Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 4, November 2012, pp. 583–595.

Equivalent moduli of continuity, Bloch's theorem for pluriharmonic mappings in $\mathbb{B}^n$

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MS received 16 June 2011; revised 2 December 2011

Abstract. In this paper, we first establish a Schwarz–Pick type theorem for pluriharmonic mappings and then we apply it to discuss the equivalent norms on Lipschitz-type spaces. Finally, we obtain several Landau's and Bloch's type theorems for pluriharmonic mappings.

Keywords. Pluriharmonic mapping; Lipschitz-type space; Bloch constant; Schwarz' lemma; equivalent norm.

1. Introduction and main results

Let $\mathbb{C}^n$ denote the complex normed (Euclidean) space of dimension n. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, the conjugate of z, denoted by $\bar{z}$, is defined by $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$. For z and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, we write

$$\langle z, w\rangle := z \cdot w = \sum_{k=1}^{n} z_k \bar{w}_k \quad \text{and} \quad |z| := \langle z, z\rangle^{1/2} = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

For $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, we set $\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$. Also, we use $\mathbb{B}^n$ to denote the unit ball $\mathbb{B}^n(0, 1)$ and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

A continuous complex-valued function f defined on a domain $G \subset \mathbb{C}^n$ is said to be *pluriharmonic* if for each fixed $z \in G$ and $\theta \in \partial\mathbb{B}^n$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta : |\zeta| < d_G(z)\}$, where $d_G(z)$ denotes the distance from z to the boundary ∂G of G. It follows from Theorem 4.4.9 of [23] that a real-valued function u defined on G is pluriharmonic if and only if u is the real part of a holomorphic function on G. Clearly, a mapping $f : \mathbb{B}^n \to \mathbb{C}$ is pluriharmonic if and only if f has a representation $f = h + \bar{g}$, where g and h are holomorphic mappings. For a pluriharmonic mapping $f : \mathbb{B}^n \to \mathbb{C}$, we introduce the notation

$$\nabla f = (f_{z_1}, \dots, f_{z_n}) \qquad \text{and} \qquad \nabla \bar{f} = (f_{\bar{z}_1}, \dots, f_{\bar{z}_n}).$$

For a proper domain G of $\mathbb{C}^n$, let $\mathcal{H}_k(G)$ denote the class of all pluriharmonic mappings $f = h + \bar{g}$ defined from G into $\mathbb{C}$ such that for any $\theta \in \partial\mathbb{B}^n$,

$$|\nabla \bar{f}(z) \cdot \theta| \leq k|\nabla f(z) \cdot \bar{\theta}|$$

for $z \in G$, where $k \in (0, 1)$ is a constant, and both h and g are holomorphic in G.

Let f be a sense-preserving harmonic mapping from $\mathbb{D}$ into $\mathbb{C}$. We say that f is a *K-quasiregular harmonic mapping* if and only if

$$\frac{\Lambda_f(z)}{\lambda_f(z)} \leq K, \quad \text{i.e.,} \quad \frac{|f_{\bar{z}}(z)|}{|f_z(z)|} \leq \frac{K-1}{K+1}$$

for $z \in \mathbb{D}$, where $\Lambda_f = |f_z| + |f_{\bar{z}}|$ and $\lambda_f = |f_z| - |f_{\bar{z}}|$.

First we improve the Schwarz–Pick type theorem for K-quasiregular harmonic mappings obtained recently by Chen (Theorem 7 of [4]).

Lemma 1. *Let f be a sense-preserving and K-quasiregular harmonic mapping on $\mathbb{D}$ with $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$\Lambda_f(z) \leq K\frac{1-|f(z)|^2}{1-|z|^2} \leq \frac{4K}{\pi}\left(\frac{\cos(|f(z)|\pi/2)}{1-|z|^2}\right), \quad z \in \mathbb{D}. \tag{1}$$

Moreover, the first inequality of (1) *is sharp when* $K = 1$.

By using Lemma 1, we obtain a Schwarz–Pick type theorem for pluriharmonic mappings which is as follows:

Theorem 1. *Let $f \in \mathcal{H}_k(\mathbb{B}^n)$ and $|f(z)| < 1$ for $z \in \mathbb{B}^n$. Then for each $\theta \in \partial\mathbb{B}^n$,*

$$|\nabla \bar{f}(z) \cdot \theta| + |\nabla f(z) \cdot \bar{\theta}| \leq K\frac{1-|f(z)|^2}{1-|z|^2}, \quad K = \frac{1+k}{1-k}.$$

Proofs of Lemma 1 and Theorem 1 will be given in §2.

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset G of $\mathbb{C}^n$, a function $f : G \to \mathbb{C}$ is said to belong to the *Lipschitz space* $\Lambda_\omega(G)$ if there is a positive constant C such that

$$|f(z) - f(w)| \leq C\omega(|z-w|) \tag{2}$$

for all $z, w \in G$. For $\delta_0 > 0$, let

$$\int_0^\delta \frac{\omega(t)}{t}\,\mathrm{d}t \leq C\omega(\delta), \qquad 0 < \delta < \delta_0 \tag{3}$$

and

$$\delta\int_\delta^{+\infty} \frac{\omega(t)}{t^2}\,\mathrm{d}t \leq C\omega(\delta), \qquad 0 < \delta < \delta_0. \tag{4}$$

A majorant ω is said to be *regular* if it satisfies conditions (3) and (4) (see [8, 22]).

Let G be a proper subdomain of $\mathbb{C}^n$. We say that a function f belongs to the *local Lipschitz space* loc $\Lambda_\omega(G)$ if there is a constant $C > 0$ satisfying (2) for all $z, w \in G$

with $|z-w| < \frac{1}{2}d_G(z)$. Moreover, G is said to be a Λ_ω-*extension domain* if $\Lambda_\omega(G) = \text{loc}\,\Lambda_\omega(G)$. The geometric characterization of Λ_ω-extension domains was first given by Gehring and Martio [11]. Later, Lappalainen [16] extended it to the general case, and proved that G is a Λ_ω-extension domain if and only if each pair of points $z, w \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d_G(z))}{d_G(z)}\,\mathrm{d}s(z) \le C\omega(|z-w|) \tag{5}$$

with some fixed positive constant $C = C(G,\omega)$, where $\mathrm{d}s$ stands for the arclength measure on γ. Furthermore, Lappalainen (Theorem 4.12 of [16]) proved that Λ_ω-extension domains exist only for majorants ω satisfying the inequality (3).

For $z_1, z_2 \in G \subset \mathbb{C}^n$, let

$$d_{\omega,G}(z_1,z_2) := \inf \int_\gamma \frac{\omega(d_G(z))}{d_G(z)}\,\mathrm{d}s(z),$$

where the infimum is taken over all rectifiable curves $\gamma \subset G$ joining z_1 and z_2. We say that $f \in \Lambda_{\omega,\inf}(G)$ whenever

$$|f(z_1) - f(z_2)| \le C d_{\omega,G}(z_1,z_2) \qquad \text{for } z_1, z_2 \in G,$$

where C is a positive constant which depends only on f (see [13]).

Dyakonov [8] characterized the holomorphic functions in Λ_ω in terms of their modulus. Later in Theorems A and B of [22], Pavlović came up with a relatively simple proof of the results of Dyakonov. Recently, many authors considered this topic and generalized the work of Dyakonov to pseudo-holomorphic functions and real harmonic functions of several variables for some special majorants $\omega(t) = t^\alpha$, where $\alpha > 0$ (see [2, 9, 12, 14, 15, 18–21]). By applying Theorem 1, we extend Theorems A and B of [22] to the case of pluriharmonic mappings.

Theorem 2. *Let ω be a majorant satisfying (3), and let G be a Λ_ω-extension. If $f \in \mathcal{H}_k(G)$ and is continuous up to the boundary ∂G, then*

$$f \in \Lambda_\omega(G) \Longleftrightarrow |f| \in \Lambda_\omega(G) \Longleftrightarrow |f| \in \Lambda_\omega(G, \partial G),$$

where $\Lambda_\omega(G,\partial G)$ denotes the class of continuous functions f on $G \cup \partial G$ which satisfy (2) with some positive constant C, whenever $z \in G$ and $w \in \partial G$.

Theorem 3. *Let ω be a majorant satisfying (3). If $f \in \mathcal{H}_k(G)$, then*

$$f \in \Lambda_{\omega,\inf}(G) \Longleftrightarrow |f| \in \Lambda_{\omega,\inf}(G).$$

We remark that Theorems 2 and 3 are the generalizations of Theorems 1 and 2 of [7] and respectively.

To state our final result, we need some preparations. First we recall that a mapping $f : \Omega \to \mathbb{C}^n$ is said to be *vector-valued pluriharmonic* if every component of f is pluriharmonic. Let $H(\mathbb{B}^n, \mathbb{C}^n)$ denote the set of all pluriharmonic mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$. Obviously, a mapping $f \in H(\mathbb{B}^n, \mathbb{C}^n)$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where g and h are holomorphic mappings $\mathbb{B}^n$ into $\mathbb{C}^n$. It is convenient to identify each point $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ with an $n\times 1$ column matrix so that

$$z=\begin{pmatrix} z_1\\ \vdots\\ z_n\end{pmatrix},\quad \mathrm{d}z=\begin{pmatrix} \mathrm{d}z_1\\ \vdots\\ \mathrm{d}z_n\end{pmatrix}\quad\text{and}\quad \mathrm{d}\bar{z}=\begin{pmatrix} \mathrm{d}\bar{z}_1\\ \vdots\\ \mathrm{d}\bar{z}_n\end{pmatrix}.$$

For a $f=(f_1,\ldots,f_n)\in H(\mathbb{B}^n,\mathbb{C}^n)$, we denote by $\partial f/\partial z_j$ the column vector formed by $\partial f_1/\partial z_j,\ldots,\partial f_n/\partial z_j$, and

$$f_z=\left(\frac{\partial f}{\partial z_1}\ \cdots\ \frac{\partial f}{\partial z_n}\right):=\left(\frac{\partial f_i}{\partial z_j}\right)_{n\times n},$$

the $n\times n$ matrix formed by these column vectors, namely, by the complex gradients $\nabla f_1,\ldots,\nabla f_n$. Similarly,

$$f_{\bar{z}}=\left(\frac{\partial f}{\partial \bar{z}_1}\ \cdots\ \frac{\partial f}{\partial \bar{z}_n}\right):=\left(\frac{\partial f_i}{\partial \bar{z}_j}\right)_{n\times n},$$

the $n\times n$ matrix formed by the column vectors $\partial f/\partial\bar{z}_j$ for $j\in\{1,\ldots,n\}$. For an $n\times n$ matrix A, we introduce the operator norm

$$|A|=\sup_{x\neq 0}\frac{|Ax|}{|x|}=\max\{|A\theta|:\ \theta\in\partial\mathbb{B}^n\}.$$

For pluriharmonic mappings $f:\mathbb{B}^n\to\mathbb{C}^n$, we use the following standard notations (cf. [6]):

$$\Lambda_f(z)=\max_{\theta\in\partial\mathbb{B}^n}|f_z(z)\theta+f_{\bar{z}}(z)\bar{\theta}|\qquad\text{and}\qquad \lambda_f(z)=\min_{\theta\in\partial\mathbb{B}^n}|f_z(z)\theta+f_{\bar{z}}(z)\bar{\theta}|.$$

Let $f=(f_1,\ldots,f_n)\in H(\mathbb{B}^n,\mathbb{C}^n)$. For $j\in\{1,\ldots,n\}$, we let $z=(z_1,\ldots,z_n)$, $z_j=x_j+iy_j$ and $f_j(z)=u_j(z)+iv_j(z)$, where u_j and v_j are real pluriharmonic functions from $\mathbb{B}^n$ into $\mathbb{R}$. We denote the real Jacobian matrix of f by

$$J_f=\begin{pmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n}\\
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n}\\
 & & & \vdots & & & \\
\frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n}\\
\frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n}
\end{pmatrix}.$$

Let $\mathbb{B}^{2n}_{\mathbb{R}}$ denote the unit ball of $\mathbb{R}^{2n}$. Then (see [6])

$$\Lambda_f=\max_{\theta\in\partial\mathbb{B}^{2n}_{\mathbb{R}}}|J_f\theta|\qquad\text{and}\qquad \lambda_f=\min_{\theta\in\partial\mathbb{B}^{2n}_{\mathbb{R}}}|J_f\theta|.$$

We use $b_h^p(\mathbb{B}^n, \mathbb{C}^n)$ to denote the *pluriharmonic Bergman space* consisting of all pluriharmonic mappings $f \in H(\mathbb{B}^n, \mathbb{C}^n)$ such that

$$\|f\|_{b^p} = \left(\int_{\mathbb{B}^n} |f(z)|^p \, \mathrm{d}V(z)\right)^{1/p} < \infty \qquad \text{or}$$
$$\|f\|_{b_N^p} = \left(\int_{\mathbb{B}^n} |f(z)|^p \, \mathrm{d}V_N(z)\right)^{1/p} < \infty,$$

where $p \in (0, \infty)$, $n \geq 2$, dV denotes the Lebesgue volume measure on $\mathbb{C}^n$ and $\mathrm{d}V_N$ denotes the normalized Lebesgue volume measure on $\mathbb{B}^n$. Obviously, if $f \in H(\mathbb{B}^n, \mathbb{C}^n)$ and f is bounded, then $f \in b_h^p(\mathbb{B}^n, \mathbb{C}^n)$.

Theorem 4. *Let $r \in (0, 1)$ and $f \in b_h^p(\mathbb{B}^n, \mathbb{C}^n)$ with $\|f\|_{b_N^p} \leq M$, $f(0) = 0$ and* $\det J_f(0) = \alpha > 0$. *Then f is injective in $\mathbb{B}^n(0, r\rho(r))$ with*

$$\rho(r) = \frac{\alpha \pi^{2n+1}}{4m(4M(r))^{2n}}$$

and $f(\mathbb{B}^n(0, r\rho(r)))$ contains a univalent ball with the radius

$$R \geq \max_{0<r<1} \left\{ \frac{\alpha \pi^{4n} r}{8m(4M(r))^{4n-1}} \right\},$$

where

$$M(r) = \frac{M}{r(1-r)^{2n/p}} \qquad \text{and} \qquad m = 2\sqrt{2}\left(\frac{3+\sqrt{17}}{(\sqrt{5-\sqrt{17}})(1+\sqrt{17})}\right) \approx 4.2. \tag{6}$$

We remark that Theorem 4 is a generalization of Theorem 5 of [6]. We now recall that a holomorphic function $f : \mathbb{B}^n \to \mathbb{C}^n$ is *convex* in $\mathbb{B}^n$ if it is one-to-one and the range $f(\mathbb{B}^n)$ is a convex domain.

Theorem 5. *Suppose $f = h + \bar{g} \in H(\mathbb{B}^n, \mathbb{C}^n)$, $f(0) = 0$, $|f_{\bar{z}}(0)| = 0$ and* $\det f_z(0) = I_n$, *where h is a convex biholomorphic mapping and g is a holomorphic mapping. If for any $z \in \mathbb{B}^n$, $|f_{\bar{z}}(z)| \leq |f_z(z)|$, then f is univalent in $\mathbb{B}^n(0, \rho_1)$, where*

$$\rho_1 = \frac{1}{m_2 + m_3} \quad \text{with } m_2 \approx 9.444 \text{ and } m_3 = 6.75.$$

Moreover, the range $f(\mathbb{B}^n(0, \rho_1))$ contains a univalent ball with center 0 *and radius at least R_1, where $R_1 = \frac{\rho_1}{2}$.*

The precise values of m_2 and m_3 are given in the proof of Theorem 5.

A continuous mapping $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called *quasiregular* if $f \in W^1_{n,\mathrm{loc}}(\Omega)$ and

$$|f'(x)|^n \leq K J_f(x) \text{ for almost every } x \in \Omega,$$

where K (≥ 1) is a constant, $f \in W^1_{n,\mathrm{loc}}(\Omega)$ means that the distributional derivatives $\partial f_j / \partial x_k$ of the coordinates f_j of f are locally in L^n and $J_f(x)$ denotes the Jacobian of f (cf. [25]).

DEFINITION 1

A pluriharmonic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ is said to be a (K, K_1)-*pluriharmonic mapping* if for each $z \in \mathbb{B}^n$ and $\theta \in \partial\mathbb{B}^n$,

$$|f_z(z)|^n \leq K|\det f_z(z)| \qquad \text{and} \qquad K_1|f_{\overline{z}}(z)\theta| \leq |f_z(z)\overline{\theta}|, \tag{7}$$

where K (≥ 1) and K_1 (> 1) are constants.

Obviously, every (K, K_1)-pluriharmonic mapping $f : \mathbb{B}^n \to \mathbb{C}^n$ is called Wu K-mapping if $f_{\overline{z}} \equiv 0$ (see [26]). In fact, holomorphic K-quasiregular mappings are referred to as Wu $K^{1-\frac{1}{n}}$-mappings (cf. [5, 26]).

For a holomorphic mapping f from the unit ball $\mathbb{B}^n$ into $\mathbb{C}^n$, $\mathbb{B}^n(a, r)$ is called a *Schlicht ball* of f if there is a subregion $\Omega \subset \mathbb{B}^n$ such that f maps Ω biholomorphically onto $\mathbb{B}^n(a, r)$. We denote by B_f the least upper bound of radii of all Schlicht balls contained in $f(\mathbb{B}^n)$ and call this the Bloch radius of f. The classical theorem of Bloch for holomorphic functions in the unit disk fails to extend to general holomorphic mappings in the ball of $\mathbb{C}^n$ (see [24, 26]). However, in 1946, Bochner [3] proved that Bloch's theorem does hold for a class of real harmonic quasiregular mappings. Recently, Chen and Gauthier [6] proved that Bloch's theorem also holds for a class of pluriharmonic K-mappings.

In this paper, our last aim is to prove the existence of Bloch's constant for a new class of pluriharmonic mappings. Our result is also a generalization of Theorem 6 of [5]. We now state a version of Bloch's theorem for a class of (K, K_1)-quasiregular pluriharmonic mappings.

Theorem 6. *Suppose f is a (K, K_1)-quasiregular pluriharmonic mapping of $\mathbb{B}^n$ into $\mathbb{C}^n$ with $|J_f(0)| = 1$. Then $f(\mathbb{B}^n)$ contains a Schlicht ball with radius at least*

$$B_f \geq \max_{0<t<1} \left\{ \frac{\pi^{4n} t}{8m(4M(t))^{4n-1}} \right\}, \quad \textit{with} \;\; M(t) = \frac{K^{\frac{1}{n}}(1+K_1)}{tK_1} \log\left(\frac{1}{1-t}\right),$$

where m is defined as in Theorem 4.

Proofs of Theorems 2 and 3 are given in §3 while the proofs of Theorems 4, 5 and 6 are given in §4.

2. Schwarz–Pick lemma for pluriharmonic mappings in $\mathbb{B}^n$

Let Ω be a domain in $\mathbb{C}$ and $\rho > 0$ a conformal metric in Ω. The Gaussian curvature of the domain is given by $K_\rho = -(1/(2\rho))\Delta \log \rho$. We denote by $\lambda(z)|\mathrm{d}z|^2$ the hyperbolic metric in $\mathbb{D}$, where $\lambda(z) = 4/(1 - |z|^2)^2$.

Lemma A (*Ahlfors-Schwarz lemma*). *If $\rho > 0$ is a C^2-function* (*metric density*) *in $\mathbb{D}$ and Gaussian curvature $K_\rho \leq -1$, then $\rho \leq \lambda$* (*cf.* [1]).

Proof of Lemma 1. By assumption, we observe that f is an open mapping, and so $|f_z(z)| \neq 0$ in $\mathbb{D}$. Let

$$\rho(z) = \frac{4}{(K+1)^2} \lambda(f(z))|f_z(z)|^2, \quad z \in \mathbb{D}.$$

Then $ds^2 = \rho(z)|dz|^2$. Simple calculations yield

$$\begin{aligned}\Delta \log \rho(z) &= \Delta \log\left[\frac{4}{(K+1)^2}\lambda(f(z))|f_z(z)|^2\right]\\ &= 4\,(\log(\lambda(f(z))))_{z\bar{z}}\\ &= \frac{8|f_z(z)|^2}{(1-|f(z)|^2)^2}\left[1+\frac{|f_{\bar{z}}(z)|^2}{|f_z(z)|^2}+2\mathrm{Re}\left(\frac{f^2(z)\bar{f}_z(z)\bar{f}\bar{z}(z)}{|f_z(z)|^2}\right)\right]\\ &= \frac{(K+1)^2\rho(z)}{2}\left[1+\frac{|f_{\bar{z}}(z)|^2}{|f_z(z)|^2}+2\mathrm{Re}\left(\frac{f^2(z)\bar{f}_z(z)\bar{f}\bar{z}(z)}{|f_z(z)|^2}\right)\right]\\ &\geq \frac{(K+1)^2\rho(z)}{2}\left(1-\frac{|f_{\bar{z}}(z)|}{|f_z(z)|}\right)^2\\ &\geq \frac{(K+1)^2\rho(z)}{2}\left(1-\frac{K-1}{K+1}\right)^2 = 2\rho(z)\end{aligned}$$

which, together with the definition of K_ρ, gives $K_\rho(z) \leq -1$. Thus, by Lemma A, we have

$$\rho(z) = \frac{4}{(K+1)^2}\lambda(f(z))|f_z(z)|^2 \leq \lambda(z)$$

whence

$$\Lambda_f(z) \leq \frac{2K}{1+K}|f_z(z)| \leq K\frac{1-|f(z)|^2}{1-|z|^2}.$$

The proof of the lemma is complete. □

Proof of Theorem 1. For each fixed $\theta \in \partial\mathbb{B}^n$, let $F(\zeta) = f(\theta\zeta)$ in $\mathbb{D}$. Then F is harmonic and $|F(\zeta)| < 1$ on $\mathbb{D}$. It follows that

$$\begin{aligned}\Lambda_F &= |F_\zeta| + |F_{\bar{\zeta}}| = |\nabla f \cdot \bar{\theta}| + |\nabla \bar{f} \cdot \theta|\\ &\leq K(|\nabla f \cdot \bar{\theta}| - |\nabla \bar{f} \cdot \theta|) = K(|F_\zeta| - |F_{\bar{\zeta}}|)\end{aligned}$$

which implies that F is a K-quasiregular harmonic mapping in $\mathbb{D}$, where $K = \frac{1+k}{1-k}$. Hence, Lemma A shows that

$$|\nabla f(z)\cdot\bar{\theta}| + |\nabla \bar{f}(z)\cdot\theta| = \Lambda_F(\zeta) \leq K\frac{1-|F(\zeta)|^2}{1-|\zeta|^2} = K\frac{1-|f(z)|^2}{1-|z|^2},$$

where $z = \zeta\theta$. This completes the proof. □

3. Equivalent moduli of continuity for pluriharmonic mappings

Proof of Theorem 2. The implications $f \in \Lambda_\omega(G) \Rightarrow |f| \in \Lambda_\omega(G) \Rightarrow |f| \in \Lambda_\omega(G, \partial G)$ are obvious. Therefore, we only need to prove the implication: $|f| \in \Lambda_\omega(G, \partial G) \Rightarrow f \in \Lambda_\omega(G)$. In order to prove this, for a fixed $z \in G$, let

$$M_z := \sup\{|f(\zeta)| : \ |\zeta - z| < d_G(z)\}, \tag{8}$$

and we define the following function:

$$F(\eta)=\frac{f(z+d_G(z)\eta)}{M_z},\quad \eta\in\mathbb{B}^n.$$

By a simple calculation, we obtain that for $\theta\in\partial\mathbb{B}^n$,

$$|\nabla\bar{F}(\eta)\cdot\theta|=\frac{d_G(z)}{M_z}|\nabla\bar{f}(\xi)\cdot\theta|\leq\frac{kd_G(z)}{M_z}|\nabla f(\xi)\cdot\bar{\theta}|=k|\nabla F(\eta)\cdot\bar{\theta}|$$

where $\xi=z+d_G(z)\eta$. Then, $F\in\mathcal{H}_k(\mathbb{B}^n)$ and $|F(\eta)|\leq 1$ in $\mathbb{B}^n$. By Theorem 1, we have that for $\theta\in\partial\mathbb{B}^n$,

$$|\nabla F(0)\cdot\bar{\theta}|+|\nabla\bar{F}(0)\cdot\theta|\leq K(1-|F(0)|^2)$$

which in turn gives

$$d_G(z)(|\nabla f(z)\cdot\bar{\theta}|+|\nabla\bar{f}(z)\cdot\theta|)\leq 2K(M_z-|f(z)|),\quad K=\frac{1+k}{1-k}. \tag{9}$$

For a fixed $\varepsilon_0>0$, there exists a $\zeta\in\partial G$ such that $|\zeta-z|<(1+\varepsilon_0)d_G(z)$. Then, for $w\in\mathbb{B}^n(z,d_G(z))$, we have

$$\begin{aligned}|f(w)|-|f(z)| &\leq ||f(w)|-|f(\zeta)||+||f(\zeta)|-|f(z)||\\ &\leq C\omega((2+\varepsilon_0)d_G(z))+C\omega((1+\varepsilon_0)d_G(z)),\end{aligned}$$

where C is a positive constant. Now we take $\varepsilon_0=1$. Then

$$\sup_{w\in\mathbb{B}^n(z,d_G(z))}(|f(w)|-|f(z)|)\leq|f(w)|-|f(z)|\leq 5C\omega(d_G(z))$$

whence $M_z-|f(z)|\leq 5C\omega(d_G(z))$, where C is a positive constant. Thus for any $\theta\in\mathbb{B}^n$, by (9) and the last inequality, we have

$$|\nabla f(z)\cdot\bar{\theta}|+|\nabla\bar{f}(z)\cdot\theta|\leq 10CK\cdot\frac{\omega(d_G(z))}{d_G(z)}\quad\text{for } z\in G. \tag{10}$$

For points $z_1,z_2\in G$, let $\gamma\subset G$ be a rectifiable curve which joins z_1 and z_2 satisfying (5). Integrating (10) along γ, we obtain that

$$|f(z_1)-f(z_2)|\leq 10CK\int_\gamma\frac{\omega(d_G(z))}{\mathrm{d}_G(z)}\,\mathrm{d}s(z). \tag{11}$$

Therefore, (5) and (11) yield $|f(z_1)-f(z_2)|\leq C_1\cdot\omega(|z_1-z_2|)$, where C_1 is a positive constant. This completes the proof. □

Proof of Theorem 3. The implication $f\in\Lambda_{\omega,\inf}(G)\Rightarrow|f|\in\Lambda_{\omega,\inf}(G)$ is obvious. We need only to prove that $|f|\in\Lambda_{\omega,\inf}(G)\Rightarrow f\in\Lambda_{\omega,\inf}(G)$.

Assume that $|f|\in\Lambda_{\omega,\inf}(G)$ and fix $z\in G$. Then it follows from a similar reasoning as in the proof of inequality (9) that for $\theta\in\partial\mathbb{B}^n$,

$$d_G(z)(|\nabla f(z)\cdot\bar{\theta}|+|\nabla\bar{f}(z)\cdot\theta|)\leq 2K(M_z-|f(z)|), \tag{12}$$

where M_z is defined by (8). For $w \in \mathbb{B}^n(z, d_G(z))$, there exists a positive constant C such that

$$|f(w)| - |f(z)| \le C d_{\omega,G}(w, z) \le C \int_{[w,z]} \frac{\omega(d_G(\zeta))}{\mathrm{d}_G(\zeta)} \,\mathrm{d}s(\zeta), \tag{13}$$

where $[w, z]$ denotes the straight segment with endpoints w and z. We observe that if $\zeta \in [w, z]$, then one has $[w, z] \subset \mathbb{B}^n(z, d_G(z)) \subset G$ and therefore,

$$d_G(\zeta) \ge d_{\mathbb{B}^n(z,d_G(z))}(\zeta)$$

which gives

$$\frac{\omega(d_G(\zeta))}{d_G(\zeta)} \le \frac{\omega(d_{\mathbb{B}^n(z,d_G(z))}(\zeta))}{d_{\mathbb{B}^n(z,d_G(z))}(\zeta)}. \tag{14}$$

For each $w \in \mathbb{B}^n(z, d_G(z))$, (13) and (14) imply that

$$\begin{aligned}
|f(w)| - |f(z)| &\le C \int_{[w,z]} \frac{\omega(d_G(\zeta))}{d_G(\zeta)} \,\mathrm{d}s(\zeta) \\
&\le C \int_{[w,z]} \frac{\omega(d_{\mathbb{B}^n(z,d_G(z))}(\zeta))}{d_{\mathbb{B}^n(z,d_G(z))}(\zeta)} \,\mathrm{d}s(\zeta) \\
&= C \int_{[w,z]} \frac{\omega(d_G(z) - |\zeta - z|)}{d_G(z) - |\zeta - z|} \,\mathrm{d}s(\zeta) \\
&\le C \int_0^{d_G(z)} \frac{\omega(t)}{t} \,\mathrm{d}t \\
&\le C\omega(d_G(z)).
\end{aligned}$$

From the last inequality, we obtain that

$$M_z - |f(z)| \le C\omega(d_G(z)). \tag{15}$$

Again, for any $z_1, z_2 \in G$, by (12) and (15), there exists a positive constant C_1 such that $|f(z_1) - f(z_2)| \le C_1 d_{\omega,G}(z_1, z_2)$. The proof of the theorem is complete. □

4. Landau's and Bloch's theorem for pluriharmonic mappings

The following three lemmas are useful for the proof of Theorem 6.

Lemma B (*Lemma* 1 *of* [6] *or Lemma* 4 *of* [17]). *Let A be an $n \times n$ complex* (*real*) *matrix. Then for any unit vector $\theta \in \partial\mathbb{B}^n$, the inequality $|A\theta| \ge |\det A|/|A|^{n-1}$ holds.*

Lemma C (*Lemma* 4 *of* [5]). *Let A be a holomorphic mapping from $\mathbb{B}^n(0, r)$ into the space of $n \times n$ complex matrices. If $A(0) = 0$ and $|A(z)| \le M$ in $\mathbb{B}^n(0, r)$, then*

$$|A(z)| \le \frac{M|z|}{r}, \quad z \in \mathbb{B}^n(0, r).$$

Proof of Theorem 4. Fix $z \in \mathbb{B}^n$ and let $D_z = \{\zeta \in \mathbb{C}^n : |\zeta - z| < 1 - |z|\}$. Then by Jensen's inequality, for $r \in [0, 1 - |z|)$ and $p \in [1, \infty)$, we have

$$|f(z)|^p \leq \int_{\partial\mathbb{B}^n} |f(z + r\zeta)|^p \, \mathrm{d}\sigma(\zeta). \tag{16}$$

Multiplying the formula (16) by $2nr^{2n-1}$ and integrating from 0 to $1 - |z|$, we have

$$\begin{aligned}(1 - |z|)^{2n}|f(z)|^p &\leq \int_0^{1-|z|} \left[2nr^{2n-1} \int_{\partial\mathbb{B}^n} |f(z + r\zeta)|^p \, \mathrm{d}\sigma(\zeta)\right] \mathrm{d}r \\ &= \int_{D_z} |f(z)|^p \, \mathrm{d}V(z) \\ &\leq \int_{\mathbb{B}^n} |f(z)|^p \, \mathrm{d}V_N(z) \leq M^p\end{aligned}$$

which gives

$$|f(z)| \leq \frac{M}{(1 - |z|)^{2n/p}}.$$

For $\zeta \in \mathbb{B}^n$ and $r \in (0, 1)$, let $F(\zeta) = r^{-1} f(r\zeta)$. Then

$$|F(\zeta)| \leq \frac{M}{r(1 - r)^{2n/p}} = M(r) \qquad \text{and} \qquad J_F(0) = J_f(0) = \alpha.$$

Using Theorem 5 of [6], we obtain that f is injective in $\mathbb{B}^n(0, r\rho(r))$ with

$$\rho(r) = \frac{\alpha\pi^{2n+1}}{4m(4M(r))^{2n}}$$

and $f(\mathbb{B}^n(0, r\rho(r)))$ contains a univalent ball with radius

$$R \geq \max_{0<r<1} \left\{\frac{\alpha\pi^{4n} r}{8m(4M(r))^{4n-1}}\right\},$$

where m is given by (6). The proof is complete. □

For the proof of Theorem 5, we need the following lemma due to Fitzgerald and Thomas [10].

Lemma D (*Proposition* 2.2 *of* [10]). *Let f be a convex mapping from $\mathbb{B}^n$ into $\mathbb{C}^n$ with $f(0) = 0$ and $f'(0) = I_n$, the $n \times n$ identity matrix. Suppose t is a positive integer, $\theta \in \partial\mathbb{B}^n$ and $r \in (0, 1)$. Then*

$$|D_\theta^t f(r\theta)| \leq \frac{t!}{(1 - r)^{t+1}}.$$

Proof of Theorem 5. We begin to note that Lemma D gives

$$|f_z(z) - f_z(0)| \leq 1 + \frac{1}{(1 - |z|)^2} \quad \text{for } z \in \mathbb{B}^n.$$

Let $W_2(r) = [1 + (1 - r)^2]/[r(1 - r)^2]$ for $r \in (0, 1)$. Then

$$W_2(r_2) = \min_{r\in(0,1)} \{W_2(r)\}, \quad \text{with } r_2 = 1 - \sqrt[3]{1+\sqrt{2}} + \frac{1}{\sqrt[3]{1+\sqrt{2}}} \approx 0.404.$$

Denote $W_2(r_2)$ by m_2. Then $m_2 \approx 9.444$ and, by Lemma C, we have

$$|f_z(z) - f_z(0)| \le m_2|z| \ \text{ for } |z| \le r_2,$$

By hypotheses, we have

$$|f_{\bar{z}}(z) - f_{\bar{z}}(0)| \le |f_z(z)| \le \frac{1}{(1-|z|)^2} \ \text{ for } z \in \mathbb{B}^n.$$

Let $W_3(r) = 1/[r(1 - r)^2]$ for $r \in (0, 1)$. Then

$$W_3(r_3) = \min_{r\in(0,1)} \{W_3(r)\}, \ \text{ with } r_3 = 1/3.$$

We denote $W_3(r_3)$ by m_3. Then $m_3 = 6.75$ and, by Lemma C, we have

$$|f_z(z) - f_z(0)| \le m_3|z| \ \text{ for } |z| \le r_3.$$

Hence for $z \in \mathbb{B}^n(0, \rho_1)$ with $\rho_1 \le r_3$,

$$|f_z(z) - f_z(0)| \le m_2|z| \ \text{ and } \ |f_{\bar{z}}(z) - f_{\bar{z}}(0)| \le m_3|z|.$$

In order to prove the univalence of f in $\mathbb{B}^n(0, \rho_1)$, we choose two distinct points z', $z'' \in \mathbb{B}^n(0, \rho_1)$ and let $[z', z'']$ denote the segment from z' to z'' with the endpoints z' and z''. Then

$$\begin{aligned} |f(z') - f(z'')| &\ge \left| \int_{[z',z'']} f_z(0)\mathrm{d}z + f_{\bar{z}}(0)\, \mathrm{d}\bar{z} \right| \\ &\quad - \left| \int_{[z',z'']} (f_z(z) - f_z(0))\, \mathrm{d}z + (f_{\bar{z}}(z) - f_{\bar{z}}(0))\, \mathrm{d}\bar{z} \right| \\ &\ge |z' - z''|\{1 - (m_2 + m_3)\rho_1\} \\ &> 0 \end{aligned}$$

which shows that f is univalent in $\mathbb{B}^n(0, \rho_1)$. Furthermore, for any z with $|z| = \rho_1$, we have

$$\begin{aligned} |f(z) - f(0)| &\ge \left| \int_{[0,\rho_1]} f_z(0)\, \mathrm{d}z + f_{\bar{z}}(0)\, \mathrm{d}\bar{z} \right| \\ &\quad - \left| \int_{[0,\rho_1]} (f_z(z) - f_z(0))\, \mathrm{d}z + (f_{\bar{z}}(z) - f_{\bar{z}}(0))\, \mathrm{d}\bar{z} \right| \\ &\ge \frac{\rho_1}{2}. \end{aligned}$$

The proof of this theorem is complete. □

Proof of Theorem 6. Without loss of generality, we assume that f is pluriharmonic on $\overline{\mathbb{B}}^n$. Otherwise, we replace $f(z)$ by $f(sz)$ for some $s \in (0,1)$. Then there exists some $z_0 \in \mathbb{B}^n$ such that

(1) $(1-|z_0|)^n|\det f_z(z_0)| = 1$; and
(2) $(1-|z|)^n|\det f_z(z)| \le 1$ for all z in the set $\{z:\ |z_0| = r \le |z| \le 1\}$.

Hence it follows from the fact that $|\det f_z(z)| \le |\det f_z(z_0)|$ for any z in $\{z:\ |z| = r = |z_0|\}$ and the maximum principle that

$$|\det f_z(z)| \le |\det f_z(z_0)|$$

in the disk $\{z:\ |z| \le r\}$. For $\zeta \in \mathbb{B}^n$ and $t \in (0,1)$, let

$$F(\zeta) = \frac{1}{t}[f(p(\zeta)) - f(z_0)],$$

where $p(\zeta) = z_0 + t(1-r)\zeta$. Then

$$|\det F_\zeta(\zeta)| \le \frac{1}{(1-t|\zeta|)^n} \tag{17}$$

and $|\det F_\zeta(0)| = 1$. By (7) and (17), we also have

$$|F_\zeta(\zeta)| + |F_{\bar{\zeta}}(\zeta)| \le M^*|\det F_\zeta(\zeta)|^{\frac{1}{n}} \le \frac{M^*}{1-t|\zeta|},$$

where $M^* = \frac{K^{\frac{1}{n}}(1+K_1)}{K_1}$.

For $\zeta \in \mathbb{B}^n$, let $\zeta = s\theta$, where $\theta \in \partial\mathbb{B}^n$ and $s = |\zeta|$. Then

$$\begin{aligned} |F(\zeta)| &\le \int_{[0,\zeta]} |\mathrm{d}F(\zeta)| = \int_{[0,\zeta]} |F_\zeta(s\theta)\theta\, \mathrm{d}s + F_{\bar{\zeta}}(s\theta)\bar{\theta}\, \mathrm{d}s| \\ &\le M^* \int_0^1 \frac{\mathrm{d}s}{1-ts} = \frac{M^*}{t}\log\left(\frac{1}{1-t}\right) := M(t). \end{aligned}$$

Then by using Theorem 5 of [6], we have $f(\mathbb{B}^n)$ which contains a Schlicht ball with radius at least

$$B_f \ge \max_{0<t<1}\left\{\frac{\alpha\pi^{4n}t}{8m(4M(t))^{4n-1}}\right\},$$

where $M(t)$ is as in the statement and m is defined as in Theorem 4. □

Acknowledgments

This research was partly supported by NSF of China (No. 11071063). The work was carried out while the first author was visiting IIT Madras, under 'RTFDCS Fellowship'. He also would like to thank Centre for International Co-operation in Science (formerly Centre for Co-operation in Science & Technology among Developing Societies (CCSTDS)) for its support and co-operation.

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