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Frobenius pull backs of vector bundles in higher dimensions

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Abstract. We prove that for a smooth projective variety X of arbitrary dimension and for a vector bundle E over X, the Harder–Narasimhan filtration of a Frobenius pull back of E is a refinement of the Frobenius pull back of the Harder–Narasimhan filtration of E, provided there is a lower bound on the characteristic p (in terms of rank of E and the slope of the destabilizing sheaf of the cotangent bundle of X). We also recall some examples, due to Raynaud and Monsky, to show that some lower bound on p is necessary. We also give a bound on the instability degree of the Frobenius pull back of E in terms of the instability degree of E and well defined invariants of X.

Keywords. Frobenius pull backs; instability degree, vector bundles.

1. Introduction

Let X be a nonsingular projective variety defined over an algebraically closed field k of an arbitrary characteristic, and let H be a very ample line bundle on X. Let E be a torsion free sheaf on X. Then the notion of E being stable (resp. semistable) is well-known and studied. In case E is not semistable, then one has the notion of Harder–Narasimhan filtration of E. In this paper, we discuss the behaviour of Harder–Narasimhan filtrations of torsion free sheaves on X, under Frobenius pull-backs.

We recall a well-known result (Corollary 2^p in [7]) of Shepherd–Barron and Sun (Theorem 3.1 in [8]):

If X is a nonsingular projective curve and E is a semistable vector bundle on X of rank r, then

$$I(F^*E) \leq (r-1)\mu_{\max}\big(\Omega^1_X\big),$$

where $I(F^*E)$ denotes the instability degree of F^*E, as defined in § 2.

More recently Sun [9] proved:

If X is a nonsingular projective curve and E is vector bundle of rank r, then

$$I(F^*E) \leq (l-s)\,\mu_{\max}\big(\Omega^1_X\big) + s\cdot pI(E),$$

where $r-1 \geq l \geq 1$ and $s \geq 0$ are integers as defined in § 2.

Then he asked whether the same bound holds for higher dimensional varieties. Here we answer his question (with some explicit bound on p) in the following:

Theorem 1.1. *Let* $\dim X = n$ *and* $\operatorname{rank} E = r$ *and let Ω^1_X denote the cotangent bundle of X. Let $l = l(F^*E)$ and let $s = s(X, E)$. Suppose $p \geq r+n-(s+2)$. Then*

$$I(F^*E) \leq (l-s)\ \mu_{\max}\big(\Omega^1_X\big) + \epsilon \cdot pI(E),$$

where $\epsilon = \min\{1, s\}$. *Moreover, if* X *is a curve then, for any prime* p, *we have*

$$I(F^*E) \leq (l-s)(2g-2) + \epsilon \cdot pI(E),$$

where g *denotes the genus of the curve.*

In particular, it gives a sharper bound than the one given in the question of Sun.

Here we use a result of Ilangovan *et al.* [2] about low height representations and fine-tune some methods of Sun.

Now having the analog of Shephers-Barron–Sun's bounds for higher dimensions, on the instability degree of F^*E, we can generalize results of [11] (proved for curves) to higher dimensions; more precisely we have

Theorem 1.2. *Let* X *be a nonsingular projective variety of dimension* n *and let* E *be a torsion free sheaf on* X *of rank* r. *Let* $p \geq \max\{(r+n-2),\ \mu_{\max}(\Omega^1_X)(r^3/4)\}$, *where* E *is a torsion free coherent sheaf over* X. *Let*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E$$

be the HN filtration of E. *Then*

$$F^*E_1 \subset F^*E_2 \subset \cdots \subset F^*E_l \subset F^*E$$

is a subfiltration of the HN filtration of F^*E, *i.e., if*

$$0 \subset \tilde{E}_1 \subset \tilde{E}_2 \subset \cdots \subset \tilde{E}_{l_1} \subset \tilde{E}_{l_1+1} = F^*E$$

is the HN filtration of F^*E *then, for every* $1 \leq i \leq l$ *there exists* $1 \leq j_1 \leq l_1$ *such that* $F^*E_i = \tilde{E}_{j_i}$.

Moreover, under the same hypothesis on p, as in Theorem 1.2, we observe that (Corollary 2.12) if the ranks of the first and last proper subsheaves of the HN filtration of E are equal to 1 and (rank E) $-$ 1, respectively, then $I(F^{s*}E) = p^s I(E)$ for any iterated Frobenius pull back F^{s*}.

We recall some Examples 2.10 and 2.11 of Raynaud [6] and Monsky [5], respectively, to show that some lower bound on the characteristic p (in terms of both, rank E and deg X) is necessary, for Theorem 1.2 to hold.

One also observes (as proved for curves in [11]) that each normalized HN slope of a HN sheaf of $F^{s*}E$ is bounded in terms of the slope of HN sheaf (of E), to which it 'descends', and explicit invariants of X and rank of E. In particular, as in the case of curves, if $HNP_{p^s}(E)$ denotes the convex polygon associated to $F^{s*}E$ (as in [11]), then

$$\lim_{p \mapsto \infty} \text{Area } HNP_{p^s}(E) = \text{Area } HNP_{p^0}(E).$$

Various other results proved for curves in [11] are valid for higher dimension with some modification in the bounds. Since proofs are very similar to the case of curves, we have stated them without proofs.

2. Vector bundles

Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic $p > 0$. Let E be a torsion free sheaf of rank r on X. We also fix a polarization H of X.

DEFINITION 2.1

A torsion-free sheaf E is μ-semistable (with respect to the polarization H), if for all subsheaves $F \subset E$, one has

$$\mu(F) \leq \mu(E), \quad \text{where} \quad \mu(E) = (c_1(E) \cdot H^{n-1})/\text{rank}\,(E).$$

DEFINITION 2.2

For a torsion free sheaf E on X, consider the Harder–Narasimhan filtration given by

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = E. \tag{2.1}$$

Then, for $i \geq 1$,

$$\mu_i(E) = \mu(E_i/E_{i-1}),\ \mu_{\max}(E) = \mu(E_1) \quad \text{and} \quad \mu_{\min}(E) = \mu(E/E_l).$$

The *instability degree* $I(E)$ of E is defined as

$$I(E) = \mu_{\max}(E) - \mu_{\min}(E).$$

DEFINITION 2.3

If X is a projective variety defined over an algebraically closed field of characteristic $p > 0$, then the absolute Frobenius morphism $F : X \to X$ is a morphism of schemes which is an identity on the underlying set of X and on the underlying sheaf of rings $F^{\#} : \mathcal{O}_X \to \mathcal{O}_X$ which is the p-th power map.

We recall the following well-known lemma.

Lemma 2.4. *If E_1 and E_2 are two torsion free sheaves on X, then*

$$\mu_{\min}(E_1) > \mu_{\max}(E_2) \implies \text{Hom}_{\mathcal{O}_X}(E_1, E_2) = 0.$$

We recall the following result (see Proposition 1 of [7]).

Lemma 2.5. *Let E be a semistable torsion free sheaf on X such that F^*E is not semistable. Let*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F_{l+1} = F^*E$$

be the Harder–Narasimhan filtration. Then there exists a canonical connection $\nabla_{\text{can}} : F^*E \to F^*E \otimes \Omega^1_X$ such that, for every $1 \leq i \leq l$, the $\mathcal{O}_X$-homomorphisms $F_i \to (F^*E/F_i) \otimes \Omega^1_X$ induced by ∇_{can} are nontrivial.

Notation 2.6. Let $l(E)$ denote the number of nontrivial subsheaves in the HN filtration of E, e.g., if E has the HN filtration as in Definition 2.2 above, then $l(E) = l$.

Moreover $s(X, E)$ denotes the number of ∇_{can}-invariant subsheaves of F^*E, which occur properly in the HN filtration of F^*E. We recall that ∇_{can}-invariant subsheaves F^*E are precisely those which descend to a subsheaf of E. Clearly $0 \leq s(X, E) \leq l(F^*E)$.

Now we recall the following conjecture of Sun (Remark (3.13) of [9]):

$$I(F^*E) \leq (l - s)\mu_{\max}\big(\Omega^1_{\text{X}}\big) + psI(E),$$

where l and s are as in Theorem 1.1. Theorem 1.1 is a modified version of this conjecture. In particular, it implies the conjecture, if $p \geq r + n - (s + 2)$. The proof is by refining some arguments given in the proof of Theoren 3.4 in [9].

Proof of Theorem 1.1. If F^*E is semistable then it is obvious. Suppose F^*E is not semistable, then we have the Harder–Narasimhan filtration of F^*E,

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F_{l+1} = F^*E,$$

in particular $l \geq 1$. Let $0 \subset E_1 \subset \cdots \subset E_t \subset E$ be the HN filtration of E. Let

$$S = \{i \mid 1 \leq i \leq l, \text{where } F_i \text{ descends to some } E_{j_i}\}.$$

Now

$$\begin{aligned} I(F^*E) &= \sum_{i=1}^{l} \mu(F_i/F_{i-1}) - \mu(F_{i+1}/F_i) \\ &= \sum_{i \notin S} \mu(F_i/F_{i-1}) - \mu(F_{i+1}/F_i) \\ &\quad + \sum_{i \in S} \mu(F_i/F_{i-1}) - \mu(F_{i+1}/F_i) \end{aligned}$$

□

Case 1. Suppose $i \notin S$. Then we have a nonzero $\mathcal{O}_X$-linear map

$$\sigma_i : F_i \to (F^*E/F_i) \otimes \Omega_X^1.$$

Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{m+1} = \Omega_X^1$$

be the HN filtration of the cotangent bundle Ω_X^1 of X. Note that F_i/F_{i-1}, F^*E/F_i and M_i/M_{i-1} are locally free sheaves of $\mathcal{O}_U$-modules, where U is an open subscheme such that $X \setminus U$ is of codimension ≥ 2 in X.

Let j be the minimum integer such that $\sigma_i(F_i) \subseteq \frac{F^*E}{F_i} \otimes M_j$. This induces a nonzero $\mathcal{O}_X$-linear map

$$F_i \to (F^*E/F_i) \otimes M_j/M_{j-1}.$$

Hence, by Lemma 2.4,

$$\mu_{\min}(F_i) \leq \mu_{\max}((F^*E/F_i) \otimes (M_j/M_{j-1})). \tag{2.2}$$

We note that rank $(F_{i+1}/F_i) \leq r-(s+1)$. Therefore, by [2], $\frac{F_{i+1}}{F_i} \otimes \frac{M_j}{M_{j-1}}$ is semistable, as

$$p + 1 \geq r - (s + 1) + n \geq \dim\left(\frac{F_{i+1}}{F_i}\right) + \dim\left(\frac{M_j}{M_{j-1}}\right).$$

Hence

$$\left(\frac{F_{i+1}}{F_i}\otimes\frac{M_j}{M_{j-1}}\right)\ \subset\ \cdots\subset\left(\frac{F_t}{F_i}\otimes\frac{M_j}{M_{j-1}}\right)$$
$$\subset\ \left(\frac{F_{t+1}}{F_i}\otimes\frac{M_j}{M_{j-1}}\right)=\left(\frac{F^*E}{F_i}\otimes\frac{M_j}{M_{j-1}}\right)$$

is the HN filtration of $\frac{F^*E}{F_i}\otimes\frac{M_j}{M_{j-1}}$. Therefore

$$\begin{aligned}\mu_{\max}((F^*E/F_i)\otimes(M_j/M_{j-1})) &= \mu\left(\frac{F_{i+1}}{F_i}\otimes\frac{M_j}{M_{j-1}}\right)\\ &= \mu\left(\frac{F_{i+1}}{F_i}\right)+\mu\left(\frac{M_j}{M_{j-1}}\right)\\ &\leq \mu\left(\frac{F_{i+1}}{F_i}\right)+\mu_{\max}(\Omega^1_X).\end{aligned}$$

Hence, equation (2.2) implies that

$$\mu(F_i/F_{i-1})-\mu(F_{i+1}/F_i)\leq\mu_{\max}(\Omega^1_X).$$

Therefore, for $i\notin S$,

$$\sum_{i\notin S}\mu(F_i/F_{i-1})-\mu(F_{i+1}/F_i)\leq(l-s)\ \mu_{\max}(\Omega^1_X).$$

Case 2. Let $F_{i_1},\ldots,F_{i_s}$ be the subsheaves of the HN filtration of F^*E which descend to subsheaves $E_{j_1},\ldots,E_{j_s}$ of E, where $E_{j_1}\subset\cdots\subset E_{j_s}$.

Claim. Let $E_{j_0}=(0)$ and $E_{j_{s+1}}=E$, then we have

$$\mu\left(\frac{F^*E_{j_k}}{F_{(i_k)-1}}\right)-\mu\left(\frac{F_{(i_k)+1}}{F^*E_{j_k}}\right)\leq p\mu\left(\frac{E_{j_k}}{F_{j_{(k-1)}}}\right)-p\mu\left(\frac{E_{j_{(k+1)}}}{E_{(j_k)}}\right).$$

Proof of the Claim. We note that $F^*E_{j_k}/F^*E_{j_{(k-1)}}$ has the following HN filtration:

$$\frac{F^*E_{j_{(k-1)}}}{F^*E_{j_{(k-1)}}}\subset\cdots\subset\frac{F_{(i_k)-1}}{F^*E_{j_{(k-1)}}}\subset\frac{F^*E_{j_k}}{F^*E_{j_{(k-1)}}}.$$

Therefore

$$\mu_{\min}\left(\frac{F^*E_{j_k}}{F^*E_{j_{(k-1)}}}\right)=\mu\left(\frac{F^*E_{j_k}}{F_{(i_k)-1}}\right)\leq\mu\left(\frac{F^*E_{j_k}}{F^*E_{j_{(k-1)}}}\right)=p\mu\left(\frac{E_{j_k}}{E_{j_{(k-1)}}}\right),$$

where the second inequality follows because $\frac{F^*E_{j_k}}{F_{(i_k)-1}}$ is a quotient of $\frac{F^*E_{j_k}}{F^*E_{j_{(k-1)}}}$ as $F^*E_{j_{(k-1)}} \subseteq F_{(i_k)-1}$. On the other hand,

$$\frac{F_{(i_k)+1}}{F^*E_{j_k}} \subset \cdots \subset \frac{F^*E_{j_{(k+1)}}}{F^*E_{j_k}}$$

is the HN filtration of $\frac{F^*E_{j_{(k+1)}}}{F^*E_{j_k}}$. Therefore

$$\mu_{\max}\left(\frac{F^*E_{j_{(k+1)}}}{F^*E_{j_k}}\right) = \mu\left(\frac{F_{(i_k)+1}}{F^*E_{j_k}}\right) \geq \mu\left(\frac{F^*E_{j_{(k+1)}}}{F^*E_{j_k}}\right) = p\mu\left(\frac{E_{j_{(k+1)}}}{E_{j_k}}\right).$$

This proves the claim. □

Now

$$\begin{aligned}
&\sum_{i \in S=\{i_1,i_2,\ldots,i_s\}} \mu\left(\frac{F_i}{F_{i-1}}\right) - \mu\left(\frac{F_{i+1}}{F_i}\right) \\
&\leq p\left[\mu(E_{j_1}) - \mu\left(\frac{E_{j_2}}{E_{j_1}}\right) + \mu\left(\frac{E_{j_2}}{E_{j_1}}\right)\right. \\
&\qquad \left. - \mu\left(\frac{E_{j_3}}{E_{j_2}}\right) + \cdots + \mu\left(\frac{E_{j_s}}{E_{j_{s-1}}}\right) - \mu\left(\frac{E}{E_{j_s}}\right)\right] \\
&= p\left[\mu(E_{j_1}) - \mu\left(\frac{E}{E_{j_s}}\right)\right] \\
&\leq p\left[\mu_{\max}(E) - \mu_{\min}(E)\right] = pI(E).
\end{aligned}$$

For a proof of the second last inequality, one can see Proposition 3.3(3) in [9]. Now the theorem follows at once by Case 1 and Case 2.

In the case $\dim\ X = 1$, the bundle $M_j/M_{j-1} = \Omega^1_X$ is a line bundle and therefore $\frac{F_{i+1}}{F_i} \otimes \frac{M_j}{M_{j-1}}$ is semistable, for every prime p. In particular, if X is a smooth projective curve of genus $g \geq 1$ and E be a torsion free sheaf on X, then for any prime p, we have

$$I(F^*E) \leq (l-s)(2g-2) + \epsilon \cdot pI(E). \qquad \square$$

Now, modifying the proof of Theorem 3.12 of [9] with similar arguments, we get the following (here l and s are as in Theorem 3.12 of [9]):

Theorem 2.7. *Let* $\dim\ X = n$ *and* $\operatorname{rank}\ E = r$ *and let* Ω^1_X *denote the cotangent bundle of* X. *Then, for any* $p > 0$, *we have*

$$L_{\max}(E) - L_{\min}(E) \leq \frac{l-s}{p} \cdot L_{\max}(\Omega^1_X) + \epsilon \cdot I(E),$$

where $\epsilon = \min\{1, s\}$ *and*

$$L_{\max}(E) := \lim_{k\to\infty} \frac{\mu_{\max}(F^{k*}E)}{p^k}, \quad L_{\min}(E) := \lim_{k\to\infty} \frac{\mu_{\min}(F^{k*}E)}{p^k}.$$

In particular,

$$I(F^*E) \leq (r-1)L_{\max}(\Omega_X^1) + I(E), \quad \textit{if } \mu_{\max}(\Omega_X^1) > 0$$

otherwise $I(F^*E) = I(E)$.

A version of this theorem (with larger coefficients of $L_{\max}(\Omega_X^1)$ and $I(E)$) can be found in [3] (see Corollary 6.2).

The following corollary is a generalization of a Shepherd-Barron–Sun's inequality, as mentioned in the Introduction, (there it is proved for dim $X = 1$, genus $X \geq 1$ and for every prime p).

COROLLARY 2.8

With the notation as in Theorem 1.1, *if, in addition* E *is semistable then*

$$I(F^*E) \leq (r-1)\mu_{\max}(\Omega_X^1)_+.$$

Theorem 1.1 gives a sharper bound than given in Theorem 3.4 of [9]. In fact the following examples show that perhaps this is the optimal bound on the instability degree of F^*E: First we recall some results from [5] and [10]. Let X be a nonsingular plane curve of degree d. Therefore $X = \text{Proj } R$, where $R = k[x, y, z]/(h)$, with h a homogeneous polynomial of degree d and k is an algebraically closed field of characteristic p. Consider the canonical map

$$0 \longrightarrow V \longrightarrow H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow 0,$$

where $\mathcal{L}$ is the very ample bundle induced by $X \hookrightarrow \mathbf{P}^2$. Then, by Corollary 5.4 of [10], the Hilbert–Kunz multiplicity of $(X, \mathcal{L})$ (which is same as the Hilbert–Kunz multiplicity of R with respect of the ideal (x, y, z)) is given as

$$e_{\text{HK}}(X, \mathcal{L}) = \frac{3d}{4} + \frac{l_1^2}{4\text{d}p^{2s_1}},$$

where $s_1 \geq 1$ such that $F^{(s_1-1)*}V$ is semistable and $F^{s_1*}V$ is not semistable, and l_1 is an integer such that $l_1 \equiv pd \pmod 2$ and $0 \leq l_1 \leq d(d-3)$. Moreover, by the proof of Theorem 5.3 of [10], we have an interpretation of l_1 as follows: Let $\mathcal{L}_1 \subset F^{s_1*}V$ be the HN filtration of $F^{s_1*}V$. Then $\deg \mathcal{L}_1 = -(d/2)p^{s_1} + (l_1/2)$. In particular, $I(F^{s_1*}V) = l_1$.

On the other hand, Monsky [5], using a theorem of Han calculated Hilbert–Kunz multiplicity of various irreducible trinomial plane curves. Here we recall two of those examples. Let $R = k[x, y, z]/(h)$, where

(1) $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$ and $d \geq 4$ is an even integer and p is a prime number such that $p \equiv \pm(d-1) \pmod{2(d^2 - 3d + 3)}$,
(2) $h = x^d + y^d + z^d$, where d is an even integer and p is a prime number such that $p \equiv d \pm 1 \pmod{2d}$.

Note that, for any given d, there are infinitely many primes satisfying conditions in (1) and (2). Then $X = \text{Proj } R$ is a nonsingular projective plane curve with

$$e_{\text{HK}}(X, \mathcal{L}) = \frac{3d}{4} + \frac{(d(d-3))^2}{4\text{d}p^2}.$$

Now, Corollary 5.4 of [10], stated above, implies that in these two examples $s_1 = 1$ and $l_1 = d(d-3)$.

Therefore, by Theorem 5.3 of [10], we have $I(F^*V) = d(d-3) = 2g-2$ and $l = l(X, F^*V) = 1$ and $s = s(X, E) = 0$. In particular,

$$I(F^*V) = (l-s)(2g-2) + pI(V)$$

for infinitely many primes. On the other hand, in Example (2) above (see [1]), for $d = 4$, $I(F^*E) = 0$ for infinitely many primes.

Remark. Let X be a nonsingular projective variety of dimension n, over a field k. Let $i : X \hookrightarrow \mathbf{P}^{n_0}$ be a closed embedding (we can always take $n_0 = 2\dim\ X + 1$). This gives a surjective map of sheaves of $\mathcal{O}_X$-modules

$$\Omega^1_{\mathbf{P}^{n_0}_k/k} \otimes \mathcal{O}_X \longrightarrow \Omega^1_{X/k}.$$

Let M_1 be a subsheaf of Ω^1_X such that $\mu_{\max}(\Omega^1_X) = \mu(M_1)$. Then the following composite map of sheaves of $\mathcal{O}_X$-modules

$$\Omega^1_{\mathbf{P}^{n_0}_k/k} \otimes \mathcal{O}_X(2) \longrightarrow \Omega^1_{X/k}(2) \longrightarrow (\Omega^1_{X/k}/M_1)(2)$$

is surjective. In particular, the sheaf $(\Omega^1_{X/k}/M_1)(2)$ is generated by global sections as $\Omega^1_{\mathbf{P}^{n_0}_k/k}(2)$ is so. Therefore $\deg(\Omega^1_{X/k}/M_1)(2) \geq 0$. In particular, $\deg(M_1)(2) \leq \deg(\Omega^1_X)(2)$, which gives the inequality

$$\mu(M_1) \leq \frac{n}{\text{rank}\,(M_1)}\mu(\Omega^1_X) + \deg(\mathcal{O}_X(2))\frac{n - \text{rank}\,(M_1)}{\text{rank}\,(M_1)}.$$

Note that $1 \leq \text{rank}\,(M_1) \leq n$, and the other invariants on the right-hand side of the inequality depend on $\deg\ X$ (with respect to the embedding i) and invariants of X, namely $\mu(\Omega^1_X)$ and $\dim\ X$.

Remark. We recall Lemma 1.5 of [11] with a small modification.

Lemma 2.9. *Let E be a torsion free coherent sheaf on X, where* $\dim\ X \geq 1$. *Let* $r = \text{rank}\ E$. *Suppose E is not semistable. Then, for the HN filtration* (2.1) *of E, we have*

$$\mu_i(E) - \mu_{i+1}(E) \geq \frac{4}{r^2}, \quad \textit{for every}\ \ 1 \leq i \leq l.$$

Proof. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} = E$$

be the HN filtration of E. Let $d_i = \deg(E_i/E_{i-1})$ and $r_i = \text{rank}\,(E_i/E_{i-1})$. Then

$$\mu_i(E) - \mu_{i+1}(E) = \frac{d_i}{r_i} - \frac{d_{i+1}}{r_{i+1}} \geq \frac{1}{r_i r_{i+1}} \geq \frac{4}{(r_i + r_{i+1})^2} \geq \frac{4}{r^2},$$

since $r_i + r_{i+1} \leq r$. This proves the lemma. □

Proof of Theorem 1.2. Due to Corollary 2.8, the arguments, as given in the proof of Lemma 1.8 in [11], can be adapted directly to the higher dimensional variety X. Hence the theorem follows. □

However Theorem 1.2 cannot be generalized to arbitrarily small prime characteristics $p > 0$. For this, one constructs the following counterexamples from examples due to Raynaud [6] and Monsky [5].

Example 2.10. Let X be a nonsingular projective curve of genus $g = pk + 1$ defined over an algebraically closed field of char $p > 2$, where k is any positive integer. Consider the canonical map of locally free sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B \to 0.$$

Then by Theorem 4.1.1 of [6], the vector bundle B is semistable of rank $p - 1$ and $\mu(B) = g - 1$.

Moreover, by Remark 4.1.2 of [6], the Frobenius pull back F^*B has the HN filtration given as follows:

$$0 = B_p \subset B_{p-1} \subset B_{p-2} \subset \cdots \subset B_2 \subset B_1 = F^*B, \tag{2.3}$$

where $B_i/B_{i+1} \simeq \Omega_X^{\otimes i}$, for all $1 \leq i \leq p-1$. Now we take a line bundle L on X of degree $d = 2k(p-1)$. Let $V = L \oplus B$. Then $0 \subset L \subset V$ is the HN filtration of V as

$$\mu(L) = 2k(p-1) > pk = g - 1 = \mu(B) = \mu(V/L)$$

and L and $V/L \simeq B$ are semistable vector bundles. On the other hand, one can check that the filtration

$$\begin{aligned} 0 \subset V_{p-1} &= F^*L \oplus B_{p-1} \subset V_{p-2} \\ &= F^*L \oplus B_{p-2} \subset \cdots \subset V_1 = F^*L \oplus B_1 = F^*V \end{aligned}$$

is the HN filtration of F^*V.

Example 2.11. Now we come back to Monsky's example of trinomial curves $h = x^d + y^d + z^d$ (see the discussion following Corollary 2.8 above), with conditions on d and p as before. Here, for the Syzygy bundle V, the HN filtration of F^*V is given by $0 \subset \mathcal{L}_1 \subset F^*V$, where

$$\deg\ \mathcal{L}_1 = -\frac{\mathrm{d}p}{2} + \frac{d(d-3)}{2} \quad \text{and} \quad \mu(F^*V) = -\frac{\mathrm{d}p}{2}.$$

We choose a line bundle $\mathcal{L}_0$ such that $\deg\ \mathcal{L}_0 = -\frac{d}{2} + \delta$, where δ is an integer such that $1 \leq \delta \cdot p \leq (d(d-3))/2$, e.g., if $p \geq 7$ is a prime number, then $d = p + 1$ and $\delta = 1$ will satisfy all these conditions. Let $W = \mathcal{L}_0 \oplus V$. Then it is easy to check that $0 \subset \mathcal{L}_0 \subset W$ is the HN filtration of the vector bundle W and

$$0 \subset \mathcal{L}_1 \subset \mathcal{L}_1 \oplus F^*\mathcal{L}_0 \subset F^*V \oplus F^*\mathcal{L}_0 = F^*W$$

is the HN filtration of F^*W. Therefore HN filtration of F^*W is not a refinement of the pull back of HN filtration of W.

In particular, the statement of Theorem 1.2 is not true in general, even for curves, for smaller (compared to the genus of the curve or rank of the vector bundle) characteristics.

COROLLARY 2.12

Let $p \geq \max\{(r+n-2), \quad \mu_{\max}(\Omega^1_X)\frac{r^3}{4}\}$, *where* E *is a torsion free sheaf, of rank* r, *over* X. *Suppose* E *is not semistable. Let*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E$$

be the HN filtration of E. *Then*

$$I(F^*E) \leq \mu_{\max}(\Omega^1_X)(\operatorname{rank}\left(\frac{E}{E_l}\right) + \operatorname{rank}(E_1) - 2) + pI(E).$$

In particular, if E *has HN filtration such that* $\operatorname{rank}(E_1) = \operatorname{rank}(E/E_l) = 1$ *then, for every* $s \geq 1$, *we have*

$$I(F^{s*}E) = p^s I(E).$$

Proof. By Theorem 1.2, the HN filtration of F^*E is of the form

$$\begin{aligned}0 &\subset E_{01} \subset \cdots \subset E_{0t_0} \subset \cdots \subset F^*E_i \subset E_{i1} \subset \cdots \subset E_{it_i} \subset F^*E_{i+1}\\ &\subset \cdots \subset F^*E.\end{aligned}$$

Therefore, for every $0 \leq i \leq l$, the sheaf E_{i+1}/E_i is semistable and

$$0 \subset \frac{E_{i,1}}{F^*E_i} \subset \frac{E_{i,2}}{F^*E_i} \subset \cdots \subset \frac{E_{i,t_i}}{F^*E_i} \subset \frac{F^*E_{i+1}}{F^*E_i}$$

is the HN filtration of F^*E_{i+1}/F^*E_i. Therefore, by Theorem 1.1,

$$\mu_{\max}(F^*E_1) - \mu_{\min}(F^*E_1) \leq \mu_{\max}(\Omega^1_X)(\operatorname{rank}(E_1) - 1) \tag{2.4}$$

Similarly,

$$\mu_{\max}\left(\frac{F^*E}{F^*E_l}\right) - \mu_{\min}\left(\frac{F^*E}{F^*E_l}\right) \leq \mu_{\max}(\Omega^1_X)(r - \operatorname{rank}(E_l) - 1). \tag{2.5}$$

But, by construction, it follows that

$$\mu_{\max}(F^*E_1) = \mu_{\max}(F^*E) \qquad \text{and} \qquad \mu_{\min}\left(\frac{F^*E}{E_l}\right) = \mu_{\min}(F^*E).$$

Therefore, by equations (2.4) and (2.5), we get

$$\begin{aligned}\mu_{\max}(F^*E) - \mu_{\min}(F^*E) \;&\leq\; \mu_{\max}(\Omega^1_X)\left(r - \operatorname{rank}\left(\frac{E_l}{E_1}\right) - 2\right)\\ &\quad + \mu_{\min}(F^*E_1) - \mu\left(\frac{E_{l,1}}{F^*E_l}\right).\end{aligned} \tag{2.6}$$

But

$$\mu_{\min}(F^*E_1) \leq \mu(F^*E_1) = p\mu(E_1) = p\mu_{\max}(E)$$

and

$$\mu\left(\frac{E_{l,1}}{F^*E_l}\right) = \mu_{\max}\left(\frac{F^*E}{F^*E_l}\right) \geq \mu\left(\frac{F^*E}{F^*E_l}\right) = p\mu\left(\frac{E}{E_l}\right) = p\mu_{\min}(E).$$

Therefore the right-hand side of the equality in equation (2.6) gives

$$\leq \mu_{\max}(\Omega_X^1)(r - \text{rank}\left(\frac{E_l}{E_1}\right) - 2) + p(\mu_{\max}(E) - \mu_{\min}(E)).$$

Now if rank $(E_1) =$ rank $(E/E_l) = 1$, then $I(F^*E) = pI(E)$. Since the HN filtration of $F^{s*}E$ is a refinement of the Frobenius pull back of the HN filtration of $F^{(s-1)*}E$, the first subsheaf and the last quotient sheaf in the HN filtration of $F^{(s-1)*}E$ are of rank $= 1$, for every $s \geq 1$. Hence $I(F^{s*}E) = p^s I(E)$. This proves the corollary. □

Remark. In fact, Lemma 2.15 below implies that, for any $s \geq 1$, the normalized HN slopes of $F^{s*}(E)$ can be estimated in terms of the HN slopes of E and a bounded constant. The proofs of Lemma 2.15, Propositions 2.17 and 2.18 are along the same lines as in Lemma 1.14, Propositions 1.16 and 2.2 of [11], respectively; we omit the details.

DEFINITION 2.13

Let E be a torsion free sheaf on X. A subsheaf $F_j \neq 0$ occurring in the HN filtration of $F^{s*}E$ is said to *almost descend* to a sheaf E_i occurring in the HN filtration of E if $F_j \subseteq F^{s*}E_i$ and E_i is the smallest subsheaf in the HN filtration of E, with this property.

Remark 2.14. Henceforth we assume that the characteristic p satisfies

$$p \geq \max\{r + n - 2, \mu_{\max}(\Omega_X^1)(r^3/4)\}.$$

Lemma 2.15. *Let E be a torsion free sheaf on X of rank r. Let $F_j \neq 0$ be a subsheaf in the HN filtration of $F^{s*}E$, which almost descends to a sheaf E_i occurring in the HN filtration of E. Then*

$$\frac{\mu_j(F^{s*}E)}{p^s} = \mu_i(E) + \frac{C}{p},$$

where $|C| \leq 2|\mu_{\max}(\Omega_X^1)|(r-1)$.

Notation 2.16. We fix a torsion free sheaf V on X of rank r with the HN filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} = V.$$

Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_t \subset F_{t+1} = F^{k*}V \tag{2.7}$$

be the HN filtration of $F^{k*}V$, and let

$$r_i(F^{k*}V) = \text{rank}\left(\frac{F_i}{F_{i-1}}\right) \qquad \text{and} \qquad a_i(F^{k*}V) = \frac{\mu_i(F^{k*}V)}{p^k}.$$

Moreover, we choose an integer $s \geq 0$ such that $F^{s*}(V)$ has a strongly semistable HN filtration and we denote

$$\tilde{a}_i(V) = a_i(F^{s*}(V)) \text{ and } \tilde{r}_i(V) = r_i(F^{s*}(V)).$$

Note that, by Theorem 1.2, these numbers are independent of the choice of such an s.

PROPOSITION 2.17

With the notation as above and the hypothesis on p as in Remark 2.14, *if a subsheaf F_j of the HN filtration of $F^{k*}V$ almost descends to a subsheaf E_i of the HN filtration of V then, for any $m \geq 1$,*

$$a_j(F^{k*}V)^m = \mu_i(V)^m + \frac{C}{p},$$

where $|C| \leq 4|\mu_{\max}(\Omega_X^1)|(r-1)(\max\{2|\mu_1(V)|, \ldots, 2|\mu_{l+1}(V)|, 2\}^{m-1})$.

PROPOSITION 2.18

Let $f : X_A \longrightarrow$ Spec A be a projective morphism of Noetherian schemes, smooth of relative dimension n, where A is a finitely generated $\mathbb{Z}$-algebra and is an integral domain. Let $\mathcal{O}_{X_A}(1)$ be an f-very ample invertible sheaf on X_A. Let V_A be a torsion free sheaf on X_A. For $s \in$ Spec A, let $V_s = V_A \otimes_A \overline{k(s)}$ be the induced torsion free sheaf on the smooth projective variety $X_s = X_A \otimes_A \overline{k(s)}$. Let $s_0 =$ Spec $Q(A)$ be the generic point of Spec A. *Then*,

(1) *for any $k \geq 0$ and $m \geq 0$, we have*

$$\lim_{s \to s_0} \sum_j r_j(F^{k*}V_s)a_j(F^{k*}V_s)^m = \sum_i r_i(V_{s_0})\mu_i(V_{s_0})^m.$$

(2) *Similarly*,

$$\lim_{s \to s_0} \sum_j \widetilde{r}_j(V_s)\widetilde{a}_j(V_s)^m = \sum_i r_i(V_{s_0})\mu_i(V_{s_0})^m,$$

where in both the limits, s *runs over closed points of* Spec A.

3. Some more generalities

Let X be a smooth projective variety over an algebraically closed field k. Let H be an ample line bundle on X.

Analogous to Theorem 1.2, which is given for vector bundles, we prove the following result for principal G-bundles, in the light of Proposition 3.4 of [4] (we follow the same notation as in [4]):

Let $E \to X$ be a principal G-bundle, where X is a smooth projective variety of dimension n, over a field of characteristic $p > 0$. Let P denote the Behrend parabolic of the principal G-bundle $E \to X$, and let P' denote the Behrend parabolic of the principal G-bundle $F^*E \to X$. Then $P' \subseteq P$ if

$$p \geq \max\left\{(\operatorname{rank}\mathfrak{g} + \dim X - 2),\ \frac{\mu_{\max}\left(\Omega_X^1\right)}{4}(\operatorname{rank}\mathfrak{g})^3,\right.$$
$$\left. 2\dim(G/Z(G)), 4h(G)\right\}.$$

We can replace G by $G/Z(G)$. Let $\mu_{\max}(\Omega_X^1) \leq 0$. Then, for $L =$ Levi subgroup of P, the prinicipal L-bundle $E_L \to X$ is strongly semistable, by Theorem 4.1 of [4]. Hence, by definition, P is the strong Behrend parabolic. In particular $P' = P$. So we can assume that $\mu_{\max}(\Omega_X^1) > 0$.

Therefore, by Proposition 3.4 of [4], for the Lie algebras $\mathfrak{p}$ and $\mathfrak{p}'$ associated to P and P' respectively, we have

$$E(\mathfrak{g})_0 = E_P(\mathfrak{p}) \quad \text{and} \quad (F^*E)(\mathfrak{g})_0 = (F^*E)_{P'}(\mathfrak{p}'),$$

where

$$\begin{aligned} 0 &\subset E(\mathfrak{g})_{-r} \subset \cdots \subset E(\mathfrak{g})_{-1} \subset E(\mathfrak{g})_0 \subset E(\mathfrak{g})_1 \\ &\subset \cdots \subset E(\mathfrak{g})_s = E(\mathfrak{g}) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 0 &\subset U_{-m_1} \subset U_{-m_1+1} \subset \cdots \subset U_0 = (F^*E)(\mathfrak{g})_0 \\ &\subset \cdots \subset U_{m_2+1} = (F^*E)(\mathfrak{g}) \end{aligned} \tag{3.2}$$

are the HN filtrations of the vector bundles $E(\mathfrak{g})$ and $(F^*E)(\mathfrak{g})$ respectively, such that

$$\mu(E(\mathfrak{g})_i/E(\mathfrak{g})_{i-1}) < 0,\ \text{for } i \geq 1 \quad \text{and} \quad \mu(E(\mathfrak{g})_i/E(\mathfrak{g})_{i-1}) \geq 0,\ \text{for } i \leq 0,$$

and similarly

$$\mu(U_j/U_{j-1}) < 0, \quad \text{for} \quad j \geq 1 \quad \text{and} \quad \mu(U_j/U_{j-1}) \geq 0, \quad \text{for} \quad j \leq 0.$$

As $p \geq \max\{(\operatorname{rank}\mathfrak{g}+\dim X-2), \frac{1}{4}\mu_{\max}(\Omega_X^1)(\operatorname{rank}\mathfrak{g})^3\}$, by Theorem 1.2, the Frobenius pull back of the filtration (3.1) is a subfiltration of (3.2), i.e., for $i \in \{-r, \ldots, s-1\}$, there exists $j \in \{-m_1, \ldots, m_2\}$ such that $F^*E(\mathfrak{g})_i = U_j$.

Claim. $U_0 \subseteq F^*E(\mathfrak{g})_0$.

Proof of the Claim. Suppose $U_0 \nsubseteq F^*E(\mathfrak{g})_0$. Then let $i > 0$ be the least integer such that $U_0 \subseteq F^*E(\mathfrak{g})_i$. This gives a nonzero map of bundles

$$U_0 \to F^*E(\mathfrak{g})_i/F^*E(\mathfrak{g})_{i-1}.$$

Therefore, by Lemma 2.4, we have

$$\mu_{\min}(U_0) \leq \mu_{\max}(F^*E(\mathfrak{g})_i/F^*E(\mathfrak{g})_{i-1}).$$

We note that

$$\mu(F^*E(\mathfrak{g})_0/F^*E(\mathfrak{g})_{-1}) = p\mu(E(\mathfrak{g})_0/E(\mathfrak{g})_{-1}),$$

where, by Proposition 3.6 of [4], for the nil radical $\mathfrak{n}$ of $\mathfrak{p}$, we have $E(\mathfrak{g})_0/E(\mathfrak{g})_{-1} = E_P(\mathfrak{p}/\mathfrak{n})$ and therefore this is a vector bundle of degree 0. By a similar argument, we have $\mu(U_0/U_{-1}) = 0$.

But

$$\mu_{\max}\left(\frac{F^*E(\mathfrak{g})_i}{F^*E(\mathfrak{g})_{i-1}}\right) \leq \mu_{\max}\left(\frac{F^*E(\mathfrak{g})_1}{F^*E(\mathfrak{g})_0}\right) < \mu_{\min}\left(\frac{F^*E(\mathfrak{g})_0}{F^*E(\mathfrak{g})_{-1}}\right)$$
$$\leq \mu\left(\frac{F^*E(\mathfrak{g})_0}{F^*E(\mathfrak{g})_{-1}}\right) = 0,$$

where the first and second inequalities follow because, as mentioned before, the Frobenius pull back of the filtration (3.1) is a subfiltration of the HN filtration (3.2). This implies that $\mu(U_0/U_1) = \mu_{\min}(U_0) < 0$, which is a contradiction. Hence the claim. Now

$$U_0 = (F^*E)(\mathfrak{g})_0 \subseteq F^*E(\mathfrak{g})_0$$

implies that

$$(F^*E)_{P'}(\mathfrak{p}') \subseteq F^*E_P(\mathfrak{p}).$$

This implies that $\mathfrak{p}' \subseteq \mathfrak{p}$ and therefore $P' \subseteq P$. This completes the proof. □

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