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Variational problem with complex coefficient of a nonlinear Schrödinger equation

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Abstract. An optimal control problem governed by a nonlinear Schrödinger equation with complex coefficient is investigated. The paper studies existence, uniqueness and optimality conditions for the control problem.

Keywords. Variational problem; optimal control.

1. Introduction

Optimal control problems for partial differential equations are currently of much interest. An extensive literature in this area is devoted to parabolic equations [10–12,15,16,23]. The problem of quantum systems control is a scientific problem of present interest related to modern physical technologies. Determining the quantum-mechanichal potential is one of the basic problems of quantum mechanics. Given simplifying assumptions, this potential is determined on the basis of intuitive concepts [3,8,10,11]. In this paper, we apply the method of variational computation to solving the inverse optimal control problem in determining the quantum-mechanical potential in the nonlinear Schrödinger equation (NLSE) while the control is a part of the potential, with complex coefficients. The optimal control problem for a system governed by the Schrödinger equation arise in various branches of quantum-mechanical and modern physics and chemistry [4–8].

Standard results of boundary value problem for the nonlinear Schrödinger equation can be found in [1,13,22]. The results of these studies is not sufficient to investigate considering the optimal control problem. Because, coefficients of the differential equations has been chosen such as bounded and measurable functions in these studies. On the other hand, we chose unbounded coefficients and larger classes of controls.

2. The optimal control problem

This section is devoted to formulation of the optimal control problem which is the subject of our investigations. The estimates for the solution of the problem is given. Let $\Omega_t =$

$(0,l) \times (0,t)$, $\Omega = \Omega_T$, and the space $C^k([0,T],B)$ be a Banach space that consist of all functions with values in the Banach space B that are defined $k \geq 0$ times continuous differentiable in $[0,T]$. Let $l > 0, T > 0$ be given positive numbers and $x \in (0,l)$, $t \in (0,T)$. The norm is defined by

$$\| u \|_{C^k([0,T],B)} = \sum_{m=0}^{k} \max_{0 \leq t \leq T} \left\| \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} \right\|_B$$

in the space $C^k([0,T],B)$ for $u = u(t) \in C^k([0,T],B)$. The spaces $W_2^1(0,l)$, $\overset{\circ}{W}{}_2^1(0,l)$, $W_2^2(0,l)$, $\overset{\circ}{W}{}_2^2(0,l)$, $W_2^{1,0}(\Omega)$, $\overset{\circ}{W}{}_2^{1,0}(\Omega)$, $W_2^{0,1}(\Omega)$, $W_2^{2,1}(\Omega)$ can be defined as in [17], [2] where $\overset{\circ}{W}{}_2^2(0,l) = W_2^2(0,l) \cap \overset{\circ}{W}{}_2^1(0,l)$, $\overset{\circ}{W}{}_2^{2,1}(\Omega) = W_2^{2,1}(\Omega) \cap \overset{\circ}{W}{}_2^{1,0}(\Omega)$. This paper is concerned with finding the minimum of the cost functional

$$J_\alpha(v) = \| \psi(.,T) - y \|_{L_2(0,l)}^2 + \alpha \| v - w \|_{L_2(0,l)}^2 \tag{1}$$

in the set of admissible controls

$$V = \{v = v(x) : v \in L_2(0,l), \| v \|_{L_2(0,l)} \leq b_0, b_0 = \text{ constant}\}$$

under the conditions

$$i\frac{\partial \psi}{\partial t} + a_0 \frac{\partial^2 \psi}{\partial x^2} - a(x)\psi - v(x)\psi + a_1|\psi|^2\psi = f(x,t) \tag{2}$$

with initial and boundary conditions

$$\psi(x,0) = \varphi(x), \ x \in (0,l), \tag{3}$$

$$\psi(0,t) = \psi(l,t) = 0, \ \ t \in (0,T), \tag{4}$$

where $i^2 = -1$, $a_0 > 0$, $\alpha > 0$, $b_0 > 0$ are the given numbers, $a(x)$ is a bounded and measurable function that satisfies the additional restrictions

$$0 < a(x) \leq \mu, \quad \overset{\circ}{\forall} x \in (0,l), \mu = \text{constant}. \tag{5}$$

Also, φ, $f(x,t)$ are given functions and they satisfy the conditions

$$\varphi \in \overset{\circ}{W}{}_2^2(0,l), \quad f(x,t) \in W_2^{0,1} \tag{6}$$

respectively. $y, w \in L_2(0,l)$ are the given functions and y is a target state, $a_1 \in \mathbb{C}$ is a complex number with property

$$\operatorname{Im} a_1 > 0, \ \operatorname{Re} a_1 < 0, \ \operatorname{Im} a_1 \geq 2|\operatorname{Re} a_1|. \tag{7}$$

The problem of finding functions $\psi = \psi(x,t) \equiv \psi(x,t;v)$ from conditions (2)–(4) for each $v \in V$ is an initial-boundary value problem. The function $\psi(x,t)$ which belongs to space $B_0 \equiv C^0([0,T], \overset{\circ}{W}{}_2^2(0,l)) \cap C^1([0,T], L_2(0,l))$ is called solution of the initial-boundary problems (2)–(4) $\forall t \in [0,T]$, $\overset{\circ}{\forall} x \in (0,l)$, where the symbol $\overset{\circ}{\forall}$ signify that the given property applies for almost all values of a variable quantity.

The initial boundary value problem for nonlinear Schrödinger equation with complex coefficient is discussed in [14]. In [14], the existence and uniqueness of the solution of the initial boundary value problem are proved by Galerkin's method under the condition $2 \leq p < n = 3$, where p is the integrability order of potential and n is the dimension of space. But, we cannot say the existence and uniqueness of the solution of the problems (2)–(4) obtained from the results in [14], since $n = 1$ and $p = 2$ in our manuscript. For this reason, using the method in [20] we can easily prove that the following theorem is valid.

Theorem 1. *Assume that the functions $a(x)$, $\varphi(x)$ and $f(x,t)$ satisfy the conditions* (5)–(6) *respectively. Let a_0 be a complex number with the property* (7). *Then, the initial boundary value problem* (2)–(4) *has a unique solution from the space B_0 $\forall v \in V$ and for the solution the estimate*

$$\| \psi(.,t) \|^2_{\overset{\circ}{W}{}_2^2(0,l)} + \left\| \frac{\partial \psi}{\partial t} \right\|^2_{L_2(0,l)} \leq c_0 (\| \varphi \|^2_{\overset{\circ}{W}{}_2^2(0,l)} + \| f \|^2_{W_2^{0,1}(\Omega)} + \| \varphi \|^6_{L_2(0,l)}) \tag{8}$$

is valid $\forall t \in [0,T]$, where the constant $c_0 > 0$ is independent from t.

3. Existence and uniqueness theorems

Optimal control problems for solutions of differential equations do not always have a solution [22]. In this section we prove sufficient conditions for the existence of a solution to problems (1)–(4). We first deal with two cases $\alpha > 0$ and $\alpha \geq 0$.

Theorem 2. *Let $y \in L_2(0,l)$, $w \in L_2(0,l)$ be given functions. Assume that the functions $a(x)$, $\varphi(x)$ and $f(x,t)$ satisfy the conditions* (5), (6) *respectively. Then, there is a dense subset G of the space $L_2(0,l)$ such that the variational problem* (1)–(4) *has a unique solution $\forall \alpha > 0$ and $\forall w \in G$.*

Proof. In the first part of the proof, we shall present the continuity of the functional

$$J_0(v) = \| \psi(.,T) - y \|^2_{L_2(0,t)} \tag{9}$$

on the set V. Let $\Delta v \in L_2(0,l)$ be an increment of control on any element $v \in V$ such that $v + \Delta v \in V$. Let $\psi = \psi(x,t) \equiv \psi(x,t;v)$ be the solution of the problems (2)–(4) corresponding to $v \in V$. Then, increment of the function ψ is $\Delta\psi = \Delta\psi(x,t) \equiv \psi_\Delta - \psi(x,t)$ where the function $\psi_\Delta = \psi(x,t;v+\Delta v)$ is the solution of the initial-boundary value problem (2)–(4) corresponding to $v + \Delta v$. On the basis of assumptions and by using the conditions (2)–(4), it follows that the function $\Delta\psi = \Delta\psi(x,t)$ is a solution of the following initial boundary value problem

$$\begin{aligned} i\frac{\partial \Delta\psi}{\partial t} &+ a_0 \frac{\partial^2 \Delta\psi}{\partial x^2} - a(x)\Delta\psi - (v+\Delta v)\Delta\psi + a_1(|\psi_\Delta|^2 + |\psi|^2))\Delta\psi \\ &+ a_1 \psi_\Delta \psi \Delta\bar{\psi} = \Delta v(x)\psi(x,t;v), \ (x,t) \in \Omega, \end{aligned} \tag{10}$$

$$\Delta\psi(x,0)=0,\quad x\in(0,l),\quad \Delta\psi(0,t)=\Delta\psi(l,t)=0, t\in[0,T]. \tag{11}$$

Let us obtain an estimate for the solution of the initial boundary value problem (10), (11). For this purpose, multiplying both sides of equation (10) by $\Delta\bar{\psi}(x,t)$ and integrating over Ω_t, using integration by parts and considering conditions (11) we obtain the following equality (we use τ as dummy argument):

$$\int_{\Omega_t}\Bigg[i\frac{\partial\Delta\psi}{\partial\tau}\Delta\bar{\psi}-a_0\left|\frac{\partial\Delta\psi}{\partial x}\right|^2-a(x)\,|\Delta\psi|^2-(v(x)+\Delta v(x))|\Delta\psi|^2$$
$$+a_1(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2+a_1\psi_\Delta\psi(\Delta\bar{\psi})^2\Bigg]\mathrm{d}x\mathrm{d}\tau$$
$$=\int_{\Omega_t}\Delta v(x)\psi\Delta\bar{\psi}\mathrm{d}x\mathrm{d}\tau.$$

If we subtract the complex conjugate of the above equality from itself then we get

$$\int_{\Omega_t} i\left(\frac{\partial\Delta\psi}{\partial\tau}\Delta\bar{\psi}+\frac{\partial\Delta\bar{\psi}}{\partial\tau}\Delta\psi\right)\mathrm{d}x\mathrm{d}\tau$$
$$=-\int_{\Omega_t}2i\,\mathrm{Im}\,a_1(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2\mathrm{d}x\mathrm{d}\tau$$
$$-\int_{\Omega_t}(a_1\psi_\Delta\psi(\Delta\bar{\psi})^2-\bar{a}_1\bar{\psi}_\Delta\bar{\psi}(\Delta\psi)^2)\mathrm{d}x\mathrm{d}\tau$$
$$+2i\int_{\Omega_t}\mathrm{Im}(\Delta v(x)\psi\Delta\bar{\psi}\mathrm{d}x\mathrm{d}\tau$$

and

$$\|\Delta\psi(.,t)\|^2_{L_2(0,l)}\le 3\,\mathrm{Im}\,a_1\int_{\Omega_t}(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2\mathrm{d}x\mathrm{d}\tau$$
$$+|\mathrm{Re}\,a_1|\int_{\Omega_t}(|\psi_\Delta|^2+|\psi|^2)|\Delta\psi|^2\,\mathrm{d}x\mathrm{d}\tau$$
$$+2\int_{\Omega_t}|\Delta v|\cdot|\psi(x,\tau)|\cdot|\Delta\psi(x,\tau)|\mathrm{d}x\mathrm{d}\tau,\quad t\in[0,T]. \tag{12}$$

Using estimate (8) and the well-known inequality

$$\|h(.,t)\|_{L_\infty(0,l)}\le c_1\left\|\frac{\partial h(.,t)}{\partial x}\right\|^{\frac{1}{2}}_{L_2(0,l)}\|h(.,t)\|^{\frac{1}{2}}_{L_2(0,l)} \tag{13}$$

in [17], we obtain the inequality

$$\|\Delta\psi(.,t)\|^2_{L_2(0,l)}\le c_2\int_0^t\|\Delta\psi(.,\tau)\|^2_{L_2(0,l)}\,\mathrm{d}\tau+c_3\,\|\Delta v\|^2_{L_2(0,l)}, \tag{14}$$

where the constants $c_2 > 0$, $c_3 > 0$ does not depend on $\Delta\psi$ and Δv. By the Gronwall's inequality, we write

$$\|\Delta\psi(.,t)\|^2_{L_2(0,l)} \le c_4 \parallel \Delta v \parallel^2_{L_2(0,l)}, \quad \forall t \in [0,T], \tag{15}$$

where $c_4 > 0$ is a constant that does not depend on Δv. Now, let us evaluate the increment of the functional $J_0(v)$ $\forall v \in V$.

$$\Delta J_0(v) = J_0(v+\Delta v) - J_0(v) = 2\int_0^l \mathrm{Re}[(\psi(x,T) - y(x))(\Delta\bar{\psi}(x,T)]\mathrm{d}x$$
$$+ \|\Delta\psi(.,T)\|^2_{L_2(0,l)} \,. \tag{16}$$

Applying the Cauchy–Bunyakowski inequality to this equality and using the estimates (8), (15) we deduce following inequality:

$$|J_0(v+\Delta v) - J_0(v)| \le c_5(\|\Delta v\|_{L_2(0,l)} + \|\Delta v\|^2_{L_2(0,l)}), \tag{17}$$

where $c_5 > 0$ is a constant that does not depend Δv. Since the inequality (17) is valid $\forall v \in V$ the functional $J_0(v)$ is continuous on the set V. Thus we have proved the first part of the theorem. The set V is closed, bounded and convex on the space $L_2(0,l)$. The space $L_2(0,l)$ is the smooth convex space [24]. Furthermore we know that the functional $J_0(v)$ is continuous on V and lower bounded ($J_0(v) \ge 0$ $\forall v \in V$). Hence the conditions of theorem in [9] holds. Therefore, according to this theorem, there is a dense set $G \subset L_2(0,l)$ such that the optimal control problem (1)–(4) has a unique solution $\forall w \in G$, $\forall\alpha > 0$. Theorem 2 is proved. □

The next theorem establishes that the optimal control problem (1)–(4) has at least one solution for any $\alpha \ge 0$.

Theorem 3. *Assume that $\alpha \ge 0$ and the hypotheses of Theorem* (2) *holds. Then, the optimal control problem* (1)–(4) *has at least one solution.*

Proof. Let $\{v^m\} \subset V$ be a minimizing sequence for the functional $J_\alpha(v)$ and $\lim_{m\longrightarrow\infty} J_\alpha(v^m) = J_{\alpha^*} = \inf_{v\in V} J_\alpha(v)$. Let us denote solution of the initial value problem (2)–(4) as $\psi_m = \psi(x,t) \equiv \psi(x,t;v^m)$ for any $v^m \in V$. From the fact that $v^m \in V$, $m = 1,2,\ldots,$ and according to Theorem 1, we know that problems (2)–(4) have a unique solution $\psi_m = \psi(x,t) \equiv \psi(x,t;v^m)$, for each $m = 1,2,\ldots$. Moreover, the following estimate is valid for the solution of problems (2)–(4):

$$\|\psi_m(.,t)\|_{\overset{\circ}{W}{}^2_2(0,l)} + \left\|\frac{\partial\psi_m(.,t)}{\partial t}\right\|_{L_2(0,l)} \le c_6, \quad \forall t \in [0,T] \tag{18}$$

where $c_6 > 0$ is a constant that does not depend m. Also, c_6 represents the right-hand side of the estimate (8). According to the definition, the set V is closed, bounded and convex on the Hilbert space $L_2(0,l)$. Therefore the set V is weakly compact and closed on the space $L_2(0,l)$ so we can chose a subsequence of $\{v^m\}$ such as weakly converging to $v \in V$. Let us denote this subsequence as $\{v^m\}$ for simplicity. Hence we can write the following limit relation:

$$\int_0^l v^m(x)q(x)\mathrm{d}x \longrightarrow \int_0^l v(x)q(x)\mathrm{d}x, \quad m \to \infty \tag{19}$$

for sequence $\{v^m\} \subset V$, for any $v \in V$ and $\forall q \in L_2(0,l)$. It follows from the estimate (18) that the sequence $\{\psi_m\}$ is smooth bounded in B_0. It follows that there exists a subsequence of $\{\psi_m\}$, which we also denote by $\{\psi_m\}$, for simplicity such that $\{\psi_m\} \to \psi(x,t)$ (weakly in the space $L_2(0,l)$). Finally, we infer the sequences $\{\psi_m\}$, $\left\{\frac{\partial\psi_m}{\partial x}\right\}$, $\left\{\frac{\partial\psi_m}{\partial t}\right\}$ $\left\{\frac{\partial^2\psi_m}{\partial x^2}\right\}$ converge weakly to ψ, $\frac{\partial\psi}{\partial x}$, $\frac{\partial\psi}{\partial t}$, $\frac{\partial^2\psi}{\partial x^2}$ in $L_2(0,l)$ respectively, for any $t \in [0,T]$, $m \to \infty$. Now, let us show that the limit function $\psi(x,t)$ is a solution of the boundary value problems (2)–(4). Since the functions $\psi_m(x,t)$, $m = 1, 2, \ldots$ are solutions of the problems (2)–(4) for any $t \in [0,T]$, $\forall g \in L_2(0,l)$ in B_0, we can write the following integral identity:

$$\int_0^l \left(i\frac{\partial\psi_m}{\partial t} + a_0\frac{\partial^2\psi_m}{\partial x^2} - a(x)\psi_m - v^m(x)\psi_m + a_1|\psi_m|^2\psi_m - f(x,t) \right)\bar{g}(x)\mathrm{d}x = 0. \tag{20}$$

According to the embedding theorem [2], the space B_0 is compactly embedded into the space $C^0([0,T], L_\infty(0,l))$ which implies that

$$\| \psi_m(.,t) - \psi(.,t) \|_{L_\infty(0,l)} \longrightarrow 0, \quad \forall t \in [0,T], \ m \to \infty. \tag{21}$$

Hence, from (21) and convergence $\{v^m\}$ we write

$$\int_0^l v^m(x)\psi_m(x,t)\bar{g}(x)\mathrm{d}x \longrightarrow \int_0^l v(x)\psi(x,t)q(x)\mathrm{d}x, \quad m \to \infty. \tag{22}$$

On the other hand, using the estimate (18) and inequality (13) we get the following inequality:

$$\| \, |\psi_m(.,t)|^2\psi_m(.,t) \, \|_{L_2(0,l)} \le c_7, \quad m = 1, 2, 3, \cdots . \tag{23}$$

According to the limit relation (21), the sequence $\psi_m(x,t)$ converges to the function $\psi(x,t)$ $\forall t \in [0,T]$. Therefore, using the lemma in pp. 530–531 of [18], we get

$$\lim_{m\to\infty} \int_0^l a_1|\psi_m(x,t)|^2\psi_m(x,t)\mathrm{d}x = \int_0^l a_1|\psi(x,t)|^2\psi(x,t)\mathrm{d}x, \quad \forall t \in [0,T]. \tag{24}$$

Hence, considering the limit relations (22), (24) and taking the limit in (20) for $m \to \infty$, we get

$$\int_0^l \left(i\frac{\partial \psi}{\partial t} + a_0 \frac{\partial^2 \psi}{\partial x^2} - a(x)\psi - v(x)\psi + a_1|\psi|^2\psi - f(x,t) \right) \bar{g}(x)\mathrm{d}x = 0 \tag{25}$$

$\forall t \in [0, T]$ and $g \in L_2(0, l)$. It follows from (25), that $\psi(x, t)$ is a solution of equation (2) $\overset{\circ}{\forall} x \in (0, l)$. Now let us prove that the limit function satisfies the the initial condition (3) and boundary conditions (4). According to the embedding theorem [2], since the space B_0 is compactly embedded into the space $C^0([0, T], L_2(0, l))$ [19], we can write the following limit relations:

$$\psi_m(x,t) \overset{\text{strongly}}{\to} \psi(x,t), \quad \text{in } L_2(0,l) \text{ for } \forall t \in [0,T], \quad m \to \infty, \tag{26}$$

$$\psi_m(x,0) \overset{\text{strongly}}{\to} \psi(x,0), \quad \text{in } L_2(0,l) \text{ for } t = 0, \quad m \to \infty. \tag{27}$$

Hence, considering the above limit relations, the initial condition $\psi_m(x, 0) = \varphi(x)$ $x \in (0, l)$ and the following inequality

$$\int_0^l |\psi(x,0) - \varphi(x)|^2 \mathrm{d}x \le 2\int_0^l |\psi(x,0) - \psi_m(x,0)|^2 \mathrm{d}x + 2\int_0^l |\psi_m(x,0) - \varphi(x)|^2 \mathrm{d}x,$$

we obtain that the function $\psi(x, t)$ satisfies the initial condition (3) $\overset{\circ}{\forall} x \in (0, l)$. Also, considering the space B_0 compactly imbedding into $C^0([0, l], L_2(0, l))$ [19] for any $\overset{\circ}{\forall} t \in (0, T)$, $\psi_m(0, t) = \psi_m(l, t)$ and the inequality

$$\int_0^T |\psi(p,t)|^2 \mathrm{d}x \le 2\int_0^T |\psi(p,t) - \psi_m(p,t)|^2 \mathrm{d}x + 2\int_0^l |\psi_m(p,t)|^2 \mathrm{d}x, \ \ p = 0, l$$

we obtain that the function $\psi(x, t)$ satisfies the boundary conditions (4) $\overset{\circ}{\forall} t \in (0, T)$. So, we show that the functions $\psi = \psi(x, t)$ is a solution of the boundary value problems (2)–(4) corresponding to limit function of the sequence $\{v^m\} \subset V$ for any $v \in V$, i.e. $\psi = \psi(x, t) = \psi(x, t; v)$, where $\psi = \psi(x, t)$ is the limit function of the sequence $\{\psi_m(x, t)\}$. Since, the solution of the problems (2)–(4) is unique we write $\psi \in B_0$. Since the space B_0 is compactly imbedded into the space $L_2(0, l)$, the limit relation $\psi_m \longrightarrow \psi$ is valid in $L_2(0, l)$. Taking into consideration that the norm functions of the spaces

$L_2(0,l)$ are lower semi-continuous functionals and $\alpha \geq 0$, we obtain that the functional $J_\alpha(v)$ is lower semicontinuous. Therefore, the relation

$$J_{\alpha^*} \leq J_\alpha(v) \leq \underline{\lim}_{m\longrightarrow\infty} J_\alpha(v^m) = J_{\alpha^*}$$

is valid. Hence, it is clear that $v \in V$ is a solution of the variational problems (1)–(4). Theorem 3 is proved. □

4. Differentiability of the functional

In this section, we will prove differentiability of the cost functional. Let us consider the following boundary value problem which is called conjugate problem of problems (2)–(4):

$$i\frac{\partial \eta}{\partial t} + a_0\frac{\partial^2 \eta}{\partial x^2} - a(x)\eta - v(x)\eta + 2\bar{a}_1|\psi|^2\eta + a_1\psi^2\bar{\eta} = 0, \quad (x,t) \in \Omega \quad (28)$$

$$\eta(x,T) = -2i(\psi(x,T) - y(x)), \quad x \in (0,l) \quad (29)$$

$$\eta(0,t) = \eta(l,t) = 0, \quad t \in [0,T] \quad (30)$$

where the function $\psi = \psi(x,t) \equiv \psi(x,t;v)$ is the solution of the initial boundary value problems (2)–(4) for any $v \in V$. In this case, by a solution of this boundary value problem we will understand the function $\eta = \eta(x,t) \equiv \eta(x,t;v)$ from B_0 which satisfies the conditions (28)–(30) $\forall t \in [0,T]$ and $\overset{\circ}{\forall} x \in (0,l)$.

Theorem 4. *Assume that the hypotheses of Theorem* 1 *hold and let* $y \in \overset{\circ}{W}{}_2^2(0,l)$ *be given function. Then, the boundary value problem* (28)–(30) *has a unique solution and the estimate*

$$\|\eta(.,t)\|^2_{\overset{\circ}{W}{}_2^2(0,l)} + \left\|\frac{\partial \eta}{\partial t}\right\|^2_{L_2(0,l)} \leq c_8(\|\varphi\|^2_{\overset{\circ}{W}{}_2^2(0,l)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|\varphi\|^6_{\overset{\circ}{W}{}_2^1(0,l)} + \|y\|^2_{\overset{\circ}{W}{}_2^2(0,l)}), \ \forall t \in [0,T] \quad (31)$$

is valid for this solution, where $c_8 > 0$ *is a constant that does not depend on* t.

Using Galerkin's method we can easily prove Theorem 4 as the proof of Theorem 1. Let us define the function

$$H(x,\psi(x,.),v(x),\bar{\eta}(x,.)) = {}_0^T \mathrm{Re}(\psi(x,t)\bar{\eta}(x,t))dt \times v(x) - \alpha\,(v(x) - w(x))^2.$$

This function is called Hamilton–Pontryagin's function for the optimal control problems (1)–(4).

Let us evaluate the increment of the functional $J_\alpha(v)$ $\forall v \in V$. Let the function $\Delta v \in L_2(0,l)$ be an increment given to any $v \in V$ such that $v + \Delta v \in V$. Then, using (1) and (16) we can write the increment of the functional $J_\alpha(v)$ for any $v \in V$ as the following:

$$\begin{aligned}\Delta J_\alpha(v) &= J_\alpha(v+\Delta v) - J_\alpha(v) = 2\int_0^l \operatorname{Re}[(\psi(x,T) - y(x)]\Delta\bar{\psi}(x,T)\mathrm{d}x \\ &\quad + 2\alpha\int_0^l (v(x) - w(x))\Delta v(x)\mathrm{d}x + \|\Delta\psi(.,T)\|^2_{L_2(0,l)} \\ &\quad + \alpha \parallel \Delta v \parallel^2_{L_2(0,l)},\end{aligned} \tag{32}$$

where $\Delta\psi = \Delta\psi(x,t) \equiv \psi(x,t;v+\Delta v) - \psi(x,t;v)$ is a solution of the problems (10)–(11) for $v \in V$.

Theorem 5. *Assume that the hypotheses of Theorem* 4 *hold and let $w \in L_2(0,l)$ be a given function. Then, the functional $J_\alpha(v)$ is Frechet-differentiable on the set V and the following formula is valid for its gradient*:

$$J'_\alpha(v) = -\frac{\partial H}{\partial v} = -\int_0^T \operatorname{Re}(\psi(x,t)\bar{\eta}(x,t)\mathrm{d}t + 2\alpha(v(x) - w(x)), \tag{33}$$

where $\psi = \psi(x,t)$ is the solution of the initial boundary value problems (2)–(4) *and $\eta = \eta(x,t)$ is the solution of the conjugate problem.*

Proof. Firstly, let us show that the following equality is valid for the first term of the right-hand side of (32):

$$\begin{aligned}2\int_0^l \operatorname{Re}[(\psi(x,T) - y(x)]\Delta\bar{\psi}(x,T)\mathrm{d}x &= -\int_\Omega \operatorname{Re}(\psi(x,t)\bar{\eta}(x,t))\Delta v(x)\mathrm{d}x\mathrm{d}t \\ &\quad + \tilde{R}(\Delta v),\end{aligned} \tag{34}$$

where

$$\begin{aligned}\tilde{R}(\Delta v) = &-\int_\Omega \operatorname{Re}(\Delta\psi\bar{\eta})\Delta v(x)\mathrm{d}x\mathrm{d}t + \int_\Omega \operatorname{Re} a_1(|\psi_\Delta|^2 - |\psi|^2)\operatorname{Re}(\Delta\psi\bar{\eta})\mathrm{d}x\mathrm{d}t \\ &- \int_\Omega \operatorname{Re}(a_1|\Delta\psi|^2)\psi\bar{\eta}\mathrm{d}x\mathrm{d}t \\ &- \int_\Omega \operatorname{Im} a_1(|\psi_\Delta|^2 - |\psi|^2)\operatorname{Im}(\Delta\psi\bar{\eta})\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{35}$$

As the solution of the initial boundary value problems (2)–(4) belongs to B_0, the function $\Delta\psi = \Delta\psi(x,t) \equiv \psi(x,t;v+\Delta v) - \psi(x,t;v)$ that is a solution of the problems (10), (11) satisfies the following integral identity:

$$\int_\Omega \left[i\frac{\partial \Delta\psi}{\partial t} + a_0 \frac{\partial^2 \Delta\psi}{\partial x^2} - a(x)\Delta\psi - (v(x) + \Delta v(x))\Delta\psi \right.$$

$$\left. + a_1(|\psi_\Delta|^2 + |\psi|^2)\Delta\psi + a_1\psi_\Delta\psi\Delta\bar{\psi} \right] \bar{\phi}(x,t)\mathrm{d}x\mathrm{d}t$$

$$= {}_\Omega \Delta v(x)\psi(x,t)\bar{\phi}(x,t)\mathrm{d}x\mathrm{d}t \tag{36}$$

and the conditions

$$\Delta\psi(x,0) = 0, \quad x \in (0,l), \quad \Delta\psi(0,t) = \Delta\psi(l,t) = 0, \ \forall t \in [0,T], \tag{37}$$

$\forall \phi \in L_2(\Omega)$. Also, the function $\eta(x,t)$ that is a solution of the conjugate problem satisfies the integral identity

$$\int_\Omega \left(i\frac{\partial \eta}{\partial t} + a_0 \frac{\partial^2 \eta}{\partial x^2} - a(x)\eta - v(x)\eta + 2\overline{a_1}|\psi|^2\eta + a_1\psi^2\bar{\eta} \right) \overline{\phi_1}(x,t)\mathrm{d}x\mathrm{d}t = 0$$

$\forall \phi_1 = \phi_1(x,t) \in L_2(\Omega)$ and the conditions (29) and (30). Let us take the function $\Delta\psi \in B_0$ instead of the function $\phi_1 = \phi_1(x,t)$ in this integral identity. Using the formula of partial integration, the initial and boundary conditions (29), (30), (37), we obtain the equality

$$\int_\Omega \left[\left(-i\frac{\partial \Delta\bar{\psi}}{\partial t} + a_0 \frac{\partial^2 \Delta\bar{\psi}}{\partial x^2} - a(x)\Delta\bar{\psi} - v(x)\Delta\bar{\psi} \right. \right.$$

$$\left. \left. + 2\overline{a_1}|\psi|^2\Delta\bar{\psi} \right) \eta + a_1\psi^2\Delta\bar{\psi}\bar{\eta} \right] \mathrm{d}x\mathrm{d}t$$

$$= -{}_0^l 2\,(\psi(x,T) - y(x))\,\Delta\bar{\psi}(x,T)\mathrm{d}x.$$

The complex conjugate of this equality is the equality

$$\int_\Omega \left[\left(i\frac{\partial \Delta\psi}{\partial t} + a_0 \frac{\partial^2 \Delta\psi}{\partial x^2} - a(x)\Delta\psi - v(x)\Delta\psi + 2a_1|\psi|^2\Delta\psi \right) \bar{\eta} \right.$$

$$\left. + \overline{a_1}\bar{\psi}^2\Delta\psi\eta \right] \mathrm{d}x\mathrm{d}t$$

$$= -{}_0^l 2\left(\bar{\psi}(x,T) - \bar{y}(x)\right)\Delta\psi(x,T)\mathrm{d}x. \tag{38}$$

Also, let us make the function $\eta(x,t)$ instead of the function $\phi = \phi(x,t)$ in the integral identity (36). Then, the following equality is obtained:

$$\int_{\Omega} \Big[i\frac{\partial \Delta\psi}{\partial t} + a_0 \frac{\partial^2 \Delta\psi}{\partial x^2} - a(x)\Delta\psi - (v(x) + \Delta v(x))\Delta\psi$$

$$+ a_1(|\psi_\Delta|^2 + |\psi|^2)\Delta\psi + a_1\psi_\Delta\psi\Delta\bar{\psi} \Big] \bar{\eta}(x,t)\mathrm{d}x\mathrm{d}t$$

$$= {}_{\Omega}\Delta v(x)\psi(x,t)\bar{\eta}(x,t)\mathrm{d}x\mathrm{d}t.$$

Subtracting the equality (38) from this equality, we obtain the equality

$$\begin{aligned} &2{}_0^l(\bar{\psi}(x,T) - \bar{y}(x))\Delta\psi(x,T)\mathrm{d}x \\ &= -{}_{\Omega}\psi(x,t)\bar{\eta}(x,t)\Delta v(x)\mathrm{d}x\mathrm{d}t - {}_{\Omega}\Delta\psi(x,t)\bar{\eta}(x,t)\Delta v(x)\mathrm{d}x\mathrm{d}t \\ &\quad + {}_{\Omega}a_1(|\psi_\Delta(x,t)|^2 - |\psi(x,t)|^2)\Delta\psi(x,t)\bar{\eta}(x,t)\mathrm{d}x\mathrm{d}t \\ &\quad + {}_{\Omega}[a_1\psi_\Delta(x,t)\psi(x,t)\Delta\bar{\psi}(x,t)\bar{\eta}(x,t) \\ &\quad - \overline{a_1}(\bar{\psi}(x,t))^2\Delta\psi(x,t)\eta(x,t)]\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Summing this equality with its complex conjugate, we get the equality (34).

If we use the equality (34) in (33), we get the formula

$$\Delta J_\alpha(v) = -\int_{\Omega} \mathrm{Re}(\psi\bar{\eta})\Delta v(x)\mathrm{d}x\mathrm{d}t + 2\alpha\int_0^l (v(x) - w(x))\Delta v(x)\mathrm{d}x + R(\Delta v),$$

where $R(\Delta v) = \tilde{R}(\Delta v) + \|\Delta\psi(.,T)\|^2_{L_2(0,l)} + \alpha \parallel \Delta v \parallel^2_{L_2(0,l)}$.

Now, let us show that $R(\Delta v) = o(\parallel \Delta v \parallel_{L_2(0,l)})$. For this purpose, let us evaluate each of the three terms in the formula $R(\Delta v)$. From (12), for $t = T$, it is written as

$$\|\Delta\psi(.,T)\|_{L_2(0,l)} \le c_9 \, \|\Delta v\|_{L_2(0,l)}. \tag{39}$$

From (35), we can easily write the inequality

$$\begin{aligned} |\tilde{R}| &\le {}_{\Omega}\,|\Delta\psi|\,|\eta|\,|\Delta v|\,\mathrm{d}x\mathrm{d}t + 3\,|a_1|\,{}_{\Omega}\,(|\psi_\Delta| + |\psi|)\,|\Delta\psi|^2\,|\eta|\,\mathrm{d}x\mathrm{d}t \\ &\le \sqrt{T}\,\|\Delta\psi\|_{L_2(\Omega)} \max_{0\le t\le T} \|\eta\|_{L_\infty(0,l)}\,\|\Delta v\|_{L_2(0,l)} \\ &\quad + 3\,|a_1|\,\|\Delta\psi\|^2_{L_2(\Omega)}\,(\|\psi_\Delta\|_{L_\infty(\Omega)} + \|\psi\|_{L_\infty(\Omega)})\,\|\eta\|_{L_\infty(\Omega)}. \end{aligned}$$

In this inequality, using the inequalities

$$\|\eta\|_{L_\infty(\Omega)} \le c_{10}\,\|\eta\|_{C^0([0,T],\overset{\circ}{W}{}_2^2(0,l))}, \tag{40}$$

$$\|\psi_\Delta\|_{L_\infty(\Omega)} \le c_{11}\,\|\psi_\Delta\|_{C^0([0,T],\overset{\circ}{W}{}_2^2(0,l))}, \tag{41}$$

$$\|\psi\|_{L_\infty(\Omega)} \le c_{12} \|\psi\|_{C^0([0,T],\mathring{W}_2^2(0,l))}, \tag{42}$$

the estimate (8) for ψ_Δ and ψ, the estimate (31) for η, we easily obtain the inequality

$$|\tilde{R}| \le c_{13}(\|\Delta v\|_{L_2(0,l)} \|\Delta\psi\|_{L_2(\Omega)} + \|\Delta\psi\|^2_{L_2(\Omega)}).$$

If we apply the estimate (12) for $\Delta\psi$ to this inequality, the estimate

$$|\tilde{R}| \le c_{14} \|\Delta v\|^2_{L_2(0,l)}$$

is obtained, where $c_{14} > 0$ is independent of Δv. Then, from this estimate and (39), we get the estimate

$$|R(\Delta v)| \le c_{15} \|\Delta v\|^2_{L_2(0,l)},$$

where $c_{15} > 0$ is independent of Δv. Thus, it is obtained that the relation

$$R(\Delta v) = o(\|\Delta v\|_{L_2(0,l)})$$

is valid. Using this relation in (32), we can easily write

$$\Delta J_\alpha(v) = -\int_\Omega \mathrm{Re}(\psi\bar{\eta})\Delta v(x)\mathrm{d}x\mathrm{d}t + 2\alpha\int_0^l (v(x) - w(x))\Delta v(x)\mathrm{d}x + o(\|\Delta v\|_{L_2(0,l)}).$$

Considering the Hamilton–Pontryagin's function, we can write the increment of the functional $J_\alpha(v)$,

$$\Delta J_\alpha(v) = \int_0^l \left(\frac{-\partial H(x,\psi(x,t),v(x),\bar{\eta}(x,t))}{\partial v}\right)\Delta v(x)\mathrm{d}x + o(\|\Delta v\|_{L_2(0,l)}). \tag{43}$$

From the definition of Frechet-derivative of the functional and (43), we can say that the functional $J_\alpha(v)$ is Frechet-differentiable on the set V and is valid for formula (33) and its gradient.

Now, we focus on the necessary optimality conditions of first order.

Theorem 6. *Suppose that the hypotheses of Theorem* 4 *hold and let* $v^* = v^*(x) \in V$ *be any solution of the optimal control problems* (1)–(4). *In this case*, $\forall v \in V$ *the inequality*

$$\int_0^l \left[\int_0^T \mathrm{Re}(\psi^*(x,t)\bar{\eta}^*(x,t))\mathrm{d}t - 2\alpha(v^*(x) - w(x))\right](v(x) - v^*(x))\mathrm{d}x \le 0$$

is valid, where the functions $\psi^*(x,t) \equiv \psi(x,t;v^*)$ *and* $\eta^*(x,t) \equiv \eta(x,t;v^*)$ *are solutions of the boundary value problems* (2)–(4) *and conjugate problems* (28), (30) *for* $v^* \in V$, *respectively.*

Proof. Firstly, let us show that the gradient J'_α is continuous on the set V, i.e. $\|\Delta v\|_{L_2(0,l)} \to 0$ implies that $\|J'_\alpha(v+\Delta v) - J'_\alpha(v)\|_{L_2(0,l)} \to 0\ \forall v \in V$. Using the formula (33), we can write

$$J'_\alpha(v+\Delta v) - J'_\alpha(v) = -\int_0^T [\mathrm{Re}(\psi_\Delta(x,t)\Delta\bar{\eta}) + \mathrm{Re}(\Delta\psi(x,t)\bar{\eta}]\,\mathrm{d}t + 2\alpha\Delta v(x), \tag{44}$$

where $\Delta\psi(x,t)$ is the solution of the boundary value problems (10), (11) and $\Delta\eta = \Delta\eta(x,t) = \eta(x,t;v+\Delta v) - \eta(x,t;v)$ is the solution of following boundary value problems:

$$i\frac{\partial\Delta\eta}{\partial t} + a_0\frac{\partial^2\Delta\eta}{\partial x^2} - a(x)\Delta\eta - (v+\Delta v)\Delta\eta + 2\bar{a}_1|\psi_\Delta|^2\Delta\eta + a_1\psi_\Delta^2\Delta\bar{\eta}$$

$$= -2\bar{a}_1\bar{\psi}\eta\Delta\psi - 2\bar{a}_1\psi_\Delta\eta\Delta\bar{\psi} - a_1\psi\bar{\eta}\Delta\psi - a_1\psi_\Delta\bar{\eta}\Delta\psi + \Delta v\eta,\ (x,t) \in \Omega \tag{45}$$

$$\Delta\eta(x,T) = -2i\Delta\psi(x,T),\ \ x \in (0,l), \tag{46}$$

$$\Delta\eta(0,t) = \Delta\psi(l,t) = 0,\quad t \in [0,T], \tag{47}$$

where the functions $\psi(x,t)$ and $\psi_\Delta(x,t)$ are the solutions of the problems (2)–(4) corresponding to v and $v+\Delta v$, respectively. The functions $\eta(x,t)$ and $\eta_\Delta(x,t)$ are the solutions of the conjugate problems (28), (30) corresponding to v and $v+\Delta v$, respectively.

Now, let us obtain an estimation for $\Delta\eta(x,t)$. For this purpose, multiplying both sides of equation (45) with $\Delta\bar{\eta}(x,t)$, we integrate over $\tilde{\Omega}_t = (0,l)\times(t,T)$. Then, we get the equality

$${}_{\tilde{\Omega}_t}\Bigg[i\frac{\partial\Delta\eta}{\partial t}\Delta\bar{\eta} - a_0\left|\frac{\partial\Delta\eta}{\partial x}\right|^2 - a(x)\,|\Delta\eta|^2$$

$$- (v+\Delta v)\,|\Delta\eta|^2 + 2\bar{a}_1|\psi_\Delta|^2\,|\Delta\eta|^2 + a_1\psi_\Delta^2\,(\Delta\bar{\eta})^2\Bigg]\mathrm{d}x\mathrm{d}\tau$$

$$= {}_{\tilde{\Omega}_t}\Delta v\eta\Delta\bar{\eta}\mathrm{d}x\mathrm{d}\tau - {}_{\tilde{\Omega}_t}[2\bar{a}_1\bar{\psi}\eta\Delta\psi\Delta\bar{\eta}$$

$$+ 2\bar{a}_1\psi_\Delta\eta\Delta\bar{\psi}\Delta\bar{\eta} - a_1\psi\bar{\eta}\Delta\psi\Delta\bar{\eta} - a_1\psi_\Delta\bar{\eta}\Delta\psi\Delta\bar{\eta}]\mathrm{d}x\mathrm{d}\tau.$$

If we subtract its complex conjugate from this equality, we obtain the equality

$$
\begin{aligned}
&{}_{\tilde{\Omega}_t} i\frac{\partial}{\partial t}\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau - 4i\,\mathrm{Im}\,a_1{}_{\tilde{\Omega}_t}\left|\psi_\Delta\right|^2\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&= -{}_{\tilde{\Omega}_t}[a_1\psi_\Delta^2\,(\Delta\bar{\eta})^2 - \bar{a}_1\bar{\psi}_\Delta^2\,(\Delta\eta)^2]\mathrm{d}x\mathrm{d}\tau \\
&\quad -{}_{\tilde{\Omega}_t}[2\bar{a}_1\bar{\psi}\eta\Delta\psi\Delta\bar{\eta} - 2a_1\psi\bar{\eta}\Delta\bar{\psi}\Delta\eta]\mathrm{d}x\mathrm{d}\tau \\
&\quad -{}_{\tilde{\Omega}_t}[2\bar{a}_1\psi_\Delta\eta\Delta\bar{\psi}\Delta\bar{\eta} - 2a_1\bar{\psi}_\Delta\bar{\eta}\Delta\psi\Delta\eta]\mathrm{d}x\mathrm{d}\tau \\
&\quad +{}_{\tilde{\Omega}_t}[a_1\psi\bar{\eta}\Delta\psi\Delta\bar{\eta} - \bar{a}_1\bar{\psi}\eta\Delta\bar{\psi}\Delta\eta]\mathrm{d}x\mathrm{d}\tau \\
&\quad +{}_{\tilde{\Omega}_t}[a_1\psi_\Delta\bar{\eta}\Delta\psi\Delta\bar{\eta} - \bar{a}_1\bar{\psi}_\Delta\eta\Delta\bar{\psi}\Delta\eta]\mathrm{d}x\mathrm{d}\tau \\
&\quad +{}_{\tilde{\Omega}_t}[\Delta v\eta\Delta\bar{\eta} - \Delta v\bar{\eta}\Delta\eta]\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
$$

Considering the condition (46) and the inequality $2\left|a_1\right| \le 2\left(\left|\mathrm{Im}\,a_1\right| + \left|\mathrm{Re}\,a_1\right|\right) \le 3\mathrm{Im}\,a_1$, we get the inequality

$$
\begin{aligned}
&\|\Delta\eta(.,t)\|^2_{L_2(0,l)} + \mathrm{Im}\,a_1{}_{\tilde{\Omega}_t}\left|\psi_\Delta\right|^2\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau \le 4\,\|\,\Delta\psi(.,T)\,\|^2_{L_2(0,l)} \\
&\quad + 6\left|a_1\right|{}_{\tilde{\Omega}_t}\left|\psi\right|\left|\eta\right|\left|\Delta\psi\right|\left|\Delta\eta\right|\mathrm{d}x\mathrm{d}\tau + 6\left|a_1\right|{}_{\tilde{\Omega}_t}\left|\psi_\Delta\right|\left|\eta\right|\left|\Delta\psi\right|\left|\Delta\eta\right|\mathrm{d}x\mathrm{d}\tau \\
&\quad + 2{}_{\tilde{\Omega}_t}\left|\Delta v\right|\left|\eta\right|\left|\Delta\eta\right|\mathrm{d}x\mathrm{d}\tau,\ \forall t \in [0,T].
\end{aligned}
$$

If we apply the inequality $\epsilon-$ Cauchy to the right-hand side of this inequality and choose $\epsilon = \frac{\mathrm{Im}\,a_1}{6|a_1|}$ we get the inequality

$$
\begin{aligned}
&\|\Delta\eta(.,t)\|^2_{L_2(0,l)} + \frac{\mathrm{Im}\,a_1}{2}{}_{\tilde{\Omega}_t}\left|\psi_\Delta\right|^2\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\le 4\|\Delta\psi(.,T)\|^2_{L_2(0,l)} + \left(3a_1 + \frac{18\left|a_1\right|^2}{\mathrm{Im}\,a_1}\right){}_{\tilde{\Omega}_t}\left|\eta\right|^2\left|\Delta\psi\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad +3\left|a_1\right|{}_{\tilde{\Omega}_t}\left|\psi\right|^2\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau + {}_{\tilde{\Omega}_t}\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau + {}_{\tilde{\Omega}_t}\left|\Delta v\right|^2\left|\eta\right|^2 \mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$\forall t \in [0,T]$. In this inequality, using inequalities (40), (42), the estimates (8), (15), (31) and Gronwall's lemma, we obtain the inequality

$$
\|\Delta\eta(.,t)\|^2_{L_2(0,l)} + \frac{\mathrm{Im}\,a_1}{2}{}_{\tilde{\Omega}_t}\left|\psi_\Delta\right|^2\left|\Delta\eta\right|^2 \mathrm{d}x\mathrm{d}\tau \le c_{16}\,\|\,\Delta v\,\|^2_{L_2(0,l)},\ \forall t \in [0,T],
$$

where the constant $c_{16} > 0$ is independent from Δv and t. Considering the formula (44) we get

$$
\begin{aligned}
\|J'_\alpha(v+\Delta v) - J'_\alpha(v)\|_{L_2(0,l)} &\le \left\| \int_0^T \left|\psi_\Delta\right| \cdot \left|\Delta\eta\right| \mathrm{d}t \right\|_{L_2(0,l)} \\
&\quad + \left\| \int_0^T \left|\eta\right| \cdot \left|\Delta\psi\right| \mathrm{d}t \right\|_{L_2(0,l)} + 2\alpha\,\|\Delta v\|_{L_2(0,l)}\,. \quad (48)
\end{aligned}
$$

Finally, if we use the inequalities

$$\left\| \int_0^T |\psi_\Delta(.,t)| \cdot |\Delta\eta(.,t)| \mathrm{d}t \right\|_{L_2(0,l)} \leq c_{17} \| \Delta v \|_{L_2(0,l)}$$

and

$$\left\| \int_0^T |\eta(.,t)| \cdot |\Delta\psi(.,t)| \mathrm{d}t \right\|_{L_2(0,l)} \leq c_{18} \| \Delta v \|_{L_2(0,l)}$$

in (48) we write $\| J'_\alpha(v + \Delta v) - J'_\alpha(v) \|_{L_2(0,l)} \leq c_{19} \| \Delta v \|_{L_2(0,l)}$, where the constant $c_{19} > 0$ is independent of Δv. This implies that $\| J'_\alpha(v + \Delta v) - J'_\alpha(v) \|_{L_2(0,l)} \to 0$ as $\|\Delta v\|_{L_2(0,l)} \to 0$, i.e., the functional J_α is continuous differentiable on the set V. From the fact that V is a convex set in $L_2(0,l)$ and J_α is Frechet-differentiable, then we know that hypotheses of the theorem in [21] holds, that is, if $J_\alpha(v)$ has a minimum value at $v^* \in V$ for every $v \in V$, then $(J'_\alpha(v^*), v - v^*)_{L_2(0,l)} \geq 0$. Thus, using the inequality for $v = v^*$ in (33) we complete the proof. □

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