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Enumerating set partitions according to the number of descents of size d or more

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Abstract. Let $P(n, k)$ denote the set of partitions of $\{1, 2, \ldots, n\}$ having exactly k blocks. In this paper, we find the generating function which counts the members of $P(n, k)$ according to the number of descents of size d or more, where $d \geq 1$ is fixed. An explicit expression in terms of Stirling numbers of the second kind may be given for the total number of such descents in all the members of $P(n, k)$. We also compute the generating function for the statistics recording the number of ascents of size d or more and show that it has the same distribution on $P(n, k)$ as the prior statistics for descents when $d \geq 2$, by both algebraic and combinatorial arguments.

Keywords. Set partitions; descents; partition statistic; combinatorial proof.

1. Introduction

A *partition* Π of the set $[n] := \{1, 2, \ldots, n\}$ is any collection $B_1, B_2, \ldots, B_k$ of nonempty disjoint subsets whose union is $[n]$. (From now on, we will use the term *partition* when referring to a partition of a set.) The elements of a partition are called *blocks*. We assume that $B_1, B_2, \ldots, B_k$ are listed in increasing order according to their minimum elements, that is, $\min B_1 < \min B_2 < \cdots < \min B_k$. The set of all partitions of $[n]$ with k blocks is denoted by $P(n, k)$. The cardinality of $P(n, k)$ is the well-known Stirling number of the second kind [10], which is usually denoted by $S_{n,k}$ and satisfies the recurrence $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$.

Recall that a partition $\Pi = B_1, B_2, \ldots, B_k$ of $[n]$ may be written in the *canonical sequential form* $\pi = \pi_1\pi_2 \cdots \pi_n$, where $i \in B_{\pi_i}$ for all i (see, e.g. [9,5]). For example, if $\Pi = \{1, 4\}, \{2, 5, 7\}, \{3\}, \{6\}$ is a partition of [7], then its canonical sequential form is $\pi = 1231242$ and in such a case we write $\Pi = \pi$. From now on, we will identify each partition with its canonical sequential form. Note that $\pi = \pi_1\pi_2 \cdots \pi_n \in P_{n,k}$ is a *restricted growth function* from $[n]$ to $[k]$ (see, for example [9] for details), meaning that it satisfies the following three properties: (i) $\pi_1 = 1$, (ii) π is onto $[k]$, and (iii) $\pi_{i+1} \leq \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all i, $1 \leq i \leq n - 1$.

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a partition of $[n]$ (or, more generally, any word of length n). In accordance with the terminology for permutations (see, for example p. 267 of [4]), we will say that π has a *descent at i* if $\pi_i > \pi_{i+1}$ and an *ascent at i* if $\pi_i < \pi_{i+1}$, where $1 \le i \le n-1$. Furthermore, if $d \ge 1$, then we will say that π has a *descent of size d or more at i* if $\pi_i \ge \pi_{i+1} + d$ and an *ascent of size d or more at i* if $\pi_i + d \le \pi_{i+1}$. We will denote the numbers of descents, descents of size d or more, ascents, and ascents of size d or more in π by $\mathrm{des}(\pi)$, $\mathrm{des}_d^+(\pi)$, $\mathrm{asc}(\pi)$ and $\mathrm{asc}_d^+(\pi)$, respectively. For example, there are three descents of size 2 or more in the partition $\pi = 12314122532$, as evidenced by the strings 31, 41 and 53 (at $i = 3, 5$, and 9, respectively), whence $\mathrm{des}_2^+(\pi) = 3$. Clearly, $\mathrm{des}(\pi) = \mathrm{des}_1^+(\pi)$ and $\mathrm{asc}(\pi) = \mathrm{asc}_1^+(\pi)$ for any π. For other examples of statistics on set partitions, we refer the reader to [2,3,6,7,11]. See also the related paper on general subword patterns [8].

Mansour and Munagi [7] proved that the generating function which counts the members of $P(n,k)$, where k is fixed, according to the number of descents is given by

$$\sum_{n\ge k}\left(\sum_{\pi\in P(n,k)} q^{\mathrm{des}(\pi)}\right)x^n = \left(\frac{x(1-q)}{1+x(q-1)}\right)^k \times \frac{1}{\prod_{j=1}^{k}\left(1-\frac{q}{(1+x(q-1))^j}\right)}, \qquad k \ge 1. \tag{1.1}$$

Here, we extend this result by computing the generating function which counts the members of $P(n,k)$ according to the number of descents of size d or more, where $d \ge 1$ is fixed. An explicit formula in terms of Stirling numbers of the second kind may be given for the total number of descents of size d or more in all the members of $P(n,k)$, which may be explained combinatorially. We also compute the comparable generating function for the statistic recording the number of ascents of size d or more which works out to be the same as the one for descents when $d \ge 2$. We subsequently provide a combinatorial proof that the statistics on $P(n,k)$ recording the number of descents or ascents of size d or more are equally distributed for all n and k when $d \ge 2$.

Recall that the binomial coefficient $\binom{n}{k}$ is given by $\frac{n!}{k!(n-k)!}$ if $0 \le k \le n$ and is zero if $k > n \ge 0$ or if $k < 0$. In addition, we will take $\binom{n}{k}$ to be zero if $n < 0$, for convenience.

2. The des_d^+ and asc_d^+ statistics

Note that each partition π of $[n]$ with exactly k blocks can be uniquely expressed as

$$\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)},$$

where $\pi^{(i)}$ is a word in the alphabet $[i]$ for each $i \in [k]$, which motivates us to study the aforementioned statistics first on words. Fix positive integers k and d such that $1 \le d \le k$. Let $W_{k,d} = W_{k,d}(x;q)$ denote the generating function for the number of words of length n in the alphabet $[k]$ according to the statistic des_d^+:

$$W_{k,d} = W_{k,d}(x;q) := \sum_{n\ge 0}\left(\sum_{\omega\in[k]^n} q^{\mathrm{des}_d^+(\omega)}\right)x^n.$$

We now refine $W_{k,d}$ according to the initial sequence of a word $w = w_1w_2\cdots$. If $a_1a_2\cdots a_s \in [k]^s$, then let

$$W_{k,d;a_1a_2\cdots a_s} := \sum_{n\geq s}\left(\sum_{\substack{\omega\in[k]^n\\ \omega_1\cdots\omega_s=a_1\cdots a_s}} q^{\mathrm{des}_d^+(\omega)}\right)x^n.$$

Let $W_{k,d;a_1a_2\cdots a_s}$ be zero if $a_i \notin [k]$ for some i.

We make two preliminary observations which follow from the definitions:

$$W_{k,d} = 1 + \sum_{a=1}^{k} W_{k,d;a}, \tag{2.1}$$

and

$$\begin{aligned} W_{k,d;a} &= x + \sum_{i=1}^{k} W_{k,d;ai} = x + xq\sum_{1\leq i\leq a-d} W_{k,d;i} + x\sum_{a-d+1\leq i\leq k} W_{k,d;i} \\ &= xW_{k,d} + x(q-1)\sum_{1\leq i\leq a-d} W_{k,d;i}, \end{aligned} \tag{2.2}$$

where $1 \leq a \leq k$. We will need the following lemma.

Lemma 2.1. *Let* $f_i = \frac{W_{k,d;i}}{xW_{k,d}}$. *Then* $f_a = \sum_{i\geq 0}\binom{a-1-i(d-1)}{i}(x(q-1))^i$ *for all* $a = 1, 2, \ldots, k$.

Proof. By (2.2), we have $f_a = 1 + x(q-1)\sum_{1\leq j\leq a-d} f_j$ for all $a \geq 1$. The conclusion then holds if $1 \leq a \leq d$ since $f_a = 1$ in this case. The result now follows by induction since for all $a \geq d+1$ one may write

$$\begin{aligned} f_a &= 1 + x(q-1)\sum_{1\leq j\leq a-d} f_j = 1 + x(q-1) \\ &\quad \times \sum_{1\leq j\leq a-d}\sum_{i\geq 0}\binom{j-1-i(d-1)}{i}(x(q-1))^i \\ &= 1 + \sum_{i\geq 0}(x(q-1))^{i+1}\sum_{1\leq j\leq a-d}\binom{j-1-i(d-1)}{i} \\ &= 1 + \sum_{i\geq 0}(x(q-1))^{i+1}\binom{a-1-(i+1)(d-1)}{i+1} \\ &= \sum_{i\geq 0}\binom{a-1-i(d-1)}{i}(x(q-1))^i, \end{aligned}$$

where we have used the combinatorial identity $\sum_{j=i}^{b}\binom{j}{i} = \binom{b+1}{i+1}$. □

We can now describe the generating function for the distribution of des_d^+ on words in $[k]$.

Theorem 2.2. *The generating function for the number of words of length n in the alphabet $[k]$ according to the statistic des_d^+ is given by*

$$W_{k,d} = \frac{1}{1 - \sum_{i\geq 0} \binom{k-i(d-1)}{i+1} x^{i+1}(q-1)^i}.$$

Proof. By (2.1), we have $W_{k,d} = 1 + \sum_{a=1}^{k} x W_{k,d} f_a$. Thus,

$$W_{k,d} = \frac{1}{1 - x\sum_{a=1}^{k} f_a} = \frac{1}{1 - x\sum_{i\geq 0}\sum_{a=1}^{k} \binom{a-1-i(d-1)}{i} x^i (q-1)^i}$$
$$= \frac{1}{1 - x\sum_{i\geq 0} \binom{k-i(d-1)}{i+1} x^i (q-1)^i},$$

by the identity $\sum_{j=i}^{b} \binom{j}{i} = \binom{b+1}{i+1}$. □

Example 2.3. If $d = 1$, then Theorem 2.2 gives the generating function for the number of words of length n in the alphabet $[k]$ according to the statistic des (see Theorem 2.3 of [1]):

$$\sum_{n\geq 0} \left(\sum_{\omega\in[k]^n} q^{\mathrm{des}(\omega)} \right) x^n = \frac{q-1}{q - (1 + x(q-1))^k}.$$

Now we focus on the case $d = 2$.

Theorem 2.4. *The generating function $W_{k,2}(x,q)$ is given by*

$$W_{k,2}(x,q) = \frac{q-1}{q - \sqrt{(x(1-q))^{k+1}}\, U_{k+1}\left(\frac{1}{2\sqrt{x(1-q)}}\right)},$$

where U_m is the m-th Chebyshev polynomial of the second kind.

Proof. Theorem 2.2 when $d = 2$ implies

$$W_{k,2}(x,q) = \frac{q-1}{q - \sum_{i\geq 0} \binom{k+1-i}{i} x^i (q-1)^i}.$$

Let $p_k = \sum_{i\geq 0} \binom{k+1-i}{i} x^i (q-1)^i$. Then the identity $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$ implies that p_k satisfies the recurrence relation $p_k = p_{k-1} - x(1-q)p_{k-2}$ if $k \geq 2$, with $p_0 = 1$ and $p_1 = 1 - x(1-q)$. Hence, by induction on k and since the Chebyshev polynomials of the second kind satisfy $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with $U_0(t) = 1$ and $U_1(t) = 2t$, we obtain $p_k = \sqrt{(x(1-q))^{k+1}}\, U_{k+1}\left(\frac{1}{2\sqrt{x(1-q)}}\right)$, which completes the proof. □

Combining Lemma 2.1 and Theorem 2.2, and noting $W_{k,d;a} = x f_a W_{k,d}$, yields the following explicit formula for $W_{k,d;a}$ which we will need later.

COROLLARY 2.5

If $1 \le a \le k$, *then*

$$W_{k,d;a} = \frac{\sum_{i\ge 0}\binom{a-1-i(d-1)}{i}x^{i+1}(q-1)^i}{1-\sum_{i\ge 0}\binom{k-i(d-1)}{i+1}x^{i+1}(q-1)^i}.$$

2.1 *Set partitions according to the statistic* des_d^+

Recall that the set of partitions of $[n]$ with k blocks is denoted by $P(n,k)$. Let $1 \le d \le k$ be fixed. Consider the generating function for the number of partitions of $[n]$ with k blocks according the statistic des_d^+:

$$P_{k,d}(x,q) := \sum_{n\ge k}\left(\sum_{\pi\in P(n,k)} q^{\mathrm{des}_d^+(\pi)}\right)x^n.$$

To find this, we first express $\pi \in P(n,k)$ as $\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$, where $\pi^{(j)}$ is a word in the alphabet $[j]$, and let n vary. If $1 \le j \le d$, then the subsequence $j\pi^{(j)}$ contributes a factor of $\frac{x}{1-jx}$ towards $P_{k,d}(x,q)$ as $\mathrm{des}_d^+(\pi)$ is always 0 in this case. If $d+1 \le j \le k$, then the subsequence $j\pi^{(j)}$ contributes a factor of exactly $W_{j,d;j}$ which, according to Corollary 2.5, is given by

$$\frac{\sum_{i\ge 0}\binom{j-1-i(d-1)}{i}x^{i+1}(q-1)^i}{1-\sum_{i\ge 0}\binom{j-i(d-1)}{i+1}x^{i+1}(q-1)^i}.$$

Combining these two cases yields the following result.

Theorem 2.6. *The generating function for the number of partitions of* $[n]$ *with exactly* k *blocks according to the number of descents of size* d *or more is given by*

$$P_{k,d}(x,q) = \frac{x^d}{(1-x)\cdots(1-dx)}\prod_{j=d+1}^{k} W_{j,d;j}$$
$$= \frac{x^k}{(1-x)\cdots(1-dx)}\prod_{j=d+1}^{k}\frac{\sum_{i\ge 0}\binom{j-1-i(d-1)}{i}x^{i}(q-1)^i}{1-x\sum_{i\ge 0}\binom{j-i(d-1)}{i+1}x^{i}(q-1)^i}.$$

Using the binomial theorem, one may show that our expression for $P_{k,d}(x,q)$ when $d=1$ reduces to the one in (1.1) above. Applying Theorem 2.6 in the case $d=2$, we obtain table 1.

By Corollary 2.5, we have

$$W_{k,d;a}(x,1) = \frac{x}{1-kx} \qquad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}q}W_{k,d;a}(x,q)\mid_{q=1} = \frac{(a-d)x^2}{1-kx} + \frac{\binom{k+1-d}{2}x^3}{(1-kx)^2}.$$

Table 1. The polynomial $\sum_{\pi\in P(n,k)} q^{\mathrm{des}_d^+(\pi)}$ for $1\le k\le n\le 5$ and $d=2$.

$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	$q+5$	1	
5	1	15	$7q+18$	$3q+7$	1

Theorem 2.6 then gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}q}P_{k,d}(x,q)\mid_{q=1} \;=\; & \frac{x^d}{\prod_{j=1}^{d}(1-jx)}\prod_{j=d+1}^{k}W_{j,d;j}(x,1)\\ & \times\sum_{j=d+1}^{k}\frac{\frac{\mathrm{d}}{\mathrm{d}q}W_{j,d;j}(x,q)\mid_{q=1}}{W_{j,d;j}(x;1)}=\frac{x^{k+1}}{\prod_{j=1}^{k}(1-jx)}\\ & \times\sum_{j=d+1}^{k}(j-d)\left(1+\frac{(j+1-d)x}{2(1-jx)}\right).\end{aligned}$$

Using the fact $\frac{x^k}{\prod_{j=1}^{k}(1-jx)}=\sum_{n\ge k}S_{n,k}x^n$ (see, for example p. 46 of [10]), we obtain the following result.

COROLLARY 2.7

The number of occurrences of descents of size d or more in all the partitions of $[n]$ with exactly k blocks is given by

$$\binom{k-d+1}{2}S_{n-1,k}+\sum_{j=d+1}^{k}\left(\binom{j-d+1}{2}\sum_{i=k}^{n-2}j^{n-2-i}S_{i,k}\right).$$

We are able to give a combinatorial proof of Corollary 2.7. In what follows, we will call an index i within $\pi=\pi_1\pi_2\cdots\pi_n\in P(n,k)$ such that $\pi_i\ge\pi_{i+1}+d$ a d-descent.

Combinatorial proof of Corollary 2.7

Given $\pi=\pi_1\pi_2\cdots\pi_n\in P(n,k)$, let us call a d-descent *initial* if the index i corresponds to the smallest element of some block of π and *non-initial* otherwise. We first show that $\binom{k-d+1}{2}S_{n-1,k}$ counts all initial d-descents within the members of $P(n,k)$. First choose two numbers a and b in $[k]$, where $a+d\le b$. Note that there are $1+2+\cdots+(k-d)=\binom{k-d+1}{2}$ choices for a and b. Given $\rho=\rho_1\rho_2\cdots\rho_{n-1}\in P(n-1,k)$, let m denote the smallest index j such that $\rho_j=b$. Insert the letter a just after position m within ρ so as to obtain an initial d-descent at index m within some member of $P(n,k)$. For example, if $n=9$, $k=4$, $d=2$, $a=1$ and $b=3$, then $\rho=12232344\in P(8,4)$ would give rise to an initial 2-descent between 3 and 1 in $122312344\in P(9,4)$.

We now count the non-initial d-ascents in all the members of $P(n,k)$. Given i and j, where $k \le i \le n-2$ and $d+1 \le j \le k$, consider all of the members of $P(n,k)$ which may be expressed uniquely in the form

$$\pi = \pi' j \gamma \delta, \tag{2.3}$$

where π' is a partition with $j-1$ blocks, γ is a word having length $n-i$ in the alphabet $[j]$ whose last two letters form a d-descent, and δ is possibly empty. For example, if $d=2$, $i=10$, $j=4$, and $\pi = 121231412231435 \in P(15,5)$, then $\pi' = 121231$, $\gamma = 12231$ and $\delta = 435$. The total number of non-initial d-ascents can be obtained by finding the number of partitions which may be expressed as in (2.3) for each i and j and then summing over all possible i and j. By doing so, a member of $P(n,k)$ is counted each time a d-descent occurs within it and summing over all members of $P(n,k)$ then yields the total number of d-descents, as desired. Moreover, there are $\binom{j-d+1}{2} j^{n-2-i} S_{i,k}$ members of $P(n,k)$ which may be expressed as in (2.3) since there are j^{n-2-i} choices concerning the first $n-2-i$ letters of γ, $\binom{j-d+1}{2}$ choices for the final two letters of γ (as the last letter must be less than its predecessor by at least d, whence there are $1+2+\cdots+(j-d) = \binom{j-d+1}{2}$ possibilities), and $S_{i,k}$ choices for the remaining letters $\pi' j \delta$ which necessarily comprise a partition of an i-set into k blocks. □

2.2 *Set partitions according to the statistic* asc_d^+

Here we consider the asc_d^+ statistic on partitions of $[n]$, where $d \ge 2$. Clearly, the generating function for the number of words of length n over the alphabet $[k]$ according to the statistic asc_d^+ is given by $W_{k,d}(x,q)$. Furthermore, if we define $W_{k,d|a}(x,q)$ to be the generating function for the number of words $\pi = \pi_1\pi_2\cdots\pi_n$ of length n over the alphabet $[k]$ with $\pi_n = a$ according to the statistic asc_d^+, then by the reversal operation we see that $W_{k,d|a}(x,q) = W_{k,d;a}(x,q)$.

Consider the generating function for the number of partitions of $[n]$ with k blocks according to the statistic asc_d^+ given by

$$Q_{k,d}(x,q) := \sum_{n\ge k}\left(\sum_{\pi\in P(n,k)} q^{\mathrm{asc}_d^+(\pi)}\right)x^n.$$

Write a partition π of $[n]$ with exactly k blocks as $\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$, where $\pi^{(j)}$ is a word in the alphabet $[j]$, and suppose n varies. If $0 \le j \le d-1$, then the subsequence $\pi^{(j)}(j+1)$ (when $j=0$ we define $\pi^{(0)}$ to be the empty word) contributes towards $Q_{k,d}(x,q)$ a factor of $\frac{x}{1-jx}$ as $\mathrm{asc}_d^+(\pi)$ is always 0 in this case. The subsequence $\pi^{(k)}$ contributes a factor of $W_{k,d}(x,q)$ towards $Q_{k,d}(x,q)$. If $d \le j \le k-1$, then the subsequence $\pi^{(j)}(j+1)$ contributes

$$x(1+qW_{j,d|1}(x,q)+\cdots+qW_{j,d|j+1-d}(x,q) \\ +W_{j,d|j+2-d}(x,q)+\cdots+W_{j,d|j}(x,q)),$$

which is equivalent to

$$xW_{j,d}(x,q)+x(q-1)\sum_{i=1}^{j+1-d} W_{j,d;i}(x,q).$$

Combining these cases, and using Lemma 2.1 and Theorem 2.2, yields

$$\begin{aligned}
Q_{k,d}(x,q) &= \frac{x^d W_{k,d}(x,q)}{(1-x)(1-2x)\cdots(1-(d-1)x)} \\
&\quad\times \prod_{j=d}^{k-1}\left(xW_{j,d}(x,q)+x(q-1)\sum_{a=1}^{j+1-d} W_{j,d;a}(x,q)\right) \\
&= \frac{x^k\prod_{j=d}^{k} W_{j,d}(x,q)}{\prod_{j=1}^{d-1}(1-jx)} \\
&\quad\times \prod_{j=d}^{k-1}\left(1+(q-1)\sum_{a=1}^{j+1-d}\sum_{i\geq 0}\binom{a-1-i(d-1)}{i}x^{i+1}(q-1)^i\right) \\
&= \frac{x^k\prod_{j=d}^{k} W_{j,d}(x,q)}{\prod_{j=1}^{d-1}(1-jx)} \\
&\quad\times \prod_{j=d}^{k-1}\left(1+\sum_{i\geq 0}\binom{j-(i+1)(d-1)}{i+1}x^{i+1}(q-1)^{i+1}\right) \\
&= \frac{x^k}{\prod_{j=1}^{d-1}(1-jx)}\,\frac{\prod_{j=d}^{k-1}\left(1+\sum_{i\geq 1}\binom{j-i(d-1)}{i}x^i(q-1)^i\right)}{\prod_{j=d}^{k}\left(1-\sum_{i\geq 0}\binom{j-i(d-1)}{i+1}x^{i+1}(q-1)^i\right)},
\end{aligned}$$

which gives the following result.

Theorem 2.8. *Let $d \geq 2$. The generating function for the number of partitions of $[n]$ with exactly k blocks according to the number of ascents of size d or more is given by*

$$Q_{k,d}(x,q) = \frac{x^k\prod_{j=d}^{k-1}\left(1+\sum_{i\geq 1}\binom{j-i(d-1)}{i}x^i(q-1)^i\right)}{\prod_{j=1}^{k}\left(1-\sum_{i\geq 0}\binom{j-i(d-1)}{i+1}x^{i+1}(q-1)^i\right)}.$$

Remark. The reasoning above shows that the $d = 1$ case of the last formula is the generating function for the number of partitions of $[n]$ with exactly k blocks according to the number of ascents, where ascents of the form $a(a+1)$ in which both a and $a+1$ correspond to minimal elements of their respective blocks are not counted.

2.3 *Equidistribution of descents and ascents*

Comparison of Theorems 2.6 and 2.8 reveals the following result.

Theorem 2.9. *If $d \geq 2$, then the members of $P(n,k)$ having exactly m descents of size d or more are equinumerous with those of $P(n,k)$ having exactly m ascents of size d or more for all $m \geq 0$. Furthermore, the members of $P(n,k)$ having exactly m descents are* *equinumerous with those having exactly m ascents, where ascents are restricted so as not to involve two consecutive minimal elements.*

In this section, we shall provide a combinatorial proof of Theorem 2.9. In what follows, we will call a descent (resp. ascent) of size d or more a *d-descent* (resp. *d-ascent*). A *run* will refer to a maximal sequence of consecutive letters of a given kind. If m and n are positive integers, then let $[m, n] = \{m, m+1, \ldots, n\}$ if $m \leq n$, with $[m, n] = \varnothing$ if $m > n$.

Combinatorial proof of Theorem 2.9

We prove only the first statement in the theorem. The second, which is easier, follows by slightly modifying the argument for the first. Let n, k, d be given positive integers, where $n \geq k \geq d$ and $d \geq 2$. Suppose λ is a fixed partition on the letters in $[k - d + 1, k]$. To λ, we will sequentially add copies of the letters $k - d, k - d - 1, \ldots, 1$ to obtain members of $P(n, k)$. Let us say two partitions $\rho_1, \rho_2 \in P(n, k)$ are *d-equivalent* if the partitions on $[k - d + 1, k]$ resulting when one covers all of the letters in $[k - d]$ occurring in ρ_1 and ρ_2 are the same. For example, if $n = 12$, $k = 4$ and $d = 2$, then $\alpha = 121314321342$ is 2-equivalent to $\beta = 123241332141$ since the same partition on $\{3, 4\}$ results, namely, 34334. Denote by $P_\lambda(n, k)$ the d-equivalence class comprising all members of $P(n, k)$ for which the partition λ arises when the letters of $[k - d]$ are covered. Note that $P(n, k) = \cup_\lambda P_\lambda(n, k)$, where λ ranges over all the partitions on $[k - d + 1, k]$ of length at most $n - k + d$.

Suppose $\pi = \pi_1\pi_2 \cdots \pi_n \in P(n, k)$ and that a d-ascent occurs at index i for some i within π. We shall say such an ascent is *of type t* if $t = \pi_j$, where j is the *smallest* index ℓ such that the following three conditions hold:

(i) $1 \leq \ell \leq i$;
(ii) $\pi_{i+1} - \pi_\ell \geq d$; and
(iii) $\pi_\ell > \max\{\pi_{\ell+1}, \pi_{\ell+2}, \ldots, \pi_i\}$.

(Here we take the max of an empty set of positive integers to be zero, for convenience.) Note that j is guaranteed to exist since $\ell = i$ satisfies (i)–(iii), the third condition vacuously.

Next suppose a d-descent occurs at index i for some i within π. We shall say such a descent is *of type t* if $t = \pi_j$, where j is now the *largest* index ℓ such that the following three conditions hold:

(i) $i + 1 \leq \ell \leq n$;
(ii) $\pi_i - \pi_\ell \geq d$; and
(iii) $\pi_\ell > \max\{\pi_{i+1}, \pi_{i+2}, \ldots, \pi_{\ell-1}\}$.

Note that j exists since $\ell = i + 1$ satisfies (i)–(iii). Observe that d-ascents and d-descents within $\pi \in P(n, k)$ may be of type t for any $t \in [k - d]$. For example, if

$$\pi = 12134332125511234233 \in P(20, 5)$$

and $d = 2$, then π has a 2-ascent of type 3 at $i = 10$ (as $j = 7$ with $\pi_7 = 3$) and a 2-descent of type 3 at $i = 12$ (as $j = 16$ with $\pi_{16} = 3$). It also has a 2-ascent of type 1 at $i = 3$ (as $j = i$ with $\pi_3 = 1$) and a 2-descent of type 2 at $i = 17$ (as $j = i + 1 = 18$ with $\pi_{18} = 2$); note that condition (iii) for j holds vacuously in the last two cases.

Suppose $a_1, a_2, \ldots, a_{k-d}$ are non-negative integers and λ is a partition on $[k-d+1, k]$. Let $X = X_\lambda(a_1, a_2, \ldots, a_{k-d})$ (resp. $Y = Y_\lambda(a_1, a_2, \ldots, a_{k-d})$) denote the subset of $P_\lambda(n, k)$ in which there are exactly a_i d-ascents (resp. d-descents) of type i for each $i \in [k-d]$. For example, if $\pi \in P(20, 5)$ is as above and $d = 2$, then $\pi \in X_\lambda(1, 0, 1)$ and $\pi \in Y_\lambda(0, 1, 1)$, where $\lambda = 4554$. To prove the first statement in Theorem 2.9 above, it suffices to show that $|X| = |Y|$ for all choices of λ and a_i. For one may take the union of all X (as well as all Y) corresponding to those choices of λ and the a_i for which $\sum_{i=1}^{k-d} a_i = m$, where m is some fixed number.

To show $|X| = |Y|$, we consider a further refinement as follows. Given positive integers $b_1, b_2, \ldots, b_{k-d}$, let $Z = Z(b_1, b_2, \ldots, b_{k-d})$ and $W = W(b_1, b_2, \ldots, b_{k-d})$ denote the subsets of X and Y, respectively, in which there are exactly b_i occurrences of the letter i for each $i \in [k-d]$. It then suffices to show that $|Z| = |W|$ for all choices of b_i. To show this, we first add b_{k-d} copies of the letter $k-d$ to the partition $\lambda = \lambda_1\lambda_2\cdots\lambda_s$, where s denotes the length of λ. We do this by first writing a single $k-d$ directly in front of λ and then writing (possibly empty) runs of the letter $k-d$ just before each letter λ_i as well as after λ_s. Placing a non-empty run of $k-d$ just before a letter k in λ produces a single d-ascent (which will necessarily be of type $k-d$ in the final member of $P(n, k)$ generated once all of the letters in $[k-d]$ have been added to λ). Placing a run of $k-d$ just after a k produces a single d-descent (of type $k-d$). Placing a run of $k-d$ anywhere else in λ produces neither a d-ascent nor a d-descent.

Since there are to be a_{k-d} d-ascents (resp. d-descents) of type $k-d$ in the final partition produced, there must be a non-empty run of $k-d$ inserted directly before (resp. just after) exactly a_{k-d} of the letters k in λ. Since the number of $(k-d)$'s to be added is the same (namely, b_{k-d}), the number of ways to insert $k-d$ is the same for ascents as it is for descents. Let Z_1 denote the resulting set of partitions on $[k-d, k]$ each of whose members have exactly a_{k-d} d-ascents and b_{k-d} copies of $k-d$ and, similarly, let W_1 denote the set of partitions on $[k-d, k]$ formed in a comparable way but having a_{k-d} d-descents instead in its members; note that $|Z_1| = |W_1|$. Now repeat the above process, adding b_{k-d-1} copies of $k-d-1$ to each member of Z_1 or W_1 so as to produce, respectively, exactly a_{k-d-1} d-ascents or d-descents of type $k-d-1$, letting Z_2 and W_2 denote the respective sets of partitions on $[k-d-1, k]$ so produced.

In general, consider adding b_i copies of the letter i so as to create either a_i d-ascents or a_i d-descents where $1 \le i \le k-d$. We add these letters i, sequentially starting with $i = k-d$ and working down, in the $(k-d-i+1)$-st step adding them to each member of the sets Z_{k-d-i} and W_{k-d-i} so as to yield new sets $Z_{k-d-i+1}$ and $W_{k-d-i+1}$, respectively, where $Z_0 = W_0 = \{\lambda\}$. When the process terminates after $k-d$ steps, it will be seen that $Z_{k-d} = Z$ and $W_{k-d} = W$. Note that Z_{k-d-i} and W_{k-d-i} consist of partitions on $[i+1, k]$. Let $\rho = \rho_1\rho_2\cdots\rho_u \in Z_{k-d-i}$, where $u = (b_{i+1} + \cdots + b_{k-d}) + |\lambda|$. Note that ρ has a_j d-ascents of type j for $i+1 \le j \le k-d$. We write the letter i before ρ and then insert $b_i - 1$ copies of i elsewhere in ρ, including at the end. Inserting a non-empty run of i into ρ just before a letter in the set $[i+d, k]$ which is *not* the larger number in a d-ascent generates a single additional d-ascent (which will be of type i in the resulting member of $P_\lambda(n, k)$ after all the letters have been added). Inserting a non-empty run of i elsewhere within ρ (i.e. either at the end or just before a letter in $[i+1, i+d-1]$ or just before a letter in $[i+d, k]$ which is currently the larger number in a d-ascent within ρ) does not produce an additional d-ascent. Note that we are to generate a_i d-ascents of type i by adding $b_i - 1$ copies of i to $i\rho$ as described.

Then exactly a_i of

$$(b_{i+d} + \cdots + b_{k-d}) + |\lambda| - (a_{i+1} + \cdots + a_{k-d})$$

possible positions within ρ are to contain a non-empty run of the letter i, while the remaining

$$(b_{i+1} + \cdots + b_{i+d-1}) + (a_{i+1} + \cdots + a_{k-d}) + 1$$

positions within ρ may or may not contain such a run (note that ρ has a total of $b_{i+1} + \cdots + b_{k-d} + |\lambda|$ letters and hence the total number of positions in which to insert a run of i is one more than this). By similar reasoning, the same can be said concerning the addition of the letter i into any $\gamma \in W_{k-d-i}$ so as to create exactly a_i d-descents (of type i). Thus, we see $\frac{|Z_{k-d-i+1}|}{|Z_{k-d-i}|} = \frac{|W_{k-d-i+1}|}{|W_{k-d-i}|}$ for all $i \in [k-d]$ since each element of Z_{k-d-i} (resp. W_{k-d-i}) is seen to give rise to the same number of elements of $Z_{k-d-i+1}$ (resp. $W_{k-d-i+1}$). Multiplying, we then have $|Z| = |Z_{k-d}| = |W_{k-d}| = |W|$, as desired, which completes the proof. □

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