Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 4, November 2012, pp. 519–524.

Discreteness criteria based on a test map in $PU(n, 1)$

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MS received 5 July 2011; revised 12 May 2012

Abstract. The discreteness of isometry groups in complex hyperbolic space is a fundamental problem. In this paper, the discreteness criteria of a n-dimensional subgroup G of $SU(n, 1)$ are investigated by using a test map which may not be in G.

Keywords. Test map; discreteness criterion; non-elementary group.

1. Introduction

It is well-known that the discreteness of non-elementary subgroups of Möbius groups is a fundamental problem. It has been investigated by many researchers. In 1976, Jørgensen established the following well-known result (see [8]).

Theorem A. *A non-elementary subgroup G of Möbius transformations acting on $\bar{R}^2$ is discrete if and only if for each f and g in G, the two-generator group $\langle f, g\rangle$ is discrete.*

The result implies that the discreteness of a non-elementary group depends on the discreteness of all its subgroups of rank two. It is a remarkable problem of how to weaken the condition of Jørgensen's theorem. In 1990, Isachenko [7] obtained the following theorem which shows that for a non-elementary group G of $SL(2, C)$ the discreteness of all two-generator subgroups, whose generators are both loxodromic, is enough to secure the discreteness of the group G.

Theorem B. *Let G be a non-elementary subgroup of $PSL(2, C)$. Then G is discrete if and only if for two arbitrary loxodromic elements f and g in G the group $\langle f, g\rangle$ is discrete.*

The higher-dimensional generalization of Theorems A and B has been studied extensively. However, Abikoff and Hass in [1] constructed an example which implies that Theorems A and B do not apply to spaces in general. They pointed out that some additional conditions must be added and obtained a space version of Theorem A under the n-dimensional condition, which means that the involved group G does not have any

G-invariant proper hyperbolic subspace. In addition, Martin [9] established Theorem A for a finitely generated non-elementary group in $M(\bar{R}^n)$ under the condition of uniformly bounded torsion. Fang and Nai [4] established space versions of Theorems A and B on the condition that there does not exist an elliptic sequence in the involved group converging to identity.

In 2004, Chen [2] first introduced another method by which one can use a fixed Möbius transformation as a test map to test the discreteness of a non-elementary subgroup G of $\mathrm{Isom}(H^n)$. It is interesting that the test map may not be in the group G. One of the main results in [2] is the following.

Theorem C. *Let G be a n-dimensional subgroup of* $\mathrm{Isom}(H^n)$ *and h a nontrivial Möbius transformation with finite order on* $\bar{R}^n$, *where* $h(\bar{R}^{n-1}) = \bar{R}^{n-1}$ *and* $\mathrm{fix}(h) \neq \bar{R}^{n-1}$. *If for each* $g \in G$ *the two-generator group* $\langle h, g\rangle$ *is discrete, then G is discrete.*

In 2009, Yang [11] obtained several different discreteness criteria in $SL(2, C)$ by using a test map. One of these criteria is as follows.

Theorem D. *Let G be a non-elementary subgroup of* $SL(2, C)$ *and f a nontrivial Möbius transformation. If for each loxodromic element* $g \in G$ *the two-generator group* $\langle f, g\rangle$ *is discrete, then G is discrete.*

Viewing the hyperbolic plane as a complex hyperbolic 1-space $\mathbf{H}^1_{\mathbf{C}}$, it is natural to generalize these results mentioned above to high dimensional complex hyperbolic space. But it is known that Theorem D does not apply to spaces in general. In this paper, we will consider the generalizations of the discreteness criteria based on a test map in higher dimensional complex hyperbolic space for a non-elementary group under the n-dimensional condition. Here G is called a n-dimensional group if it does not leave a point in $\partial\mathbf{H}^n_C$ or a proper totally geodesic submanifold of $\mathbf{H}^n_C$ invariant. In fact, a n-dimensional subgroup of $SU(n, 1)$ is either discrete or dense in $SU(n, 1)$ (Corollary 4.5.1 of [3]). We will use this to construct a sequence in a discrete and non-elementary group which violates Jørgensen's inequality in the proofs of our theorems.

Our main results are the following theorems. In these theorems, the element f is used as the test map, which may not be in G.

Theorem 1.1. *Let G be a n-dimensional and non-elementary subgroup of* $SU(n, 1)$, *and f a non-elliptic element in* $SU(n, 1)$. *If for any loxodromic element* $g \in G$ *the two-generator group* $\langle f, g\rangle$ *is discrete, then G is discrete.*

Theorem 1.2. *Let G be a n-dimensional and non-elementary subgroup of* $SU(n, 1)$, *and f a regular elliptic element with finite order k* $(k \geq 3)$ *in* $SU(n, 1)$. *If for each loxodromic element* $g \in G$ *the two-generator group* $\langle f, g\rangle$ *is discrete, then G is discrete.*

Theorem 1.3. *Let G be a n-dimensional and non-elementary subgroup of* $SU(n, 1)$, *and f a non-elliptic element in* $SU(n, 1)$. *If for each regular element* $g \in G$ *the two-generator group* $\langle f, g\rangle$ *is discrete, then G is discrete.*

Theorem 1.4. *Let G be a n-dimensional and non-elementary subgroup of* $SU(n, 1)$, *and f a regular elliptic element of finite order k* $(k \geq 3)$ *in* $SU(n, 1)$. *If for any regular element g the two-generator group* $\langle f, g\rangle$ *is discrete, then G is discrete.*

2. Preliminaries

First, we recall some notations and terminologies. For more information on $\mathbf{H}^n_{\mathbb{C}}$, the readers can refer to [6, 10]. Let $A = (a_{ij})$ be a complex matrix. Its Hermitian transpose is the complex matrix $A^* = (\bar{a}_{ji})$. Let $\mathbb{C}^{n,1}$ be the vector space $\mathbb{C}^{n+1}$ equipped with the Hermitian form $\langle \cdot, \cdot \rangle$:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = z_1 \overline{w_1} + z_2 \overline{w_2} + \cdots + z_n \overline{w_n} - z_{n+1} \overline{w_{n+1}},$$

where $\mathbf{z}, \mathbf{w} \in \mathbb{C}^{n+1}$ and $\mathbf{w}^*$ are the Hermitian transpose of $\mathbf{w}$ and J is the first Hermitian matrix. Complex hyperbolic n-space $\mathbf{H}^n_{\mathbb{C}}$ and its ideal boundary $\partial \mathbf{H}^n_{\mathbb{C}}$ are respectively the projective images, in complex projective plane $\mathbb{CP}^n$, of

$$\begin{aligned} V_- &= \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}, \\ V_0 &= \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}. \end{aligned}$$

Denote $\mathbf{H}^n_{\mathbb{C}} \cup \partial \mathbf{H}^n_{\mathbb{C}}$ by $\overline{\mathbf{H}^n_{\mathbb{C}}}$. The projection

$$(z_1, z_2, \ldots, z_{n+1}) \to (z_1/z_{n+1}, z_2/z_{n+1}, \ldots, z_n/z_{n+1})$$

maps V_-, V_0 respectively to the open unit ball and unit sphere in $\mathbb{C}^n$. Denote the holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$ by $PU(n, 1)$. For a non-trivial element g in $PU(n, 1)$, g is called *loxodromic* if it has exactly two fixed points in $\partial \mathbf{H}^n_{\mathbb{C}}$, *parabolic* if it has exactly one fixed point in $\partial \mathbf{H}^n_{\mathbb{C}}$, *elliptic* if it has at least a fixed point in $\mathbf{H}^n_{\mathbb{C}}$. Particularly, an elliptic is called *regular* if it has only one fixed point in $\mathbf{H}^n_{\mathbb{C}}$. A subgroup G in $PU(n, 1)$ is non-elementary if there does not exist a finite G-orbit in $\mathbf{H}^n_{\mathbb{C}}$.

The holomorphic isometry group acts on $\mathbb{CP}^n$ by matrix multiplication. Denote the unitary group of matrix with determinant 1 by $SU(n, 1)$. It is known that $PU(n, 1) = SU(n, 1)/\{\omega^j I | j = 0, 1, \ldots, n-1\}$, where ω is the n-th root of unity. For any element $g \in SU(n, 1)$, $\langle g\mathbf{z}, g\mathbf{w} \rangle = \mathbf{w}^* g^* J g \mathbf{z} = \langle \mathbf{z}, \mathbf{w} \rangle$ and $g^* J g = J$. Denote the fixed point set of $g \in SU(n, 1)$ by $\mathrm{fix}(g)$. Define the norm $\|g\|$ of $g \in SU(n, 1)$:

$$\|g\| = \left(\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} |g_{ij}|^2 \right)^{1/2}.$$

Jørgensen inequality plays an important role in many different methods of proving the discreteness of a group. However, there are several versions about the generalization of Jørgensen inequality. In order to prove our results, we will adopt the version obtained by Friedland and Hersonsky in [5]. For the convenience of the reader, we give a sketch by using our notations here. Let $\mathscr{A}$ be a normed algebra with an involution. Obviously, Hermitian transpose $*$ is the involution defined by Friedland and Hersonsky (see page 597 of [5]). Denote the set $\{x : x^* J x = J, x \in \mathscr{A}\}$ by $O(J)$.

Lemma 2.1 (*Theorem* 2.6 *of* [5]). *If for any a and b in $O(J)$ the group $\langle a, b \rangle$ is discrete and non-elementary, then*

$$\max\{\|[a, b] - I\|, \|a - I\|\} \geq \tau, \max\{\|b - I\|, \|a - I\|\} \geq \tau.$$

Here τ is the unique positive solution of the cubic equation

$$2\tau(1 + \tau)^2 = 1, \quad \tau > 0.2971.$$

In addition, the following lemmas are essential for the proofs of our results.

Lemma 2.2 [10]. *The loxodromic elements in* $SU(n,1)$ *form an open set.*

Lemma 2.3 (*Theorem* 6.2.1 *of* [6]). *The regular elliptic elements in* $SU(n,1)$ *form an open set.*

3. Proofs of theorems

Proof of Theorem 1.1. For the purpose of contradiction, we suppose that G is not discrete. Then G is dense in $SU(n,1)$ according to Corollary 4.5.1 of [3]. By Lemma 2.2, there exists a sequence $\{g_m\}$ in G consisting of distinct loxodromic elements with distinct fixed points such that $g_m \to I$ $(m \to \infty)$. Then for large enough m, we have

$$\max\{\|g_m - I\|, \|[g_m, f] - I\|\} < \tau.$$

Thus by Lemma 2.1 and hypothesis, the two-generator group $\langle f, g_m\rangle$ is discrete and elementary, and

$$\mathrm{fix}(f) \cap \mathrm{fix}(g_m) \neq \emptyset.$$

Consequently, we can find at least three positive integers m_1, m_2, m_3 such that

$$\mathrm{fix}(g_{m_i}) \cap \mathrm{fix}(g_{m_j}) = \emptyset, \qquad (i, j = 1, 2, 3 \text{ and } i \neq j)$$

and

$$\mathrm{fix}(f) \cap \mathrm{fix}(g_{m_k}) \neq \emptyset, \qquad k = 1, 2, 3.$$

This implies that f has at least three distinct fixed points. It is impossible for a non-elliptic element.

Proof of Theorem 1.2. Suppose that G is not discrete. Then G is dense in $SU(n,1)$ by Corollary 4.5.1 of [3]. By Lemma 2.2, we can find a sequence $\{g_m\}$ in G of distinct loxodromic elements with distinct fixed points such that $g_m \to I$ $(m \to \infty)$. Then for similar reasons mentioned in the proof of Theorem 1.1, the two-generator group $\langle f, g_m\rangle$ is discrete and elementary for sufficiently large m. Thus, f fixes or exchanges the fixed point of g_m.

By the density of G in $SU(n,1)$ and by Lemma 2.2, we can find an infinite set of loxodromic elements in G, no two elements of which have a common fixed point. So, we can find a loxodromic element $\varphi \in G$ such that $\mathrm{fix}(\varphi) \cap \mathrm{fix}(f^2) = \emptyset$. Denote the attractive (resp. repelling) fixed point of φ by a_φ (resp. r_φ), and choose accordingly an open neighborhood U in $\overline{H^n_C}$ such that $a_\varphi \in U$ and $\mathrm{fix}(g_m) \cap U = \emptyset$. It is known that $\varphi^r(c) \to a_\varphi$ $(r \to \infty)$ for any point c $(c \neq r_\varphi)$. Hence for large enough r we have $\varphi^r(\mathrm{fix}(f^2)) \subset U$.

On the other hand, for large enough m, we have

$$\max\{\| [\varphi^r f \varphi^{-r}, g_m] - I \|, \| g_m - I \|\} < \tau.$$

By Lemma 2.1 and hypothesis, the group $\langle \varphi^r f \varphi^{-r}, g_m\rangle$ is elementary and discrete for large enough m. Thus, we have

$$\mathrm{fix}(g_m) \cap \mathrm{fix}(\varphi^r f^2 \varphi^{-r}) \neq \emptyset,$$

that is,

$$\mathrm{fix}(g_m) \cap \varphi^r(\mathrm{fix}(f^2)) \neq \emptyset.$$

This means

$$\mathrm{fix}(g_m) \cap U \neq \emptyset.$$

This is a desired contradiction.

Proof of Theorem 1.3. Suppose that each group $\langle f, g\rangle$ generated by f and any regular element $g \in G$ is discrete, yet G itself is not discrete. Then G is dense in $SU(n,1)$ by Corollary 4.5.1 of [3]. By Lemma 2.3, we can find a sequence $\{g_m\}$ in G consisting of distinct regular elliptic elements with distinct fixed points such that $g_m \to I$ $(m \to \infty)$. Then

$$\max\{\| g_m - I \|, \| [g_m, f] - I \|\} < \tau.$$

Thus, by Lemma 2.1 and hypothesis, the group $\langle f, g_m\rangle$ is elementary and discrete for large enough m. Without loss of generality, we assume that the group $\langle f, g_m\rangle$ is elementary and discrete for each m. Hence

$$\mathrm{fix}(f) \cap \mathrm{fix}(g_m^2) \neq \emptyset.$$

This implies that f has more than two fixed points. It is impossible if f is a non-elliptic element.

Proof of Theorem 1.4. Suppose that G is not discrete for the purpose of contradiction. Then G is dense in $SU(n,1)$. By Lemma 2.3, we can find a sequence $\{g_m\}$ in G consisting of regular elliptic elements with distinct fixed points. For similar reasons mentioned above, the group $\langle f, g_m\rangle$ is elementary and discrete for large enough m. Hence

$$\mathrm{fix}(f^2) \cap \mathrm{fix}(g_m^2) \neq \emptyset.$$

By the density of G in $SU(n,1)$ and by Lemma 2.2, we can find an infinite set of loxodromic elements in G, no two elements of which have a common fixed point. So, we can find a loxodromic element h such that

$$\mathrm{fix}(h) \cap \mathrm{fix}(f^2) = \emptyset.$$

If we denote the attractive (resp. repelling) fixed point of h by a_h (resp. r_h), then an open neighborhood U of a_h can be given in $\overline{\mathbf{H}^n_{\mathbb{C}}}$ such that

$$\mathrm{fix}(g_m^2) \cap U = \emptyset.$$

Since $h^r(c) \to a_h$ $(r \to \infty)$ for any point c $(c \neq r_h)$, we have $h^r(\mathrm{fix}(f^2)) \subset U$ for large enough r.

On the other hand, for large enough m, we have

$$\max\{\| [h^r f h^{-r}, g_m] - I \|, \| g_m - I \|\} < \tau.$$

Then the group $\langle g_m, h^r f h^{-r}\rangle$ is discrete and elementary by Lemma 2.1 and hypothesis. Thus, we have

$$\mathrm{fix}(g_m^2) \cap \mathrm{fix}(h^r f^2 h^{-r}) \neq \emptyset,$$

that is,

$$\text{fix}(g_m^2) \cap h^r(\text{fix}(f^2)) \neq \emptyset.$$

This means

$$\text{fix}(g_m^2) \cap U \neq \emptyset.$$

This is a desired contradiction.

Acknowledgements

The authors would like to thank the referee for his/her valuable comments and suggestions. This work is supported by NSF of China (No. 10801107) and NSF of Guangdong Province (Nos S2011010000735 and 9452902001003278).

References

[1] Abikoff W and Hass A, Nondiscrete groups of hyperbolic motions, *Bull. London Math. Soc.* **22** (1990) 233–238

[2] Chen M, Discreteness and convergence of Möbius groups, *Geom. Dedicate* **104** (2004) 61–69

[3] Chen S S and Greenberg L, Hyperbolic spaces, in: Contributions to analysis (a collection of papers dedicated to Lipman Bers) (1974) (New York: Academic Press) pp. 49–87

[4] Fang A and Nai B, On the discreteness and convergence in n-dimensional Möbius groups, *J. London Math. Soc. (2)* **61** (2000) 761–773

[5] Friedland S and Hersonsky S, Jørgensen's inequality for discrete groups in normed algebras, *Duke Math. J.* **69** (1993) 593–614

[6] Goldman W M, Complex hyperbolic geometry (1999) (New York: Oxford Univ. Press)

[7] Isachenko N A, Systems of generators of subgroups of $PSL(2, C)$, *Siberian Math. J.* **31** (1990) 162–165

[8] Jørgensen T, On discrete groups of Möbius transformations, *Am. J. Math.* **98** (1976) 739–749

[9] Martin G J, On discrete Möbius groups in all dimensions, *Acta Math.* **163** (1989) 253–289

[10] Xie B, Jiang Y and Wang H, Discreteness of subgroups of $PU(1, n; C)$, *Proc. Japan Acad.* **A84** (2008) 78–80

[11] Yang S, Test maps and discrete groups in $SL(2, C)$. *Osaka J. Math.* **46** (2009) 403–409