

## On dominator colorings in graphs

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**Abstract.** A dominator coloring of a graph  $G$  is a proper coloring of  $G$  in which every vertex dominates every vertex of at least one color class. The minimum number of colors required for a dominator coloring of  $G$  is called the dominator chromatic number of  $G$  and is denoted by  $\chi_d(G)$ . In this paper we present several results on graphs with  $\chi_d(G) = \chi(G)$  and  $\chi_d(G) = \gamma(G)$  where  $\chi(G)$  and  $\gamma(G)$  denote respectively the chromatic number and the domination number of a graph  $G$ . We also prove that if  $\mu(G)$  is the Mycielskian of  $G$ , then  $\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$ .

**Keywords.** Dominator coloring; dominator chromatic number; chromatic number; domination number.

### 1. Introduction

By a graph  $G = (V, E)$ , we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of  $G$  are denoted by  $n = |V|$  and  $m = |E|$  respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

Graph coloring and domination are two major areas in graph theory that have been well studied. An excellent treatment of fundamentals of domination is given in the book by Haynes *et al.* [13] and survey papers on several advanced topics are given in the book edited by Haynes *et al.* [14].

Let  $G = (V, E)$  be a graph. Let  $v \in V$ . The *degree* of a vertex  $v$  in a graph  $G$  is defined to be the number of edges incident with  $v$  and is denoted by  $\deg v$ . A vertex of degree zero in  $G$  is an *isolated vertex* and a vertex of degree one is a *pendant vertex* or a *leaf*. Any vertex which is adjacent to a pendant vertex is called a *support vertex*. The *open neighborhood*  $N(v)$  and the *closed neighborhood*  $N[v]$  of  $v$  are defined by  $N(v) = \{u \in V : uv \in E\}$  and  $N[v] = N(v) \cup \{v\}$ . A subset  $S$  of  $V$  is said to be an *independent set* if no two vertices in  $S$  are adjacent. The *independence number*  $\beta_0(G)$  is the maximum cardinality of an independent set in  $G$ . A subset  $S$  of  $V$  is called a *dominating set* of  $G$  if every vertex in  $V - S$  is adjacent to a vertex in  $S$ , in which case we

also say that  $S$  is a *dominator* of  $V - S$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set in  $G$ .

A *proper coloring* of a graph  $G$  is an assignment of colors to the vertices of  $G$  in such a way that no two adjacent vertices receive the same color. The *chromatic number*  $\chi(G)$  is the minimum number of colors required for a proper coloring of  $G$ . The *clique number*  $\omega(G)$  of a graph  $G$  is the maximum order among the complete subgraphs of  $G$ . For a set  $S$  of vertices of  $G$ , the *induced subgraph* is the maximal subgraph of  $G$  with vertex set  $S$  and is denoted by  $\langle S \rangle$ . Thus two vertices of  $S$  are adjacent in  $\langle S \rangle$  if and only if they are adjacent in  $G$ . A graph  $G$  is called *perfect* if  $\chi(H) = \omega(H)$  for every induced subgraph  $H$  of  $G$ .

A graph  $G$  is called a *split graph* if its vertex set can be partitioned into a clique and an independent set. The *corona*  $G_1 \circ G_2$  of two graphs  $G_1$  and  $G_2$  is defined to be the graph  $G$  obtained by taking one copy of  $G_1$  and  $|V(G_1)|$  copies of  $G_2$ , and then joining the  $i$ -th vertex of  $G_1$  to every vertex in the  $i$ -th copy of  $G_2$ .

Hedetniemi *et al.* [15, 16] introduced the concepts of dominator partition and dominator coloring of a graph.

#### DEFINITION 1.1

A vertex  $v \in V$  is a dominator of a set  $S \subseteq V$  if  $v$  dominates every vertex in  $S$ . A partition  $\pi = \{V_1, V_2, \dots, V_k\}$  is called a dominator partition if every vertex  $v \in V$  is a dominator of at least one  $V_i$ . The dominator partition number  $\pi_d(G)$  equals the minimum  $k$  such that  $G$  has a dominator partition of order  $k$ . If we further require that  $\pi$  be a proper coloring of  $G$ , then we have a dominator coloring of  $G$ . The dominator chromatic number  $\chi_d(G)$  is the minimum number of colors required for a dominator coloring of  $G$ .

Since every vertex is a dominator of itself, the partition  $\{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$  into singleton sets is a dominator coloring. Thus, every graph of order  $n$  has a dominator coloring of order  $n$  and therefore the dominator chromatic number  $\chi_d(G)$  is well defined. Gera *et al.* [12] also studied this concept. Some basic results on dominator colorings are given in [5, 10–12]. In this paper we present further results on dominator colorings.

We need the following theorems.

**Theorem 1.2 [10].** *Let  $G$  be a connected graph of order  $n \geq 2$ . Then  $\chi_d(G) = 2$  if and only if  $G$  is a complete bipartite graph of the form  $K_{a,b}$ , where  $1 \leq a \leq b \leq n$  and  $a + b = n$ .*

**Theorem 1.3 [10].** *Let  $G$  be a connected graph of order  $n$ . Then  $\chi_d(G) = n$  if and only if  $G$  is the complete graph  $K_n$ .*

**Theorem 1.4 [5].** *Let  $T$  be a tree of order  $n \geq 2$ . Then  $\gamma(T) + 1 \leq \chi_d(T) \leq \gamma(T) + 2$ .*

**Theorem 1.5 [10].** *For the cycle  $C_n$ , we have*

$$\chi_d(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 4 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n = 5 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{otherwise.} \end{cases}$$

**Theorem 1.6 [12].** For the path  $P_n$ ,  $n \geq 2$ , we have

$$\chi_d(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n = 2, 3, 4, 5, 7 \\ \left\lceil \frac{n}{3} \right\rceil + 2, & \text{otherwise.} \end{cases}$$

If  $\{V_1, V_2, \dots, V_{\chi_d}\}$  is a  $\chi_d$ -coloring of  $G$  and if  $v_i \in V_i$ , then  $S = \{v_1, v_2, \dots, v_{\chi_d}\}$  is a dominating set of  $G$ . Also if  $D$  is a  $\gamma$ -set of  $G$ , then  $\mathcal{C} \cup \{\{v\} : v \in D\}$  where  $\mathcal{C}$  is a proper coloring of  $G - D$  gives a dominator coloring of  $G$ . These observations lead to the following bounds for  $\chi_d(G)$ .

**Theorem 1.7 [10].** Let  $G$  be a connected graph. Then  $\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$ .

**COROLLARY 1.8**

For any bipartite graph  $G$ ,  $\gamma(G) \leq \chi_d(G) \leq \gamma(G) + 2$ .

**PROPOSITION 1.9 [12]**

For a connected graph  $G$  of order  $n \geq 3$ ,  $\chi_d(G) \leq n - \beta_0(G) + 1$ , and this bound is sharp.

**PROPOSITION 1.10 [12]**

If  $G$  is a disconnected graph with components  $G_1, G_2, \dots, G_k$  with  $k \geq 2$ , then  $\max_{i \in \{1, 2, \dots, k\}} \chi_d(G_i) + k - 1 \leq \chi_d(G) \leq \sum_{i=1}^k \chi_d(G_i)$ , and these bounds are sharp.

*Observation 1.11.* For any graph  $G$ ,  $\omega(G) \leq \chi(G) \leq \chi_d(G)$ .

## 2. Basic results

Theorem 1.3 shows that the complete graph  $K_n$  is the only connected graph of order  $n$  with  $\chi_d(G) = n$ . We start with a simple generalization of this result.

**PROPOSITION 2.1**

Let  $G$  be a graph of order  $n$ . Then  $\chi_d(G) = n$  if and only if  $G = K_a \cup (n - a)K_1$ , where  $1 \leq a \leq n$ .

*Proof.* Suppose  $\chi_d(G) = n$ . Clearly every component of  $G$  is complete. If  $G$  contains two nontrivial components  $G_1$  and  $G_2$ , choose  $u \in V(G_1)$  and  $v \in V(G_2)$ . Then  $\{\{u, v\}\} \cup \{\{x\} : x \in V(G) - \{u, v\}\}$  is a dominator coloring of  $G$ , which is a contradiction. Hence  $G = K_a \cup (n - a)K_1$ , where  $1 \leq a \leq n$ .

The converse is obvious. □

We now proceed to characterize graphs with  $\chi_d(G) = n - 1$ .

**Theorem 2.2.** Let  $G$  be a connected graph of order  $n$ . Then  $\chi_d(G) = n - 1$  if and only if one of the following holds:

- (i)  $G \neq K_n$  and  $K_{n-1}$  is a subgraph of  $G$ .

- (ii)  $V(G) = V_1 \cup \{u, v\}$ , where  $\langle V_1 \rangle = K_{n-2}$ ,  $\deg u = 1$ ,  $uv \in E(G)$  and  $v$  is nonadjacent to at least one vertex in  $V_1$ .

*Proof.* Let  $G$  be a connected graph of order  $n$  with  $\chi_d(G) = n - 1$  and let  $V(G) = \{v_1, v_2, \dots, v_n\}$ . By Proposition 1.9, we have  $\chi_d(G) \leq n - \beta_0(G) + 1$  and hence  $\beta_0(G) = 2$ . If there exist three disjoint  $\beta_0$ -sets, say  $\{v_1, v_2\}$ ,  $\{v_3, v_4\}$  and  $\{v_5, v_6\}$ , then  $\{\{v_1, v_2\}, \{v_3, v_4\}\} \cup \{\{v_i\} : 5 \leq i \leq n\}$  is a dominator coloring of  $G$  and hence  $\chi_d(G) \leq n - 2$ , which is a contradiction. Hence there exist at most two disjoint  $\beta_0$ -sets in  $G$ . We consider two cases.

*Case i.* Any two  $\beta_0$ -sets in  $G$  are not disjoint.

Let  $\{v_1, v_2\}$  be a  $\beta_0$ -set in  $G$ . Clearly the induced subgraph  $H = \langle \{v_3, v_4, \dots, v_n\} \rangle$  is complete. We now claim that either  $v_1$  or  $v_2$  is adjacent to every vertex of  $H$ .

Suppose there exist  $v_i, v_j \in V(H)$  such that  $v_1v_i, v_2v_j \notin E(G)$ . Since  $\beta_0(G) = 2$ , it follows that  $i \neq j$  and hence  $\{v_1, v_i\}, \{v_2, v_j\}$  are two disjoint  $\beta_0$ -sets in  $G$ , which is a contradiction. Hence we may assume that  $v_2$  is adjacent to every vertex in  $H$ . Hence  $\langle V(H) \cup \{v_2\} \rangle$  is isomorphic to  $K_{n-1}$  and  $G$  is of the form (i).

*Case ii.* There exist two disjoint  $\beta_0$ -sets in  $G$ , say  $\{v_1, v_2\}$  and  $\{v_3, v_4\}$ .

Since  $\chi_d(G) = n - 1$ , it follows that  $\mathcal{C} = \{\{v_1, v_2\}, \{v_3, v_4\}\} \cup \{\{v_i\} : 5 \leq i \leq n\}$  is not a dominator coloring of  $G$ . We may assume without loss of generality that the vertex  $v_1$  does not dominate any color class. Hence  $\deg v_1 = 1$  and let  $v_1v_3 \in E(G)$ . Now  $H_1 = \langle \{v_2, v_4, v_5, \dots, v_n\} \rangle = K_{n-2}$  and hence  $G$  is of the form (ii).

Conversely, if a graph  $G$  satisfies condition (i) or (ii), it is easy to see that  $\chi_d(G) = n - 1$ .  $\square$

**Theorem 2.3.** *Let  $G$  be a graph of order  $n$ . Then  $\chi_d(G) = n - 1$  if and only if one of the following is true.*

- (i)  $G$  contains exactly one nontrivial component  $G_1$  of order  $n_1$  with  $\chi_d(G_1) = n_1 - 1$ .  
(ii)  $G$  contains exactly two nontrivial components say  $G_1$  and  $G_2$ , where  $G_1 = K_{n_1}, n_1 \geq 2$  and  $G_2 = K_2$ .

*Proof.* Suppose  $\chi_d(G) = n - 1$  and let  $G_1, G_2, \dots, G_k$  be the set of all components of  $G$  of order  $n_1, n_2, \dots, n_k$  respectively. Now we claim that  $G$  contains at most one noncomplete component. Suppose  $G_1$  and  $G_2$  are noncomplete. Then  $\chi_d(G_1) \leq n_1 - 1$  and  $\chi_d(G_2) \leq n_2 - 1$  and hence  $\chi_d(G) \leq \sum_{i=1}^k \chi_d(G_i) < n - 1$ , which is a contradiction.

*Case i.*  $G_1$  is not complete.

Then  $G_2, G_3, \dots, G_k$  are complete. Let  $u$  and  $v$  be two nonadjacent vertices in  $G_1$ . If  $n_2 \geq 2$ , let  $w \in V(G_2)$ . Then  $\{\{u, v, w\}\} \cup \{\{v_i\} : v_i \notin \{u, v, w\}\}$  is a dominator coloring of  $G$  and hence  $\chi_d(G) < n - 1$ , which is a contradiction. Thus  $G_2 = G_3 = \dots = G_k = K_1$ . Further since  $\chi_d(G) = n - 1$ , it follows that  $\chi_d(G_1) = n_1 - 1$ .

*Case ii.* Every component of  $G$  is complete.

It follows from Proposition 2.1 that  $G$  contains at least two nontrivial components, say  $G_1$  and  $G_2$ . We now claim that  $G_3 = G_4 = \dots = G_k = K_1$ . If  $n_3 \geq 2$ , choose  $u \in V(G_1), v \in V(G_2)$  and  $w \in V(G_3)$ . Then  $\{\{u, v, w\}\} \cup \{\{v_i\} : v_i \notin \{u, v, w\}\}$  is a dominator coloring of  $G$  and hence  $\chi_d(G) < n - 1$ , which is a contradiction. Hence  $G_3 = G_4 = \dots = G_k = K_1$ . Now if  $n_1 \geq 3$  and  $n_2 \geq 3$ , choose  $u, v \in V(G_1)$  and

$u', v' \in V(G_2)$ . Then  $\{\{u, u'\}, \{v, v'\}\} \cup \{\{v_i\} : v_i \notin \{u, u', v, v'\}\}$  is a dominator coloring of  $G$  and hence  $\chi_d(G) < n - 1$ , which is a contradiction. Thus  $G$  is of the form (ii).

The converse is obvious. □

The next result concerns the complement  $\bar{G}$  of a graph  $G$ , where  $V(\bar{G}) = V(G)$  and two vertices  $u$  and  $v$  are adjacent in  $\bar{G}$  if and only if they are not adjacent in  $G$ .

**Theorem 2.4.** *Let  $G$  be a connected graph of order  $n \geq 2$ . Then  $4 \leq \chi_d(G) + \chi_d(\bar{G}) \leq 2n$ . Further  $\chi_d(G) + \chi_d(\bar{G}) = 4$  if and only if  $G = K_2$  and  $\chi_d(G) + \chi_d(\bar{G}) = 2n$  if and only if  $G = K_n$ .*

*Proof.* Since  $2 \leq \chi_d(G) \leq n$ , the inequalities are trivial. Also  $\chi_d(G) + \chi_d(\bar{G}) = 2n$  if and only if  $\chi_d(G) = \chi_d(\bar{G}) = n$  and hence it follows from Theorem 1.3 that  $G = K_n$ . Also  $\chi_d(G) + \chi_d(\bar{G}) = 4$  if and only if  $\chi_d(G) = \chi_d(\bar{G}) = 2$  and hence it follows from Theorem 1.2 that  $G = K_2$ .

The converse is obvious. □

**PROPOSITION 2.5**

*Let  $G$  be a connected graph of order  $n$ . Then  $\chi_d(G) + \chi_d(\bar{G}) = 2n - 1$  if and only if  $G = K_n - e$ .*

*Proof.* Suppose  $\chi_d(G) + \chi_d(\bar{G}) = 2n - 1$ . If  $\chi_d(G) = n$ , then it follows from Theorem 1.3 that  $G = K_n$  and  $\chi_d(\bar{G}) = n$ , which is a contradiction. Hence  $\chi_d(G) = n - 1$  and  $\chi_d(\bar{G}) = n$ . Hence it follows from Proposition 2.1 that  $\bar{G} = K_2 \cup (n - 2)K_1$ , so that  $G = K_n - e$ .

The converse is obvious. □

*Observation 2.6.* In [11] it has been conjectured that for the  $n$ -dimensional hypercube  $Q_n$ ,  $\chi_d(Q_n) = 2 + 2^{n-2}$ . This conjecture is false. It has been proved in [1] that  $\gamma(Q_7) = 16$ . Hence  $\chi_d(Q_7) \leq \gamma(Q_7) + \chi(Q_7) = 18$ .

**3. Graphs with  $\chi_d(G) = \chi(G)$**

For any graph  $G$ , we have  $\chi_d(G) \geq \chi(G)$ . In this section we investigate graphs for which  $\chi_d(G) = \chi(G)$ . In particular, we characterize unicyclic graphs, split graphs and complements of bipartite graphs with  $\chi_d(G) = \chi(G)$ .

It follows from Theorem 1.2 that for a tree  $T$  of order  $n \geq 2$ ,  $\chi_d(T) = \chi(T)$  if and only if  $T = K_{1,n-1}$ . In the following theorem we characterize unicyclic graphs with  $\chi_d = \chi$ .

**Theorem 3.1.** *Let  $G$  be a connected unicyclic graph. Then  $\chi_d(G) = \chi(G)$  if and only if  $G$  is isomorphic to  $C_3$  or  $C_4$  or  $C_5$  or the graph obtained from  $C_3$  by attaching any number of leaves at one or two vertices of  $C_3$ .*

*Proof.* Let  $G$  be a unicyclic graph with  $\chi_d(G) = \chi(G)$ . Let  $C$  be the unique cycle in  $G$ . If  $C$  is an even cycle, then  $\chi_d(G) = \chi(G) = 2$  and hence it follows Theorem 1.2 that  $G = C_4$ . Suppose  $C$  is an odd cycle, so that  $\chi_d(G) = \chi(G) = 3$ .

Suppose there exists a support vertex  $v$  not on  $C$ . Since there exists a  $\chi_d$ -coloring of  $G$  in which  $\{v\}$  is a color class, it follows that  $\chi_d(G) \geq 4$ , which is a contradiction. Hence any support vertex lies on  $C$  and any vertex not on  $C$  is a leaf. Since there exists a  $\chi_d$ -coloring of  $G$  in which every support vertex appears as a singleton color class, it

follows that the number of support vertices is at most two. Hence if  $C$  is of length three, then  $G$  is isomorphic to  $C_3$  or the graph obtained from  $C_3$  by attaching any number of leaves at one or two vertices of  $C_3$ . Suppose the length of  $C$  is at least 5. If there exists a support vertex  $v$  on  $C$ , then there exists a  $\chi_d$ -coloring  $\{\{v\}, V_1, V_2\}$  of  $G$  such that  $V_1$  contains all the leaves of  $G$ . Now, there exists a vertex  $w$  such that  $w \in V_2$ ,  $w$  lies on  $C$  and  $w$  is not adjacent to  $v$ . Clearly  $w$  does not dominate any color class of  $G$ , which is a contradiction. Thus  $G$  has no support vertices and hence  $G = C$ . Now, it follows from Theorem 1.5 that  $G = C_5$ .

The converse is obvious.  $\square$

Chellali and Maffray [5] have obtained a characterization of split graphs  $G$  with  $\chi_d(G) = \gamma(G) + 1$ . In the following theorem we prove that  $\chi_d(G) = \omega(G)$  or  $\omega(G) + 1$  for any split graph  $G$ . Arumugam *et al.* [2] have used this theorem to prove that the dominator coloring problem is NP-complete even for split graphs.

**Theorem 3.2.** *Let  $G$  be a split graph with split partition  $(K, I)$  and  $|K| = \omega$ . Then  $\chi_d = \omega$  or  $\omega + 1$ . Furthermore  $\chi_d = \omega$  if and only if there exists a dominating set  $D$  of  $G$  such that  $D \subseteq K$  and every vertex  $v$  in  $I$  is nonadjacent to at least one vertex in  $K - D$ .*

*Proof.* The coloring of  $G$  given by  $\mathcal{C} = \{\{v\} : v \in K\} \cup \{I\}$  is a dominator coloring of  $G$  and hence  $\chi_d \leq \omega + 1$ . Thus  $\chi_d = \omega$  or  $\omega + 1$ . Now suppose  $\chi_d = \omega$ . Let  $\mathcal{C} = \{V_1, V_2, \dots, V_\omega\}$  be a dominator coloring of  $G$ . Hence  $|V_i \cap K| = 1$ . Let  $D = \{x : \{x\} \in \mathcal{C}\}$  and let  $v \in I$ . Any color class dominated by  $v$  is of the form  $\{x\}$ , where  $x \in D$ . Hence it follows that  $D$  is a dominating set of  $G$ . Also, if  $V_i \cap K = \{x\}$  where  $V_i \in \mathcal{C}$  and  $v \in V_i$ , then  $x \in K - D$  and  $v$  is nonadjacent to  $x$ .

Conversely, suppose there exists a dominating set  $D$  of  $G$  such that  $D \subseteq K$  and every vertex  $v$  in  $I$  is nonadjacent to at least one vertex in  $K - D$ . Now we assign colors  $1, 2, \dots, \omega$  to the elements of  $K$  and for any vertex  $v$  in  $I$  we choose a vertex  $x$  in  $K - D$  which is nonadjacent to  $v$  and assign the color of  $x$  to  $v$ . This gives a dominator coloring of  $G$  with  $\omega$  colors.  $\square$

**Theorem 3.3.** *Let  $G = (X \cup Y, E)$  be a bipartite graph and  $|X| \leq |Y|$ . Then  $\chi_d(\bar{G}) = \omega(\bar{G})$  or  $\omega(\bar{G}) + 1$ . Further  $\chi_d(\bar{G}) = \omega(\bar{G})$  if and only if one of the following is true.*

- (1)  $\omega(\bar{G}) > |Y|$ .
- (2)  $|X| < |Y|$  and  $N_{\bar{G}}(x) \cap Y \neq \emptyset$  for all  $x \in X$ .
- (3)  $|X| = |Y|$ ,  $N_{\bar{G}}(x) \cap Y \neq \emptyset$  for all  $x \in X$  and  $N_{\bar{G}}(y) \cap X \neq \emptyset$  for all  $y \in Y$ .

*Proof.* Since  $\bar{G}$  is a perfect graph, it follows that  $\chi(\bar{G}) = \omega(\bar{G})$ . Hence  $\omega(\bar{G}) \leq \chi_d(\bar{G})$ . Let  $\mathcal{C}$  be a  $\chi$ -coloring of  $\bar{G}$  using  $\omega(\bar{G})$  colors. Clearly  $|V_i| \leq 2$  for all  $V_i \in \mathcal{C}$ , and  $|V_i| = 2$  if and only if  $|V_i \cap X| = |V_i \cap Y| = 1$ .

*Case i.*  $\omega(\bar{G}) > |Y| \geq |X|$ .

Since  $|X| \leq |Y| < \omega(\bar{G})$ , there exist two color classes  $V_1$  and  $V_2$ , such that  $V_1 \cap X = \emptyset$  and  $V_2 \cap Y = \emptyset$ . Hence  $|V_1| = |V_2| = 1$ , so that  $\mathcal{C}$  is a dominator coloring of  $\bar{G}$ . Thus  $\chi_d(\bar{G}) = \omega(\bar{G})$ .

*Case ii.*  $\omega(\bar{G}) = |Y| > |X|$ .

In this case there exists  $y_j$  in  $Y$  such that  $\{y_j\} \in \mathcal{C}$  and hence every element of  $Y$  dominates the color class  $\{y_j\}$ . Now, suppose  $N_{\bar{G}}(x) \cap Y \neq \emptyset$  for all  $x \in X$ . Let

$y_x \in N_{\bar{G}}(x) \cap Y$ . Then  $x$  dominates the color class which contains  $y_x$ . Thus  $\mathcal{C}$  is a dominator coloring of  $\bar{G}$  and  $\chi_d(\bar{G}) = \omega(\bar{G})$ . Now suppose there exists  $x \in X$  such that  $N_{\bar{G}}(x) \cap Y = \emptyset$ . Then in any dominator coloring of  $\bar{G}$ , the color class which  $x$  dominates is of the form  $\{x_1\}$  where  $x_1 \in X$  and hence it follows that  $\chi_d(\bar{G}) \geq \omega + 1$ . Now let  $V_1 \in \mathcal{C}$  and  $x \in V_1$ . Clearly  $|V_1| = 2$ . Let  $V_1 = \{x, y\}$ . Then  $\mathcal{C}_1 = (\mathcal{C} - \{v_1\}) \cup (\{x\}, \{y\})$  is a  $\chi_d$ -coloring of  $\bar{G}$  and hence  $\chi_d(\bar{G}) \leq \omega(\bar{G}) + 1$ . Thus  $\chi_d(\bar{G}) = \omega(\bar{G}) + 1$ .

Case iii.  $\omega(\bar{G}) = |Y| = |X|$ .

Then  $|V_i| = 2$  for each  $V_i \in \mathcal{C}$ . Now if  $N_{\bar{G}}(x) \cap Y \neq \emptyset$  for all  $x \in X$  and  $N_{\bar{G}}(y) \cap X \neq \emptyset$  for all  $y \in Y$ , then  $\mathcal{C}$  itself is a dominator coloring of  $\bar{G}$  so that  $\chi_d(\bar{G}) = \omega(\bar{G})$ . Otherwise proceeding as in Case ii, we get  $\chi_d(\bar{G}) = \omega(\bar{G}) + 1$ . □

#### 4. Graphs with minimum degree 1

In this section we consider graphs with  $\delta(G) = 1$ . The following theorem gives a lower bound for  $\chi_d(G)$  in terms of the number of support vertices.

**Theorem 4.1.** *If  $G$  is a graph with  $\delta(G) = 1$  and  $k$  support vertices, then  $\chi_d(G) \geq k + 1$ , and  $\chi_d(G) = k + 1$  if and only if the set of nonsupport vertices is an independent dominating set of  $G$ .*

*Proof.* Let  $S$  denote the set of support vertices of  $G$ . Let  $v \in S$ . Then in any  $\chi_d$ -coloring of  $G$  either  $v$  or a leaf adjacent to  $v$  appears as a singleton color class and hence it follows that  $\chi_d(G) \geq k + 1$ . Now, if  $V - S$  forms an independent dominating set of  $G$ , then  $\{\{v\} : v \in S\} \cup \{V - S\}$  is a dominator coloring of  $G$ , so that  $\chi_d(G) = k + 1$ .

Conversely, suppose  $\chi_d(G) = k + 1$ . Let  $\mathcal{C}$  be a  $\chi_d$ -coloring of  $G$  such that  $\{v\}$  is a color class for every  $v \in S$ . Hence it follows that  $V - S$  is a color class in  $\mathcal{C}$  and hence  $V - S$  is an independent dominating set of  $G$ . □

#### COROLLARY 4.2

*If  $G$  is any graph of order  $n$ , then  $\chi_d(G \circ K_1) = n + 1$ .*

**Theorem 4.3.** *Let  $G$  be a graph with  $\delta(G) = 1$ . Let  $V_1$  be the set of all support vertices of  $G$  and let  $|V_1| = k$ . Then  $\chi_d(G) = k + 2$  if and only if the following conditions are satisfied:*

- (i)  $\langle V - V_1 \rangle$  is a nontrivial bipartite graph, and
- (ii) if  $V_1$  is not a dominating set, then  $\langle V - V_1 \rangle$  contains exactly one nontrivial component which is a complete bipartite graph with  $V_2 = V - N[V_1]$  as one of the partite sets.

*Proof.* Suppose  $\chi_d(G) = k + 2$ . Let  $\mathcal{C} = \{\{v\} : v \in V_1\} \cup \{C_1, C_2\}$  be a  $\chi_d$ -coloring of  $G$  such that  $C_1$  contains all the pendant vertices of  $G$ . Clearly  $\langle V - V_1 \rangle$  is a nontrivial bipartite graph with bipartition  $C_1, C_2$ . Now, suppose  $V_1$  is not a dominating set. Then  $V_2 = V - N[V_1] \neq \emptyset$  and every vertex of  $V_2$  dominates the color class  $C_2$ . It follows that  $V_2 \subseteq C_1$  or  $V_2 \subseteq C_2$ . Hence  $\langle V - V_1 \rangle$  contains exactly one nontrivial component which is a complete bipartite graph with  $V_2$  as one of the partite sets.

Conversely, suppose (i) and (ii) are satisfied. Clearly  $\chi_d(G) \geq k + 2$ . Now if  $V_1$  is a dominating set, then  $\{\{v\} : v \in V_1\} \cup \{C_1, C_2\}$ , where  $C_1, C_2$  is a bipartition of  $\langle V - V_1 \rangle$

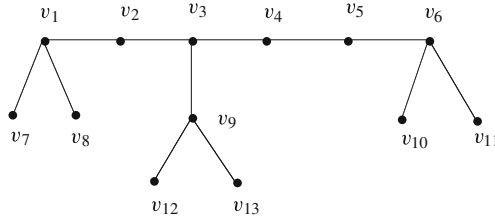


Figure 1. A tree with  $\gamma = 4$  and  $\chi_d = 5$ .

is a  $\chi_d$ -coloring of  $G$ . If  $V_1$  is not a dominating set, then  $\{\{v\} : v \in V_1\} \cup \{C_1, C_2\}$ , where  $C_2$  is the set consisting of  $V_2$  and all the isolated vertices of  $\langle V - V_1 \rangle$  is a  $\chi_d$ -coloring of  $G$ . Hence  $\chi_d(G) = k + 2$ .  $\square$

**Theorem 4.4.** *Let  $G$  be any graph with  $\delta(G) = 1$ . Then  $\chi_d(G) > \gamma(G)$ .*

*Proof.* Let  $\{V_1, V_2, \dots, V_k\}$  be a  $\chi_d$ -coloring of  $G$  in which every support vertex is a singleton color class and the set of all leaves of  $G$  is contained in one color class, say  $V_1$ . Let  $S = \{v_2, v_3, \dots, v_k\}$  where  $v_i \in V_i, 2 \leq i \leq k$ . Clearly  $S$  contains all the support vertices. We now claim that  $S$  is a dominating set of  $G$ . Let  $v \in V - S$  and let  $v$  dominate the color class  $V_i$ . If  $i > 1$ , then  $v_i$  dominates  $v$ . If  $i = 1$ , then  $v$  is either a support vertex or a leaf and hence is dominated by  $S$ . Thus  $\gamma(G) \leq |S| = \chi_d(G) - 1$ .  $\square$

**PROPOSITION 4.5**

*Let  $T$  be a tree of order  $n$ . If there exists a  $\gamma$ -set  $S$  in  $T$  such that  $V - S$  is independent, then  $\chi_d(T) = \gamma(T) + 1$ .*

*Proof.* It follows from Theorem 1.4 that  $\chi_d(T) = \gamma(T) + 1$  or  $\gamma(T) + 2$ . Let  $S = \{v_1, v_2, \dots, v_k\}$  be a  $\gamma$ -set in  $T$  such that  $V - S$  is independent. Then  $\mathcal{C} = \{\{v_i\} : 1 \leq i \leq k\} \cup \{V - S\}$  is a  $\chi_d$ -coloring of  $T$  and hence  $\chi_d(T) = \gamma(T) + 1$ .  $\square$

*Remark 4.6.* The converse of Proposition 4.5 is not true. For the tree  $T$  given in figure 1, the set  $D = \{v_1, v_4, v_6, v_9\}$  is a minimum dominating set and  $\mathcal{C} = \{\{v_1\}, \{v_6\}, \{v_9\}, \{v_3, v_5\}, \{v_2, v_4, v_7, v_8, v_{10}, v_{11}, v_{12}, v_{13}\}\}$  is a  $\chi_d$ -coloring of  $T$ . Hence  $\gamma(T) = 4, \chi_d(T) = 5$ . However, for any  $\gamma$ -set  $S$  in  $T, V - S$  is not independent.

**5. Dominator chromatic number of the Mycielskian**

In search for triangle-free graphs with arbitrarily large chromatic number, Mycielski [19] gave an elegant graph transformation. For a graph  $G = (V, E)$ , the *Mycielskian* of  $G$  is the graph  $\mu(G)$  with vertex set  $V \cup V' \cup \{u\}$ , where  $V' = \{x' : x \in V\}$  and is disjoint from  $V$ , and  $E' = E \cup \{xy' : xy \in E\} \cup \{x'u : x' \in V'\}$ . The vertices  $x$  and  $x'$  are called twins of each other and  $u$  is called the root of  $\mu(G)$ . For recent results on the Mycielskian of a graph, we refer to [3, 6–9, 17, 18]. The Mycielskian of  $C_5$  along with a dominator coloring is given in figure 2.

It is well-known that  $\chi(\mu(G)) = \chi(G) + 1$ . However for dominator colorings, we have the following theorem.



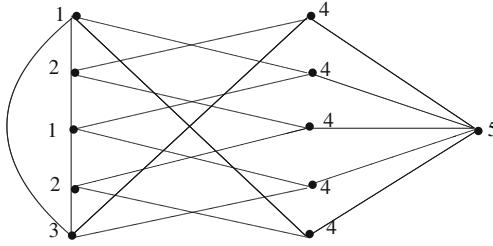


Figure 2. A dominator coloring of  $\mu(C_5)$ .

**Theorem 5.1.** For any graph  $G$ ,  $\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$ . Further if there exists a  $\chi_d$ -coloring  $\mathcal{C}$  of  $G$  in which every vertex  $v$  dominates a color class  $V_i$  with  $v \notin V_i$ , then  $\chi_d(\mu(G)) = \chi_d(G) + 1$ .

*Proof.* If  $\mathcal{C}$  is any  $\chi_d$ -coloring of  $G$ , then  $\mathcal{C} \cup \{V', \{u\}\}$  is a dominator coloring of  $\mu(G)$  and hence  $\chi_d(\mu(G)) \leq \chi_d(G) + 2$ . Now let  $\chi_d(\mu(G)) = k$ . Let  $\mathcal{C} = \{V_1, V_2, \dots, V_k\}$  be a dominator coloring of  $\mu(G)$  and let  $u \in V_1$ .

Case i.  $V_1 = \{u\}$ .

For each color  $\alpha$  that appears on a vertex of  $V'$  but not on  $V$ , we choose an arbitrary vertex  $x'$  of  $V'$  that has the color  $\alpha$  and recolor its twin  $x \in V$  with color  $\alpha$ . The restriction of this coloring to  $G$  gives a dominator coloring of  $G$  with  $k - 1$  colors.

Case ii.  $V_1 \neq \{u\}$ .

Let  $S = V_1 \cap V(G)$ . Note that no vertex in  $V(G)$  dominates the color class  $V_1$ . As in Case i, for each color  $\alpha$  that appears on a vertex of  $V'$  but not on  $V$ , we choose an arbitrary vertex  $x'$  of  $V'$  that has the color  $\alpha$  and recolor its twin  $x \in V$  with color  $\alpha$ . Now for each vertex  $v \in S$ , we recolor  $v$  with the color of its twin  $v'$ . Let  $\mathcal{C}'$  denote the restriction of this coloring to  $V(G)$ . We first show that  $\mathcal{C}'$  is a proper coloring of  $G$ . Let  $u$  and  $v$  be two adjacent vertices in  $G$ . If both  $u$  and  $v$  are in  $V - S$ , obviously they have distinct colors. If  $u \in S$  and  $v \in V - S$ , then  $v$  is adjacent to  $u'$ , and the new color of  $u$  is that of  $u'$ . Hence  $u$  and  $v$  have distinct colors.

Now, let  $v \in V(G)$  and let  $V_i$  be the color class dominated by  $v$  in  $\mathcal{C}$ . Since  $i \neq 1$ , it follows that  $v$  continues to dominate the same color class in  $\mathcal{C}'$ . Thus  $\chi_d(\mu(G)) > \chi_d(G)$ , and hence  $\chi_d(\mu(G)) = \chi_d(G) + 1$  or  $\chi_d(G) + 2$ .

Now, if there exists a  $\chi_d$ -coloring of  $G$  in which every vertex  $v$  dominates a color class  $V_i$  with  $v \notin V_i$ , then the coloring of  $\mu(G)$  obtained by assigning the color of  $v_i$  to its twin  $v'_i$  and a new color to the root  $u$ , is a dominator coloring of  $\mu(G)$ . Hence  $\chi_d(\mu(G)) = \chi_d(G) + 1$ . □

There exist graphs with  $\chi_d(\mu(G)) = \chi_d(G) + 1$  or  $\chi_d(\mu(G)) = \chi_d(G) + 2$ .

For the complete graph  $K_n$ , we have  $\chi_d(\mu(K_n)) = \chi_d(K_n) + 1 = n + 1$ . Also for the cycle  $C_6$ , we have  $\chi_d(\mu(C_6)) = \chi_d(C_6) + 1 = 5$ .

The following lemma gives an example of a graph with  $\chi_d(\mu(G)) = \chi_d(G) + 2$ .

**Lemma 5.2.**  $\chi_d(\mu(C_5)) = \chi_d(C_5) + 2$ .

*Proof.* Let  $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$ . Since  $\chi_d(C_5) = 3$ , we need to prove that  $\chi_d(\mu(C_5)) = 5$ . Suppose  $\chi_d(\mu(C_5)) = 4$ . Let  $\mathcal{C}$  be a dominator coloring of  $\mu(C_5)$  using

four colors 1, 2, 3 and 4. If all the colors appear on  $C_5$ , then the root  $u$  does not dominate a color class. Hence, we may assume that the vertices  $v_1, v_2, v_3, v_4$  and  $v_5$  receive respectively the colors 1, 2, 1, 2 and 3. If  $\{v_5\} \in \mathcal{C}$ , then  $v'_5$  receives color 4 and hence the root  $u$  receives color 1 or color 2. Hence color 4 appears on at least one of  $v'_1$  or  $v'_4$ . It follows that  $v'_5$  does not dominate a color class, a contradiction. Hence  $\{v_5\} \notin \mathcal{C}$  and  $v_5$  dominates the color class 4. Now  $v'_5$  receives color 3 and does not dominate any color class, a contradiction. Thus  $\chi_d(\mu(C_5)) = 5$ .  $\square$

## 6. Conclusion and scope

The following are some interesting problems for further investigation.

*Problem 6.1.* For which cycles  $C_n$ ,  $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$ ?

*Problem 6.2.* Characterize graphs  $G$  for which  $\chi_d(G) = \chi(G)$  or  $\chi_d(G) = \gamma(G)$ .

*Problem 6.3.* Characterize trees  $T$  for which  $\chi_d(T) = \gamma(T) + 1$ .

*Problem 6.4.* Characterize graphs  $G$  for which  $\chi_d(\mu(G)) = \chi_d(G) + 1$ .

*Problem 6.5.* We have proved in [2] that the problem of determining the dominator chromatic number is NP-complete even for split graphs. Hence designing efficient algorithms for computing  $\chi_d(G)$  for special families of graphs is an interesting problem. In particular does there exist a polynomial time algorithm for computing  $\chi_d(T)$  for trees?

*Problem 6.6.* We observe that the dominator chromatic number of a graph  $G$  may increase arbitrarily on the removal of a vertex. For example  $\chi_d(W_n) = \chi_d(C_{n-1} + K_1) = \chi(C_{n-1} + K_1) = 3$  or 4 according as  $n$  is odd or even and on removing the central vertex of  $W_n$ ,  $\chi_d$  increases arbitrarily. Hence the study of changing and unchanging of the dominator chromatic number on the removal of a vertex or an edge is an interesting problem for further investigation.

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