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Solution of the problem of the identified minimum for the tri-variate normal

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Abstract. Let $X = (X_1, X_2, X_3)$ be a non-singular tri-variate normal vector with zero means. Let $T = \min\{X_1, X_2, X_3\}$, and $I = i$ iff $T = X_i$, $i = 1, 2, 3$. The problem of the identified minimum (I, T) is then to find if its joint distribution determines uniquely X. This problem is solved here in the affirmative. To the best of our knowledge, it was first solved in the bivariate normal case (and partially in the tri-variate normal case) in 1978 in [1].

Keywords. Tri-variate normal; identified minimum; identification of parameters.

1. Introduction

In [1], the following minimum problem was discussed in the context of a probability model describing the death of an individual from one of several competing causes. Let $X_1, X_2, \ldots, X_n$ be independent random variables with continuous distribution function $T = \min\{X_1, X_2, \ldots, X_n\}$, and I is defined as $I = k$ if and only if $T = X_k$. If the X_is have a common distribution function, then it is unique determined by the distribution function of T. In [2], it was shown that when the distribution of X_i are not all the same, then the joint distribution function of the identified minimum (that is, the distribution function of (I, T)) uniquely determined each individual distribution function $F_i(x)$ of X_i, $i = 1, 2, \ldots, n$.

The problem whether the joint distribution function of the identified minimum (I, T) uniquely determines the parameters of the distribution of $(X_1, X_2, \ldots, X_n)$, when the X_is are not necessarily independent, was solved in [1], when $n = 2$ and (X_1, X_2) has a bivariate normal distribution and solved only partially in [1], when $n = 3$ and X_1, X_2, X_3 has a tri-variate normal distribution. Here, in this paper, we solve the problem completely for the tri-variate normal distribution.

2. The pdf of the identified minimum

For simplicity, we will assume in this paper that the tri-variate normal vector whose identified minimum is considered below has zero means, and distinct variances. The solution of our problem in the general case (when the means are not necessarily zeros and the variances are not necessarily distinct) is no more difficult.

Let (X_1, X_2, X_3) be a tri-variate normal random vector with zero means, and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ such that $\sigma_1^2 > \sigma_2^2 > \sigma_3^2$, and a non-singular covariance matrix Σ, where $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ for $i \neq j$, $i = \sigma_i^2$ and $i = j$. Let $T = \min\{X_1, X_2, X_3\}$, and $I = i$ if and only if $T = X_i$, $i = 1, 2, 3$. Let

$$F(t, i) = P(T \leq t, I = i).$$

Then we have

$$\begin{aligned} P(T \leq t, I = 1) &= P(X_1 \leq t, X_1 \leq X_2, X_1 \leq X_3) \\ &= \int_{-\infty}^{t} P(X_2 \geq x, X_3 \geq x | X_1 = x) f_{X_1}(x)\mathrm{d}x. \end{aligned}$$

Differentiating with respect to t we obtain

$$f_1(t) = \frac{\mathrm{d}}{\mathrm{d}t}[F(t, 1)] = f_{X_1}(t) P(X_2 \geq t, X_3 \geq t | X_1 = t).$$

The conditional density of (X_2, X_3), given $X_1 = t$ is a bivariate normal with means $\rho_{12}\frac{\sigma_2}{\sigma_1}t$, $\rho_{13}\frac{\sigma_3}{\sigma_1}t$ and variances $\sigma_2^2(1 - \rho_{12}^2)$, $\sigma_3^2(1 - \rho_{13}^2)$. Thus,

$$f_1(t) = \frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right) P\left(W_{21} \geq \frac{1 - \rho_{12}\frac{\sigma_2}{\sigma_1}}{\sigma_2\sqrt{1 - \rho_{12}^2}}t,\, W_{31} \geq \frac{1 - \rho_{13}\frac{\sigma_3}{\sigma_1}}{\sigma_3\sqrt{1 - \rho_{13}^2}}t\right),$$

where (W_{21}, W_{31}) is a bivariate normal with zero means, variances each one, and correlation $\rho_{23.1} = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2}}$ and φ is the standard normal density. Let us write

$$\frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_1\sigma_2\sqrt{1 - \rho_{12}^2}} = a_{21}, \qquad \frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_1\sigma_3\sqrt{1 - \rho_{13}^2}} = a_{31};$$

$$\frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma_1\sigma_2\sqrt{1 - \rho_{12}^2}} = a_{12}, \qquad \frac{\sigma_2 - \rho_{23}\sigma_3}{\sigma_2\sigma_3\sqrt{1 - \rho_{23}^2}} = a_{32};$$

$$\frac{\sigma_3 - \rho_{13}\sigma_1}{\sigma_1\sigma_3\sqrt{1 - \rho_{13}^2}} = a_{13}, \qquad \frac{\sigma_3 - \rho_{23}\sigma_2}{\sigma_2\sigma_3\sqrt{1 - \rho_{23}^2}} = a_{23}.$$

Then, we have

$$f_1(t) = \frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right) P(W_{21} \geq a_{21}\, t, W_{31} \geq a_{31}\, t). \tag{2.1}$$

Similarly, using similar notations, we have

$$f_2(t) = \frac{1}{\sigma_2}\varphi\left(\frac{t}{\sigma_2}\right) P(W_{12} \geq a_{12}\, t, W_{32} \geq a_{32}\, t), \tag{2.2}$$

$$f_3(t) = \frac{1}{\sigma_3}\varphi\left(\frac{t}{\sigma_3}\right) P(W_{13} \geq a_{13}\, t, W_{23} \geq a_{23}\, t). \tag{2.3}$$

Statement of the problem. We assume that the functions f_1, f_2, f_3 in (2.1)–(2.3) are given, where the parameters (the variances and the correlations) $\sigma_1^2, \sigma_2^2, \sigma_3^2, \rho_{12}, \rho_{13}, \rho_{23}$ are all unknown. The problem is to find if there is a unique set of parameters $\sigma_1^2, \sigma_2^2, \sigma_3^2, \rho_{12}, \rho_{13}, \rho_{23}$ corresponding to the three given functions f_1, f_2 and f_3.

In the next section we prove some lemmas which will be needed for the solution of the problem above.

In §4, we show that there is a positive solution to this problem (that is, there is a unique set of parameters corresponding to the given f_1, f_2 and f_3). Thus, it makes sense to estimate these parameters by usual estimation methods such as the method of moments or the method of maximum likelihood.

Before we go to the next section, let us state the well-known Mills' ratio result for a standard normal random variable Z, namely that

$$P(Z \geq t) \sim \frac{1}{t\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}t^2}, \qquad \text{as } t \to \infty,$$

(that is, as $t \to \infty$, the ratio of the sides $\to 1$). We will use this result often.

3. Some useful lemmas

If $f_1(t)$ and $f_2(t)$ are two real functions of the real variable t, then we say that f_1 and f_2 have the same order as $t \to a$, where a is either ∞ or $-\infty$, and write $f_1 \sim f_2$, as $t \to a$, if and only if

$$\lim_{t \to a} \frac{f_1(t)}{f_2(t)} = 1.$$

In Lemma 3.1 below and all other lemmas, (X, Y) is a non-singular bivariate normal vector with zero means. In only Lemma 3.1, the variances of (X, Y) are σ_1^2 and σ_2^2. In all other lemmas, the variances are each one, and the correlation is ρ.

Lemma 3.1. (*see* [3]). *Let* Σ *be the covariance matrix of* (X, Y) *and* $1\Sigma^{-1} = (\alpha_1, \alpha_2)$, *where* $\alpha_1 > 0, \alpha_2 > 0$, *and* 1 *is the vector* $(1, 1)$. *Thus, as* $t \to \infty$, *the following result holds*:

$$P(X \geq t, Y \geq t) \sim \frac{\mathrm{e}^{-\frac{1}{2}t^2(1\Sigma^{-1}1^T)}}{2\pi\alpha_1\alpha_2 t^2\sqrt{|\det\Sigma|}}. \tag{3.1}$$

Lemma 3.2. *Let* $\alpha > 0, \beta > 0, \alpha \leq \beta$ *and* $\rho < \frac{\alpha}{\beta}$. *Then as* $t \to \infty$, *we have*

$$P(X \geq \alpha t,\ Y \geq \beta t) \sim \frac{(1-\rho^2)^{\frac{3}{2}}\alpha^2\beta^2}{2\pi t^2(\alpha - \rho\beta)(\beta - \rho\alpha)}\ \mathrm{e}^{-\frac{1}{2}\left[\frac{\alpha^2+\beta^2-2\rho\alpha\beta}{1-\rho^2}\right]t^2}$$

which is $o(t^{-2}\mathrm{e}^{-\frac{1}{2}\beta^2 t^2})$ *as* $t \to \infty$.

Proof. Note that when ρ: the correlation of (X, Y), is less than $\frac{\alpha}{\beta}$, then $1\Sigma^{-1}$ is positive, where Σ is the covariance matrix of $(\frac{X}{\alpha}, \frac{Y}{\beta})$. Thus, this lemma follows from Lemma 3.1. □

Lemma 3.3. *Let $X, Y, \alpha, \beta, \rho$ be as in Lemma* 3.2. *Let $\rho > \frac{\alpha}{\beta}$. Then, as $t \to \infty$,*

$$P(X > \alpha t, Y > \beta t) \sim \frac{1}{\beta t\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\beta^2 t^2}.$$

Proof. Notice that

$$\begin{aligned}
&P(X \geq \alpha t,\ Y \geq \beta t)\\
&\quad= \int_{\beta t}^{\infty} P(X \geq \alpha t | Y = x) f_Y(x)\,\mathrm{d}x\\
&\quad= \int_{\beta t}^{\infty} f_Y(x)\,\mathrm{d}x \left[\int_{\alpha t}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2}\mathrm{d}y\right]\\
&\quad= \int_{\beta t}^{\infty} f_Y(x)\,\mathrm{d}x \left[\frac{1}{\sqrt{2\pi}}\int_{u(x)}^{\infty} \mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}y\right],
\end{aligned}$$

where $u(x) = \frac{\alpha t - \rho x}{\sqrt{1-\rho^2}}$. Since $\rho > \frac{\alpha}{\beta}$, it follows that for $x > \beta t$, $\alpha t - \rho x < t(\alpha - \rho\beta)$. This means that as $t \to \infty$, $u(x) \to -\infty$. Thus, as $t \to \infty$,

$$P(X \geq \alpha t,\ Y \geq \beta t) \sim P(Z \geq \beta t) \sim \frac{1}{\beta t\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\beta^2 t^2},$$

by the well-known Mills' ratios result, where Z is the standard normal random variable. □

Lemma 3.4. *Let $\rho = \frac{\alpha}{\beta}$ in the previous lemma, instead of the assumption $\rho > \frac{\alpha}{\beta}$. Then, for $t > 0$ we have*

$$\frac{1}{2}P(Z \geq \beta t) \leq P(X \geq \alpha t, Y \geq \beta t) \leq P(Z \geq \beta t).$$

Proof. The proof of this lemma follows easily from that of the previous lemma. □

Lemma 3.5. *Let $\alpha < 0$, $\beta > 0$, $|\alpha| < \beta$. Then, as $t \to \infty$,*

(i) $P(X \geq \alpha t, Y \geq \beta t) \sim \frac{1}{\beta t\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\beta^2 t^2}$, *when $\rho > \frac{\alpha}{\beta}$;*

(ii) *when $\rho < \frac{\alpha}{\beta}$, $P(X \geq \alpha t, Y \geq \beta t) \leq \frac{\sqrt{1-\rho^2}}{2\pi|\alpha - \beta\rho|\beta t^2}\,\mathrm{e}^{-\frac{1}{2}t^2\left[\frac{\alpha^2+\beta^2-2\rho\alpha\beta}{1-\rho^2}\right]}$;*

(iii) *when $\rho = \frac{\alpha}{\beta}$,*

$$P(X \geq \alpha t, Y \geq \beta t) \leq \frac{1}{2}P(Z \geq \beta t) \sim \frac{1}{2}\frac{1}{\beta t\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\beta^2 t^2}.$$

Proof. We prove (ii) first. As before, we can write

$$P(X \geq \alpha t, Y \geq \beta t) = \int_{\beta t}^{\infty} f_Y(x)\,\mathrm{d}x \left[\frac{1}{\sqrt{2\pi}}\int_{u(x)}^{\infty} \mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}y\right],$$

where $u(x) = \frac{\alpha t - \rho x}{\sqrt{1-\rho^2}}$. If $\rho < \frac{\alpha}{\beta}$, then $\rho < 0$ and for $x \geq \beta t$, $u(x) \geq \frac{(\alpha-\beta\rho)t}{\sqrt{1-\rho^2}} \to \infty$, as $t \to \infty$. This shows that as $t \to \infty$,

$$P(X \geq \alpha t, Y \geq \beta t) \leq P\left(Z \geq \frac{(\alpha - \beta\rho)t}{\sqrt{1-\rho^2}}\right) P\left(Z \geq \beta t\right),$$

and (ii) follows, by Mills' ratio result. The result in (iii) also follows from here since $\alpha - \beta\rho = 0$, when $\rho = \frac{\alpha}{\beta}$. To prove (i), notice that

$$P(X \geq \alpha t, Y \geq \beta t) = P(Y \geq \beta t) - P(-X \geq (-\alpha)t, Y \geq \beta t).$$

The result in (i) then follows from Lemma 3.2, when the correlation $-\rho$ of $(-X, Y)$ is less than $-\frac{\alpha}{\beta}$ (or when $\rho > \frac{\alpha}{\beta}$), as $t \to \infty$. □

Remark to Lemma 3.5. Suppose now that $t \to -\infty$ and $|\alpha| > \beta > 0$, $\alpha < 0$ in Lemma 3.5. Then we can write

$$P(X \geq \alpha t,\ Y \geq \beta t) = P(Y \geq (-\beta)(-t),\ X \geq (-\alpha)(-t)).$$

Thus, when $t \to -\infty$, $-t \to \infty$, and Lemma 3.5 again applies with (α, β) in Lemma 3.5 now replaced by $(-\beta, -\alpha)$. We can thus state the following lemma.

Lemma 3.5A. *Let $\alpha < 0,\ \beta > 0,\ |\alpha| > \beta$ and $t \to -\infty$. Then the following hold*:

(i) *when* $\rho > \frac{\beta}{\alpha}$, $P(X \geq \alpha t, Y \geq \beta t) \sim \frac{1}{\alpha t\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\alpha^2 t^2}$;

(ii) *when* $\rho < \frac{\beta}{\alpha}$, $P(X \geq \alpha t, Y \geq \beta t) \leq \frac{\sqrt{1-\rho^2}}{2\pi|\alpha\rho - \beta|\alpha t^2} \mathrm{e}^{-\frac{1}{2}t^2\left[\frac{\alpha^2+\beta^2-2\rho\alpha\beta}{1-\rho^2}\right]}$;

(iii) *when* $\rho = \frac{\beta}{\alpha}$,

$$\begin{aligned} &P(X \geq \alpha t, Y \geq \beta t) \\ &\quad \leq \frac{1}{2} P(Z \geq \alpha t) \\ &\quad \sim \frac{1}{2}\frac{1}{\alpha t\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\alpha^2 t^2}. \end{aligned}$$

□

Lemma 3.6. *Let $\alpha < 0,\ \beta > 0,\ |\alpha| > \beta$, and $t \to \infty$. Then*

$$P(X \geq \alpha t, Y \geq \beta t) \sim \frac{1}{\beta t\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\beta^2 t^2}.$$

Lemma 3.6A. *Let $\alpha < 0,\ \beta > 0,\ |\alpha| < \beta$, and $t \to -\infty$. Then*

$$P(X \geq \alpha t, Y \geq \beta t) \sim \frac{1}{|\alpha t|\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\alpha^2 t^2}.$$

We prove only Lemma 3.6. The proof of Lemma 3.6A will follow just like that of Lemma 3.5A followed from Lemma 3.5.

Proof of Lemma 3.6. Write

$$P(X \geq \alpha t, Y \geq \beta t) = P(Y \geq \beta t) - P(-X \geq (-\alpha)t, Y \geq \beta t).$$

By Lemma 3.2, as $t \to \infty$, when $-\rho < -\frac{\beta}{\alpha}$ (that is, when $\rho > \frac{\beta}{\alpha}$),

$$\begin{aligned} P(X \geq \alpha t, Y \geq \beta t) &\sim P(Y \geq \beta t) \\ &\sim \frac{1}{\beta t\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\beta^2 t^2}. \end{aligned}$$

Also, by Lemmas 3.3 and 3.4, when $-\rho \geq -\frac{\beta}{\alpha}$, as $t \to \infty$,

$$P(-X \geq (-\alpha)t, Y \geq \beta t) = o\left(\frac{1}{\beta t\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\beta^2 t^2}\right).$$

The lemma follows. □

Lemma 3.7. *Let $\alpha > 0$ or $\alpha < 0$. Then as $t \to \infty$ (or as $t \to -\infty$),*

$$P(X \geq \alpha t, Y \geq (-\alpha)t) \sim \frac{1}{|\alpha t|\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\alpha^2 t^2}.$$

Proof. Let $\alpha < 0$ and $t \to \infty$. Then

$$P(X \geq \alpha t, Y \geq (-\alpha)t) = P(Y \geq (-\alpha)t) - P(-X \geq (-\alpha)t, Y \geq (-\alpha)t).$$

The proof follows from Lemma 3.2. □

Lemma 3.8. *Let $\alpha < 0$, $\beta < 0$, $|\alpha| \leq |\beta|$, and $t \to -\infty$. Then*

(i) $\rho < \frac{\alpha}{\beta}$, $P(X \geq \alpha t, Y \geq \beta t) \sim \frac{(1-\rho^2)^{\frac{3}{2}}\,\alpha^2\beta^2\,\mathrm{e}^{-\frac{1}{2}t^2\left[\frac{\alpha^2+\beta^2-2\rho\alpha\beta}{1-\rho^2}\right]}}{2\pi t^2|(\alpha-\rho\beta)(\beta-\rho\alpha)|}$;

(ii) $\rho > \frac{\alpha}{\beta}$, $P(X \geq \alpha t, Y \geq \beta t) \sim \frac{1}{|\beta t|\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\beta^2 t^2}$;

(iii) $\rho = \frac{\alpha}{\beta}$,

$$\frac{1}{2}P(Z \geq \beta t) \leq\ P(X \geq \alpha t,\ Y \geq \beta t) \leq\ P(Z \geq \beta t).$$

Proof. The proof follows immediately from Lemmas 3.2, 3.3 and 3.4 by observing that

$$P(X \geq \alpha t,\ Y \geq \beta t) = P(X \geq (-\alpha)(-t),\ Y \geq (-\beta)(-t)).$$

Lemma 3.9. *Let $\beta > 0$ and $t \to \infty$. Then we have*

$$\begin{aligned} P(X \geq 0, Y \geq \beta t) &\sim \frac{1}{|\beta t|\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\beta^2 t^2}, \quad \textit{if } \rho > 0; \\ &\leq P(Z \geq \beta t) P\left(Z \leq \frac{-\rho\beta t}{\sqrt{1-\rho^2}}\right), \quad \textit{if } \rho < 0; \\ &\sim \frac{1}{2}\frac{1}{\beta t\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\beta^2 t^2}, \quad \textit{if } \rho = 0. \end{aligned}$$

Here Z is the standard normal random variable.

Proof. Observe that as in earlier results,

$$\begin{aligned} &P(X \geq 0, Y \geq \beta t) \\ &= \int_{\beta t}^{\infty} f_Y(x)\mathrm{d}x \left[\int_0^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2} \mathrm{d}y \right] \\ &= \int_{\beta t}^{\infty} f_Y(x)\mathrm{d}x \left[\int_{u(x)}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}y^2} \mathrm{d}y \right], \end{aligned}$$

where $u(x) = -\frac{\rho x}{\sqrt{1-\rho^2}}$. Now if $x \geq \beta t,\ \rho > 0$, then $-\rho x \leq -\rho\beta t$, so that $u(x) \to -\infty$ as $t \to \infty$. It is clear that

$$\begin{aligned} P(X \geq 0,\ Y \geq \beta t) &\sim P(Y \geq \beta t) \\ &\sim \frac{1}{\beta t\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\beta^2 t^2},\ \text{when } \rho > 0. \end{aligned}$$

The rest of the lemma is now clear. □

Lemma 3.10. *Let $\alpha > 0,\ \beta > 0, \alpha \leq \beta$. Then as $t \to -\infty$,*

$$1 - P(X \geq \alpha t,\ Y \geq \beta t) \sim \frac{1}{\alpha|t|\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\alpha^2 t^2},$$

if $\rho < \frac{\alpha}{\beta}$ (when $\alpha \leq \beta$), and also if $\rho \geq \frac{\alpha}{\beta}$ (when $\alpha < \beta$). If $\alpha = \beta$, as $t \to -\infty$,

$$1 - P(X \geq \alpha t,\ Y \geq \beta t) \sim \frac{2}{\alpha|t|\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\alpha^2 t^2}.$$

Proof. For a proof, see [4]. □

Lemma 3.11. *Let $\alpha > 0,\ \beta > 0,\ \alpha \leq \beta$. Then, as $t \to -\infty$,*

$$P(Z \geq \alpha t) - P(X \geq \alpha t,\ Y \geq \beta t) \sim \frac{1}{|\beta t|\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\beta^2 t^2},\ \textit{when } \rho < \frac{\alpha}{\beta}.$$

Proof. Write

$$\begin{aligned} P(Z \geq \alpha t) - P(X \geq \alpha t,\ Y \geq \beta t) &= P(X \geq \alpha t,\ Y \leq \beta t) \\ &= P(Y \leq \beta t) - P(X \leq \alpha t,\ Y \leq \beta t) \\ &= P(-Y \geq \beta(-t)) \\ &\quad - P(-X \geq \alpha(-t),\ -Y \geq \beta(-t)). \end{aligned}$$

The lemma now follows from Lemma 3.2. □

Let us finally state without proof the following lemma which follows from our earlier lemmas.

Lemma 3.12. *Let $\alpha < 0,\ \beta > 0$. Let us define*

$$s^* = \sup\left\{ s \,\middle|\, \lim_{t \to -\infty} \mathrm{e}^{st^2} P(X \geq \alpha t,\ Y \geq \beta t) = 0 \right\}.$$

Then the following holds:

(i) *If* $|\alpha| > \beta$, *then* $s^* \geq \frac{1}{2}\alpha^2$.

(ii) *If* $|\alpha| \leq \beta$, *then* $s^* = \frac{1}{2}\alpha^2$.

4. Solution of the 'identified minimum' problem

In this section, we show that there can only be a unique set of parameters σ_1^2, σ_2^2, σ_3^2, ρ_{12}, ρ_{13}, ρ_{23} (the three variances and the three correlations) corresponding to the given pdf funtions f_1, f_2 and f_3 in (2.1), (2.2) and (2.3).

We will use the following notations:

$$\begin{aligned} A_j &= \min\{|a_{ij}|,\ |a_{kj}|\}, \\ B_j &= \max\{|a_{ij}|,\ |a_{kj}|\}, \\ j &= 1, 2 \text{ or } 3,\ \{i,\ j,\ k\} = \{1,\ 2,\ 3\}, \\ \rho_i &\equiv \rho_{jk.i},\ \text{the correlation of}(W_{ji},\ W_{ki}). \end{aligned}$$

We have assumed, for simplicity, that

$$\sigma_1^2 > \sigma_2^2 > \sigma_3^2. \tag{4.1}$$

Note that we can always make the assumption $\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2$ by simply renaming X_1, X_2, X_3 appropriately. Also the proofs for the cases $\sigma_1^2 = \sigma_2^2 > \sigma_3^2$ and other similar cases, not considered here, should be clear from our proofs given here under the assumption (4.1).

We show below that the six parameters σ_1^2, σ_2^2, σ_3^2, ρ_{12}, ρ_{13}, ρ_{23} can be determined uniquely, when the functions f_1, f_2, f_3 are given or known, through limiting processes.

Step 1. In this step, we identify σ_1, A_1, B_1.

Notice that because of assumption (4.1), $a_{21} > 0$ and $a_{31} > 0$. As a result, it follows that

$$\frac{1}{2\sigma_1^2} = \sup\left\{s \,\middle|\, \lim_{t\to-\infty} \mathrm{e}^{st^2} f_1(t) = 0\right\}. \tag{4.2}$$

This identifies σ_1^2. To identify A_1 and B_1, we consider

$$s_o = \sup\left\{s \,\middle|\, \lim_{t\to\infty} \mathrm{e}^{st^2} f_1(t) = 0\right\}. \tag{4.3}$$

It follows from Lemma 3.2 that

$$\lim_{t\to\infty} t^2 \mathrm{e}^{s_o t^2} f_1(t) \text{ is non-zero and finite if } \rho_1 < \frac{A_1}{B_1} \tag{4.4}$$

and

$$\lim_{t\to\infty} t \mathrm{e}^{s_o t^2} f_1(t) \text{ is non-zero and finite if } \rho_1 \geq \frac{A_1}{B_1},$$

and in this case,

$$s_o = \frac{1}{2\sigma_1^2} + \frac{B_1^2}{2}. \tag{4.5}$$

In case (4.4) occurs, it follows from Lemma 3.10 that as $t \to -\infty$,

$$\frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right) - f_1(t) \sim \frac{1}{A_1|t|\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}A_1^2 t^2} \frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right). \tag{4.6}$$

Thus, (4.6) identifies A_1 since

$$\frac{1}{2\sigma^2} + \frac{A_1^2}{2} = \sup\left\{s \,\Big|\, \lim_{t\to-\infty} \mathrm{e}^{st^2}\left[\frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right) - f_1(t)\right] = 0\right\}.$$

In case (4.5) occurs, (4.3) along with (4.5) identifies B_1. Then, by Lemma 3.10, as $t \to -\infty$,

$$\frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right) - f_1(t) \sim \varphi\left(\frac{t}{\sigma_1}\right) \frac{c}{A_1|t|\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}A_1^2 t^2},$$

where c is a constant. This identifies A_1.

Thus, in case of (4.5), σ_1, A_1 and B_1, all three, have been identified. However, in case of (4.4), we have identified only σ_1 and A_1. To identify B_1 in this case, we consider, as $t \to -\infty$, the function $g(t)$ defined by

$$g(t) = \frac{1}{\sigma_1}\varphi\left(\frac{t}{\sigma_1}\right) P(Z \geq A_1 t) - f_1(t).$$

In this case (see (4.4)), Lemma 3.11 applies, and we have, as $t \to -\infty$,

$$\sup\left\{s \,\Big|\, \lim_{t\to-\infty} \mathrm{e}^{st^2} g(t) = 0\right\} = \frac{1}{2}\left[\frac{1}{\sigma_1^2} + B_1^2\right].$$

This identifies B_1, completing Step 1.

Step 2. We are given the function f_2, and we need to identify σ_2, A_2 and B_2 using our knowledge of f_2, and that of σ_1, A_1 and B_1 identified in Step 1. This step is more complicated since (4.1) does not imply that $a_{12} > 0$, though $a_{32} > 0$. Let us define

$$s_2 = \sup\left\{s \,\Big|\, \lim_{t\to-\infty} \mathrm{e}^{st^2} f_2(t) = 0\right\}. \tag{4.7}$$

Then we have

(i) If $a_{12} \geq 0$, then $s_2 = \frac{1}{2\sigma_2^2}$. Thus, for $a_{12} \geq 0$,

$$\frac{1}{\sigma_2^2} < \frac{1}{\sigma_3^2} + a_{13}^2 = \frac{1}{\sigma_1^2} + a_{31}^2.$$

When $a_{12} > 0$,

$$\frac{1}{\sigma_2^2} < \frac{1}{\sigma_2^2} + a_{12}^2 = \frac{1}{\sigma_1^2} + a_{21}^2,$$

and this means, when $a_{12} > 0$,

$$s_2 < \frac{1}{2}\left[\frac{1}{\sigma_1^2} + A_1^2\right]. \tag{4.8}$$

However when $a_{12} = 0$,

$$\frac{1}{2\sigma_2^2} = s_2 = \frac{1}{2}\left[\frac{1}{\sigma_1^2} + a_{21}^2\right] = \frac{1}{2}\left[\frac{1}{\sigma_1^2} + A_1^2\right]. \tag{4.9}$$

(ii) If $a_{12} < 0$, then by Lemmas 3.5, 3.6A and 3.7, we have

$$s_2 \geq \frac{1}{2}\left[\frac{1}{\sigma_2^2} + a_{12}^2\right] \geq \frac{1}{2}\left[\frac{1}{\sigma_1^2} + A_1^2\right]. \tag{4.10}$$

Thus when $a_{12} < 0$, by (4.10), we have two possibilities:

(i) $\quad s_2 > \frac{1}{2}\left[\frac{1}{\sigma_1^2} + A_1^2\right]$,

(ii) $\quad s_2 = \frac{1}{2}\left[\frac{1}{\sigma_2^2} + a_{12}^2\right]$.

In case of (i), by (4.7),

$$\lim_{t\to-\infty} \mathrm{e}^{\frac{1}{2}\left[\frac{1}{\sigma_1^2}+A_1^2\right]t^2} f_2(t) = 0. \tag{4.11}$$

In case of (ii), by Lemmas 3.5, 3.6A and 3.7, we also have

$$\lim_{t\to-\infty} \mathrm{e}^{\frac{1}{2}\left[\frac{1}{\sigma_1^2}+A_1^2\right]t^2} f_2(t) = \lim_{t\to-\infty} \mathrm{e}^{\frac{1}{2}\left[\frac{1}{\sigma_2^2}+a_{12}^2\right]t^2} f_2(t) = 0, \tag{4.12}$$

because of a factor $|t|$ in the denominators of the right-hand sides of Lemmas 3.5, 3.6A and 3.7. However, when $a_{12} = 0$, it follows from (4.9) that

$$\lim_{t\to-\infty} \mathrm{e}^{\frac{1}{2}\left[\frac{1}{\sigma_1^2}+A_1^2\right]t^2} f_2(t) \neq 0. \tag{4.13}$$

Thus (4.8), (4.11), (4.12) and (4.13) help us determine when $a_{12} > 0$, $a_{12} < 0$, and $a_{12} = 0$.

Since $a_{32} > 0$, when $a_{12} > 0$, we can use the method of Step 1 to identify σ_2, A_2 and B_2. If $a_{12} = 0$, σ_2 is already identified by (4.9). By Lemma 3.9, as $t \to \infty$, we have

$$f_2(t) \sim \frac{c}{\sigma_2}\varphi\left(\frac{t}{\sigma_2}\right)\frac{1}{a_{32}t\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}a_{32}^2t^2}, \tag{4.14}$$

where $c = 1$ if $\rho_2 > 0$ and $c = \frac{1}{2}$ if $\rho_2 = 0$. It is also important to note that if $\rho_2 < 0$, then as $t \to -\infty$, we have by Lemma 3.9 again,

$$\frac{1}{\sigma_2}\varphi\left(\frac{t}{\sigma_2}\right)\left[\frac{1}{2} - P(W_{12} \geq 0,\ W_{32} \geq a_{32}\, t)\right]$$

$$= \frac{1}{\sigma_2}\varphi\left(\frac{t}{\sigma_2}\right)[P(W_{12} \geq 0,\ -W_{32} \geq a_{32}(-t))]$$

$$\sim \frac{1}{\sigma_2}\varphi\left(\frac{t}{\sigma_2}\right)\frac{1}{a_{32}|t|\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}a_{32}{}^2t^2}.$$

Thus, it is clear that in case $a_{12} = 0$, we can identify a_{32} as well as σ_2.

Next, we consider the case when $a_{12} < 0$. In this case, we use Lemmas 3.5, 3.5A, 3.6 and 3.6A. We observe that exactly one of the following three possibilities must then occur. Each one of these possibilities identifies σ_2 and $A_2 = \min\{|a_{12}|, a_{32}\}$.

(1) If $|a_{12}| < a_{32}$, with s_2 as defined in (4.7), we must have by Lemmas 3.5 and 3.6A,

$$s_2 = \frac{1}{2}\left[\frac{1}{\sigma_2^2} + a_{12}^2\right], \tag{4.15}$$

$$\lim_{t\to-\infty} 2\pi\,|\,t\,|\mathrm{e}^{s_2t^2}\,f_2(t) = \frac{1}{\sigma_2|a_{12}|} \neq 0, \tag{4.16}$$

$$\lim_{t\to-\infty} t\mathrm{e}^{s_2t^2}\,f_2(t) = 0. \tag{4.17}$$

Also, defining

$$s_1 = \sup\left\{s\,\Big|\,\lim_{t\to\infty}\mathrm{e}^{st^2}\,f_2(t) = 0\right\}, \tag{4.18}$$

it follows from Lemma 3.5 that when the correlation ρ_2 of $(W_{12},\ W_{32})$ is greater than $\frac{a_{12}}{a_{32}}$, then

$$\lim_{t\to\infty} t\mathrm{e}^{s_1t^2}\,f_2(t) \neq 0 \tag{4.19}$$

and in this case (when (4.19) occurs and $\rho_2 > \frac{a_{12}}{a_{32}}$), by Lemma 3.5, we have

$$s_1 = \frac{1}{2\sigma_2^2} + \frac{1}{2}a_{32}^2. \tag{4.20}$$

Also, for $\rho_2 \leq \frac{a_{12}}{a_{32}}$, by Lemma 3.5, we have

$$s_1 \geq \frac{1}{2\sigma_2^2} + a_{32}^2. \tag{4.21}$$

(2) If $|a_{12}| > a_{32}$, we must have by Lemmas 3.6 and 3.5A,

$$s_1 = \frac{1}{2}\left[\frac{1}{\sigma_2^2} + a_{32}^2\right], \tag{4.22}$$

$$\lim_{t\to\infty} 2\pi t e^{s_1 t^2} f_2(t) = \frac{1}{\sigma_2\, a_{32}} \neq 0, \tag{4.23}$$

$$\lim_{t\to-\infty} t e^{s_1 t^2} f_2(t) = 0. \tag{4.24}$$

It also follows from Lemma 3.5A that when $\rho_2 > \frac{a_{32}}{a_{12}}$, then

$$\lim_{t\to-\infty} t e^{s_2 t^2} f_2(t) \neq 0 \tag{4.25}$$

and in this case, by Lemma 3.5A, when $\rho_2 > \frac{a_{32}}{a_{12}}$,

$$s_2 = \frac{1}{2\sigma_2^2} + a_{12}^2. \tag{4.26}$$

Also, for $\rho_2 \leq \frac{a_{32}}{a_{12}}$, by Lemma 3.5A,

$$s_2 \geq \frac{1}{2\sigma_2^2} + a_{12}^2. \tag{4.27}$$

(3) If $|a_{12}| = a_{32}$, then by Lemma 3.7, $s_1 = s_2$ and the first two results in each of (1) and (2) above then must hold. Thus, identification of σ_2, A_2 and B_2 are immediate.

Note that when possibility (1) above occurs (that is, when $s_1 > s_2$ by (4.15), (4.20) and (4.21)), $\sigma_2,\ A_2$ are both identified by the equations (4.15) and (4.16), and when $\rho_2 > \frac{a_{12}}{a_{32}}$ and $A_2 = |a_{12}|$, $B_2 = a_{32}$ is identified by (4.20). Thus, we must still identify B_2 in the case when $\rho_2 \leq \frac{a_{12}}{a_{32}}$ and $A_2 = |a_{12}|$ is identified. To do this, we consider the function h defined by

$$\begin{aligned} h(t) &= P(Z \geq a_{12}t) - P(W_{12} \geq a_{12}t, W_{32} \geq a_{32}t) \\ &= P(W_{12} \geq (-a_{12})(-t),\ -W_{32} \geq a_{32}(-t)) \end{aligned}$$

and if we define

$$s^* = \sup\left\{ s \,\middle|\, \lim_{t\to-\infty} e^{st^2} h(t) = 0 \right\},$$

then, by Lemma 3.2, when $-\rho_2 \geq -\frac{a_{12}}{a_{32}}$ (that is, when $\rho_2 \leq \frac{a_{12}}{a_{32}}$),

$$s^* = \frac{1}{2} a_{32}^2,$$

which identifies B_2.

Similarly, when possibility (2) occurs, we can also identify $\sigma_2,\ A_2$ and B_2. This completes Step 2.

Step 3. In this step, we first show that in all situations except when we have

$$\frac{1}{\sigma_1^2} + A_1^2 = \frac{1}{\sigma_2^2} + A_2^2 \quad \text{and} \quad \frac{1}{\sigma_1^2} + B_1^2 = \frac{1}{\sigma_2^2} + B_2^2, \tag{4.28}$$

we can identify uniquely a_{21}^2, a_{31}^2, a_{12}^2 and a_{32}^2. In case of the above two equalities, we shall show that we must have $a_{13}^2 = a_{23}^2$. In the proof of the above assertions, the main argument is the fact that

$$\frac{1}{\sigma_1^2} + a_{21}^2 = \frac{1}{\sigma_2^2} + a_{12}^2. \tag{4.29}$$

Let us also observe that in each of the following remaining cases (leaving aside (4.28) above)

$$\text{(i)} \qquad \frac{1}{\sigma_1^2} + A_1^2 = \frac{1}{\sigma_2^2} + A_2^2, \quad \frac{1}{\sigma_1^2} + B_1^2 \neq \frac{1}{\sigma_2^2} + B_2^2, \tag{4.30}$$

$$\text{(ii)} \qquad \frac{1}{\sigma_1^2} + A_1^2 \neq \frac{1}{\sigma_2^2} + A_2^2, \quad \frac{1}{\sigma_1^2} + B_1^2 = \frac{1}{\sigma_2^2} + B_2^2, \tag{4.31}$$

$$\text{(iii)} \qquad \frac{1}{\sigma_1^2} + A_1^2 = \frac{1}{\sigma_2^2} + B_2^2, \quad \frac{1}{\sigma_1^2} + B_1^2 \neq \frac{1}{\sigma_2^2} + A_2^2, \tag{4.32}$$

$$\text{(iv)} \qquad \frac{1}{\sigma_1^2} + A_1^2 \neq \frac{1}{\sigma_2^2} + B_2^2, \quad \frac{1}{\sigma_1^2} + B_1^2 = \frac{1}{\sigma_2^2} + A_2^2, \tag{4.33}$$

$$\text{(v)} \qquad \frac{1}{\sigma_1^2} + A_1^2 = \frac{1}{\sigma_2^2} + B_2^2, \quad \frac{1}{\sigma_1^2} + B_1^2 = \frac{1}{\sigma_2^2} + A_2^2, \tag{4.34}$$

it can be easily verified that $a_{21}^2, a_{31}^2, a_{12}^2$ and a_{32}^2 can all be identified. Let us only mention that in case (v) above , $A_1^2 = B_1^2 = a_{21}^2 = a_{31}^2$ and $A_2^2 = B_2^2 = a_{12}^2 = a_{32}^2$. In case of (i) through (iv) above, we can use (4.29) to identify a_{21}^2, a_{31}^2, a_{12}^2, a_{32}^2 uniquely. Now we consider (4.28). In this case, there are two possibilities:

$$(A_1^2, B_1^2, A_2^2, B_2^2) = (a_{21}^2, a_{31}^2, a_{12}^2, a_{32}^2) \tag{4.35}$$

or

$$(A_1^2, B_1^2, A_2^2, B_2^2) = (a_{31}^2, a_{21}^2, a_{32}^2, a_{12}^2). \tag{4.36}$$

In case of (4.35), we have

$$\frac{1}{\sigma_1^2} + a_{31}^2 = \frac{1}{\sigma_2^2} + a_{32}^2,$$

which implies that

$$\frac{1}{\sigma_3^2} + a_{13}^2 = \frac{1}{\sigma_3^2} + a_{23}^2 \quad \text{or} \quad a_{13}^2 = a_{23}^2.$$

The same is the conclusion when (4.36) occurs.

First, note that in each of the five possibilities, (4.30) through (4.34), we can identify uniquely $\sigma_1^2, \sigma_2^2, a_{21}^2, a_{31}^2, a_{12}^2, a_{32}^2$. Thus in this case, if σ_3 is also identified, then a_{13}^2 and a_{23}^2 are also identified since

$$\frac{1}{\sigma_1^2} + a_{31}^2 = \frac{1}{\sigma_3^2} + a_{13}^2, \quad \frac{1}{\sigma_2^2} + a_{32}^2 = \frac{1}{\sigma_3^2} + a_{23}^2. \tag{4.37}$$

Next, when (4.28) occurs, but (4.34) does not, we have already seen in (4.37) that $a_{13}^2 = a_{23}^2$. Suppose now that we have also identified $\frac{1}{\sigma_3^2} + a_{13}^2$. Thus considering

$$r = \min\left\{\frac{1}{\sigma_1^2} + A_1^2, \frac{1}{\sigma_2^2} + A_2^2, \frac{1}{\sigma_3^2} + A_3^2\right\},$$

it is clear that if $r = \frac{1}{\sigma_3^2} + A_3^2$, then $A_1^2 = a_{31}^2$ and $A_2^2 = a_{32}^2$, and if $r < \frac{1}{\sigma_3^2} + A_3^2$, then $A_1^2 = a_{21}^2$ and $A_2^2 = a_{12}^2$, and thus, $a_{21}^2, a_{31}^2, a_{12}^2, a_{32}^2$ are also identified uniquely.

Still the question remains: how do we identify the correlations ρ_{12}, ρ_{13} and ρ_{23} by knowing the nine parameters σ_1^2, σ_2^2, σ_3^2, a_{21}^2, a_{31}^2, a_{12}^2, a_{32}^2, a_{13}^2, a_{23}^2? We use the following method.

Suppose we find that

$$r = \frac{1}{\sigma_2^2} + a_{12}^2,$$

then

$$r = \frac{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}. \tag{4.38}$$

From Step 2, we know if $a_{12} = 0$, > 0 or < 0. By solving (4.38), we know

$$\rho_{12} = \frac{1 \pm \sqrt{(r\sigma_1^2 - 1)(r\sigma_2^2 - 1)}}{r\sigma_1\sigma_2} \tag{4.39}$$

so that

$$\sigma_2 - \rho_{12}\sigma_1 = \frac{\sqrt{r\sigma_2^2 - 1}}{r\sigma_2}\left[\sqrt{r\sigma_2^2 - 1} \mp \sqrt{r\sigma_1^2 - 1}\right]. \tag{4.40}$$

It follows that

$$a_{12} = 0 \Rightarrow \rho_{12} = \frac{\sigma_2}{\sigma_1},$$

$$a_{12} < 0 \Rightarrow \text{we take the '+' sign in (4.39)},$$

$$a_{12} > 0 \Rightarrow \text{we take the '−' sign in (4.39)},$$

Similarly, considering $\frac{1}{\sigma_1^2} + a_{31}^2 = r$, we get ρ_{13}; and we get ρ_{23} from $\frac{1}{\sigma_2^2} + a_{32}^2 = r$.

Finally, before we can complete the solution of our problem, we need to use equation (2.3) and show how we can identify the parameters σ_3^2, A_3^2 and B_3^2 in case the function f_3 in (2.3) is given and $a_{13}^2 = a_{23}^2$ in (2.3).

Let us define for $a = \pm\infty$,

$$s_a = \sup\left\{s \,\middle|\, \lim_{t\to a} e^{st^2} f_3(t) = 0\right\}. \tag{4.41}$$

When $a_{13} = 0$, it is clear that

$$s_a = \frac{1}{2\sigma_3^2}, \quad \text{for } a = \infty \text{ or } -\infty. \tag{4.42}$$

When $a_{13} > 0$ and $a_{13} = a_{23}$, then

$$s_a = \frac{1}{2\sigma_3^2}, \quad \text{for } a = -\infty \tag{4.43}$$

and by Lemma 3.2,

$$\frac{1}{2\sigma_3^2} < s_a = \frac{1}{2\sigma_3^2} + \frac{a_{13}^2}{1+\rho} \text{ for } a = \infty \tag{4.44}$$

Also, in the case when $0 < a_{13} = a_{23}$, by Lemma 3.10,

$$\frac{1}{2} a_{13}^2 = \sup\left\{ s \,\Big|\, \lim_{t\to-\infty} \mathrm{e}^{st^2}[1 - P(W_{13} \geq a_{13}t,\ W_{23} \geq a_{13}t) = 0] \right\} \tag{4.45}$$

and by Lemma 3.2,

$$\lim_{t\to\infty} t\, \mathrm{e}^{s_\infty t^2} f_3(t) = 0 \tag{4.46}$$

The same results (4.43) through (4.46) hold for $0 > a_{13} = a_{23}$, with $-\infty$ replaced by ∞ and ∞ by $-\infty$.

Now consider the last case when $a_{13} > 0$ and $a_{23} = -a_{13}$. In this case, by Lemma 3.10,

$$s_\infty = s_{-\infty} = \frac{1}{2}\left[\frac{1}{\sigma_3^2} + a_{13}^2\right] \tag{4.47}$$

and

$$\lim_{|t|\to\infty} 2\pi |\, t\, |\mathrm{e}^{s_\infty t^2} f_3(t) = \frac{1}{\sigma_3 a_{13}}. \tag{4.48}$$

In this case, (4.47) and (4.48) give σ_3^2 and a_{13}^2 uniquely in terms of s_∞ and the limit on the left in (4.48).

It is clear from the above results which of the following four cases hold: $a_{13} = 0$, $0 < a_{13} = a_{23}$, $a_{13} = a_{23} < 0$ and $a_{13} = -a_{23} > 0$; also, how in each case, we can identify σ_3^2 and a_{13}^2 in terms of f_3 through a limiting process.

Finally, as we remarked earlier, when a_{13}^2 may not equal a_{23}^2, we still need to identify σ_3^2, and this can be done using the lemmas in §3 and following the method used above. See (4.41) through (4.48). In this case, σ_1^2, σ_2^2, a_{21}^2, a_{31}^2, a_{12}^2 and a_{32}^2 are already identified, and thus, (4.37) tells us if a_{13}^2 is greater than, less than or equal to a_{23}^2. In each of these cases, σ_3^2 can be identified using (4.41) and our lemmas. We omit the details to avoid duplication.

This completes the solution of the problem.

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