \asymp q^{M+N-T} + o(q^{M + \frac{L}{3} + \frac{2T}{3}}).$$

Estimation of N_2. In order to estimate N_2, once again, we fix m and t and count the number of n with $\deg n = N$ such that $\frac{n^2 - m^g}{t^2}$ divisible by p^2 for some prime p with $\log L < \deg p \leq Q = \frac{L - T + 2\log L}{3}$. Therefore the sum over n that we seek is

$$\sum_{\log L < \deg p \leq Q} \sum_{\substack{\deg n = N \\ n^2 \equiv m^g \pmod{p^2 t^2}}} 1. \tag{15}$$

Following the same line of argument as in the estimation of N_1 we deduce that the sum in (15) is equal to

$$\sum_{\log L < \deg p \leq Q} \left(\frac{q^N \rho_m(p^2 t^2)}{|p^2 t^2|} + O(\rho_m(p^2 t^2))\right). \tag{16}$$

Since $\rho_m(p/(p,t)) \leq 2$, the main term in (16) is

$$\begin{aligned}
&q^{N-2T} \rho_m(t^2) \sum_{\log L < \deg p \leq Q} \frac{\rho_m(p/(p,t))}{|p|^2} \\
&\leq q^{N-2T} \rho_m(t^2) \sum_{\log L \leq \deg p \leq Q} \frac{2}{|p|^2} = 2q^{N-2T} \rho_m(t^2) \sum_{Y = \log L}^{Q} \sum_{\deg p = Y} \frac{1}{|p|^2} \\
&= 2q^{N-2T} \rho_m(t^2) \sum_{Y = \log L}^{Q} q^{-2Y} \sum_{\deg p = Y} 1 = 2q^{N-2T} \rho_m(t^2) \sum_{Y = \log L}^{Q} q^{-2Y} \pi(Y) \\
&\leq 2q^{N-2T} \rho_m(t^2) \sum_{Y = \log L}^{Q} q^{-2Y} q^Y / Y \\
&\leq \frac{2q^{N-2T} \rho_m(t^2)}{\log L} \sum_{Y = \log L}^{Q} q^{-Y} \leq \frac{2q^{N-2T} \rho_m(t^2)}{q^{\log L} \log L} \left(1 - \frac{1}{q}\right)^{-1} \\
&= \frac{2q^{N-2T} \rho_m(t^2)}{L \log L} \left(1 - \frac{1}{q}\right)^{-1} \ll \frac{q^{N-2T} \rho_m(t^2)}{L}.
\end{aligned}$$

From

$$\rho_m(p^2t^2) = \rho_m(t^2)\rho_m(p^2/(p,t)^2) = \rho_m(t^2)\rho_m(p/(p,t)) \leq 2\rho_m(t^2),$$

we deduce that the remainder term in (16) is

$$O\left(\rho_m(t^2) \sum_{\log L < \deg p \leq Q} 1\right). \tag{17}$$

Now,

$$\sum_{\log L < \deg p \leq Q} 1 \leq \sum_{D=\log L}^{Q} \frac{q^D}{D}.$$

It can be easily seen that

$$\sum_{D=\log L}^{Q} \frac{q^D}{D} \ll q^Q/Q.$$

Now,

$$\frac{q^Q}{Q} = \frac{q^{L/3}q^{-T/3}q^{2\log L/3}}{\frac{L}{3} - \frac{T}{3} + \frac{2\log L}{3}} = \frac{3q^{L/3}q^{-T/3}L^{2/3}}{L(1 - \frac{T}{L} + \frac{2\log L}{L})}.$$

In the end we will take T to be a constant (< 1) multiple of L. For such choice of T, we have from above that

$$\frac{q^Q}{Q} \ll q^{L/3}q^{-T/3}L^{-1/3} = o(q^{L/3}q^{-T/3}).$$

Using this estimate in (17) we deduce that the remainder term in (16) is $o(q^{L/3}q^{-T/3}\rho_m(t^2))$.

Therefore the sum over n in (15) is

$$\sum_{\log L < \deg p \leq Q} \sum_{\substack{\deg n = N \\ n^2 \equiv m^g \pmod{p^2t^2}}} 1 \ll \frac{q^{N-2T}\rho_m(t^2)}{L} + o(q^{L/3}q^{-T/3}\rho_m(t^2)). \tag{18}$$

Summing over all monic m and t in (18) with $\deg m = M$ and $\deg t = T$, and using Lemma 5 we get

$$N_2 \ll \frac{q^{M+N-T}}{L} + o(q^{M+\frac{L}{3}+\frac{2T}{3}}).$$

Estimation of N_3. If (m, n, t) is a tuple counted in N_3, then

$$n^2 - m^g = \beta p^2 t^2, \tag{19}$$

for some monic prime p with $\deg p > Q$ and some $\beta \in \mathcal{A}$. Clearly, $\deg \beta < L - 2Q = (L + 2T - 4\log L)/3$. As m, n and t are monic and pairwise relatively prime, for fixed

m and β with $\deg m = M$, and $\deg \beta < L - 2Q$, the number of monic n and t satisfying (19) is bounded by the number of solutions to the equation

$$m^g = x^2 - \beta y^2 \tag{20}$$

with x and y monic and co-prime. Assuming that such x and y exists, the ideal $(m)^g$ factors in $\mathcal{A}[\sqrt{\beta}]$ as

$$m^g = (x + y\sqrt{\beta})(x - y\sqrt{\beta}).$$

Working similarly as in Proposition 1, it can be seen that any common factor of the ideals $(x + y\sqrt{\beta})$ and $(x - y\sqrt{\beta})$ contains m^g and x. But $(m^g, x) = 1$ as x and y are co-prime, hence any common factor of $(x + y\sqrt{\beta})$ and $(x - y\sqrt{\beta})$ must be the whole ring $\mathcal{A}[\sqrt{\beta}]$. Therefore the ideals $(x + y\sqrt{\beta})$ and $(x - y\sqrt{\beta})$ are co-prime. From unique factorization of ideals of $\mathcal{A}[\sqrt{\beta}]$ we have

$$(x + y\sqrt{\beta}) = \mathfrak{a}^g \quad \text{and} \quad (x - y\sqrt{\beta}) = \bar{\mathfrak{a}}^g,$$

for some ideal $\mathfrak{a}$ and its conjugate $\bar{\mathfrak{a}}$ in $\mathcal{A}[\sqrt{\beta}]$. Thus the number of solutions in x and y to (20) is bounded by the number of factorizations of the ideal (m) into the product $\mathfrak{a}\bar{\mathfrak{a}}$. It can be easily verified that the number of such factorizations of the ideal (m) in $\mathcal{A}[\sqrt{\beta}]$ is $\leq d(m)$. Thus for fixed m and β, the number of choices for n and t satisfying (19) is $\leq d(m)$. From Proposition 2.5 of [23] it follows that $\sum_{\substack{m\text{-monic} \\ \deg m = M}} d(m) = q^M(M+1)$. Therefore N_3 is $\leq$ (number of choices of β)$\Big(\sum_{\substack{m\text{-monic} \\ \deg m = M}} d(m)\Big)$ which is

$$\begin{aligned}
&\leq (1 + q + q^2 + \cdots + q^{L-2Q}) \sum_{\substack{m\text{-monic} \\ \deg m = M}} d(m) \\
&= \frac{(q^{L-2Q+1} - 1)}{q - 1} q^M (M+1) \\
&\leq q^{L-2Q+1} q^M (M+1) \\
&= q \cdot q^{(L+2T-4\log L)/3} q^M (M+1) \\
&= q^{L/3} q^{2T/3} q^M q L^{-4/3} (M+1).
\end{aligned}$$

Noting from (3) that $M < L$, we conclude

$$N_3 \leq q^{L/3} q^{2T/3} q^M q L^{-4/3} (M+1) \leq q^{L/3} q^{2T/3} q^M q L^{-1/3} = o(q^{M + \frac{L}{3} + \frac{2T}{3}}),$$

as desired.

5. Proof of Lemma 2

Let $\mathcal{S}$ denote the set of monic tuples $(m_1, n_1, t_1; m_2, n_2, t_2)$ such that $\dfrac{n_1^2 - m_1^g}{t_1^2} = \dfrac{n_2^2 - m_2^g}{t_2^2}$ with $\deg m_i = M$, $\deg n_i = N$, $\deg t_i = T$; $(m_i, n_i) = (m_i, t_i) = 1$, and $(m_1, n_1, t_1) \neq (m_2, n_2, t_2)$. It can be seen that for a square-free f, if (m_1, n_1, t_1) and

(m_2, n_2, t_2) are solutions to equation (2) of § 3, then $(m_1, n_1, t_1; m_2, n_2, t_2) \in \mathcal{S}$. For a fixed square-free f, the number of such tuples is $\mathcal{R}(f)\big(\mathcal{R}(f)-1\big)$. Thus

$$\sum_{\deg f=L} \mathcal{R}(f)\big(\mathcal{R}(f)-1\big) \leq |\mathcal{S}|.$$

For $(m_1, n_1, t_1; m_2, n_2, t_2) \in \mathcal{S}$ we have

$$t_2^2(n_1^2 - m_1^g) = t_1^2(n_2^2 - m_2^g).$$

Rearranging we have

$$(t_1 n_2 + t_2 n_1)(t_1 n_2 - t_2 n_1) = t_1^2 m_2^g - t_2^2 m_1^g.$$

Since $\deg(t_1^2 m_2^g - t_2^2 m_1^g) \leq Mg + 2T < 3L$, for a fixed m and t, the number of choices for n_1 and n_2 is bounded by $d(t_1^2 m_2^g - t_2^2 m_1^g)$, provided $t_1^2 m_2^g \neq t_2^2 m_1^g$. However, if $t_1^2 m_2^g = t_2^2 m_1^g$, then from $(m_i, t_i) = 1$ and since g is odd, we have $t_1 = t_2$, $m_1 = m_2$, and consequently $n_1 = n_2$, contradicting the fact that $(m_1, n_1, t_1) \neq (m_2, n_2, t_2)$. Now $d(t_1^2 m_2^g - t_2^2 m_1^g) = O(q^{\epsilon L})$.

Thus summing over m_i and t_i for $i = 1, 2$ we have

$$\begin{aligned}\sum_{\deg f=L} \mathcal{R}(f)\big(\mathcal{R}(f)-1\big) &\leq \sum_{\deg m_i=M} \sum_{\deg t_i=T} d(t_1^2 m_2^g - t_2^2 m_1^g) \\ &\ll q^{\epsilon L} \sum_{\deg m_i=M} \sum_{\deg t_i=T} 1 \\ &= q^{\epsilon L+2M+2T}.\end{aligned}$$

6. Proof of Theorem 1

In this section we first determine a suitable optimal value of the parameter T so that the inequality (6) is justified.

Substituting the values of M and N from (3) in (6) and rearranging the terms we obtain

$$T/L \geq \frac{(g-2)}{4(g+1)} - \frac{\epsilon g}{2(g+1)}. \tag{21}$$

Thus in view of (21), the obvious optimal choice for T/L is

$$T/L = \frac{g-2}{4(g+1)}.$$

Therefore we take

$$T = \frac{L(g-2)}{4(g+1)}. \tag{22}$$

Now substituting the value of T from (22) in (7), we conclude that the number of solutions to equation (2) is

$$\gg q^{L\left(\frac{1}{2} + \frac{3}{2(g+1)} - \epsilon\right)}.$$

Therefore, it follows from Proposition 1 that

$$N_g(L) \gg q^{L(\frac{1}{2}+\frac{3}{2(g+1)}-\epsilon)},$$

and this completes the proof of the Theorem 1.

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