Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 3, August 2012, pp. 437–458.

The Urbanik generalized convolutions in the non-commutative probability and a forgotten method of constructing generalized convolution

BARBARA JASIULIS-GOŁDYN[1] and ANNA KULA[2]

[1]Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
[2]Institute of Mathematics, Jagiellonian University, ul. S. Łojasiewicza 6, 30-348 Kraków, Poland
E-mail: jasiulis@math.uni.wroc.pl; anna.kula@im.uj.edu.pl

MS received 8 March 2011; revised 17 November 2011

Abstract. The paper deals with the notions of weak stability and weak generalized convolution with respect to a generalized convolution, introduced by Kucharczak and Urbanik. We study properties of such objects and give examples of weakly stable measures with respect to the Kendall convolution. Moreover, we show that in the context of non-commutative probability, two operations: the q-convolution and the $(q, 1)$-convolution satisfy the Urbanik's conditions for a generalized convolution, interpreted on the set of moment sequences. The weak stability reveals the relation between two operations.

Keywords. Generalized convolution; Kendall convolution; non-commutative probability; q-deformed convolution; weakly stable measure.

1. Introduction

In classical probability the convolution is an associative and commutative operation which preserves probability measures on the real line $\mathbb{R}$. An alternative definition associates it to the sum of independent random variables – the convolution is precisely the distribution of the sum of such random variables. These two aspects give rise to two different directions of the generalization of this notion.

If we focus on the algebraic properties of the operation – associativity, commutativity, linearity (with respect to convex combinations), dilation-invariance and weak-continuity – we come to the definition of a generalized convolution, due to Urbanik, cf. [17–19]. A number of examples of such an object are known in the classical probability theory: the α-convolution, the ∞-convolution, the Kendall convolution, etc. To avoid troubles, one should however restrict the set of probability measures to those supported on $[0, +\infty)$.

If the probabilistic aspects are considered, a variety of convolutions studied in the non-commutative probablity becomes a natural generalization of the classical one. Among the most influential, we have the free convolution [24], the boolean convolution [16] and the monotone convolution [15]. They share with the classical convolution the property of expressing the addition of random variables which are 'independent', but the independence is no longer the classical one, and should be replaced by one of its non-commutative analogue. These convolutions have been intensively studied during the last few decades. However, they do not fit to the scheme of Urbanik, since none of them satisfies all the conditions of the generalized convolution.

Other examples of convolutions in the non-commutative probability are the q-*convolution*, introduced and studied by Carnovale and Koornwinder in [3], and the $(q, 1)$-*convolution*, defined by Ricard and Kula in [13]. They are related to the q-commutativity relation ($ab = qba$ with a fixed parameter $q > 0$) and the so-called braided algebras (cf. [1,6,13] for details, see for example [4] for the definition of a braided algebra and a convolution on it). Both convolutions can be well defined for measures with all moments finite and it was shown in [13] that if $q \in (0, 1)$, then they preserve moment sequences coresponding to measures on $[0, +\infty)$.

This paper is motivated by the observation that both, the q-convolution and the $(q, 1)$-convolution, satisfy the Urbanik conditions of generalized convolutions, once the definition is appropriately interpreted for an operation on the set on moment sequences. Moreover, it turns out that the relation between these two operations can be described in analogy to the notion of weakly stable measures and weakly generalized convolution. At some point we were wondering if the fact that one operation is a generalized convolution implies the same for the weakly generalized one. We found out that this problem has already been studied by Kucharczak and Urbanik in [9], but the paper seems to be less known to specialists. They showed that each weakly stable measure belonging to some domain of attraction induces the weak generalized convolution, which is again a generalized convolution. Their results fit to our observations made on the basis of the q- and $(q, 1)$-convolutions.

Our paper is divided into two sections which reflects two main parts of the article. Section 2 is devoted to the notion of generalized convolution, considered as operations on the set of probability measures on $[0, +\infty)$. We focus on the notion of the weak stability with respect to a generalized convolution and recall the results of Kucharczak and Urbanik with more detailed proofs. We complete these results with a new one concerning the regularity of the weak generalized convolution (Theorem 2.4). Next, the method of constructing new generalized convolutions is recalled and applied to the Kendall convolution (§ 2.3).

In § 3 we are looking for examples of generalized convolution in non-commutative probability: we start by explaining why the free, boolean and monotone convolutions are not the Urbanik convolutions. Next, we present the q-convolution and the $(q, 1)$-convolution (from that point we have to restrict our considerations to the set of moment sequences only) and give the interpretation of the Urbanik's definition on the set of moment sequences. Then we show that both convolutions in question satisfy it and we reveal the relations between these two objects with the aid of the notion of weak stability, analogous to the one studied in § 2. A number of examples is presented is § 3.5. We end up with some open questions about the regularity of the q-convolution and the $(q, 1)$-convolution.

Notation. Let $\mathcal{P}^+$ be the set of all probability (Borel) measures on the non-negative real half-line $[0, +\infty)$. For any $\mu \in \mathcal{P}^+$ and $a > 0$, we denote by T_a a *rescaling operator* (called also *dilation*) $\mu \mapsto T_a\mu$ defined by the formula $(T_a\mu)(A) = \mu(A/a)$ when $a \neq 0$ and $A \subset [0, +\infty)$ is a Borel set, and $T_0\mu = \delta_0$. This means that $T_a\mu$ is the distribution of the random variable aX provided μ is the distribution of X.

For $\mu, \nu \in \mathcal{P}^+$, a *scale mixture* is a measure $\mu \circ \nu$ on $[0, +\infty)$ given by the formula

$$(\mu \circ \nu)(A) = \int_0^\infty (T_a \mu)(A)\, \nu(\mathrm{d}a).$$

This is the same as to say that $\mu \circ \nu$ is the distribution of the random variable XY if the random variables X and Y are independent and have the distributions μ and ν, respectively. That is why we also call $\mu \circ \nu$ a *multiplicative convolution* of μ and ν.

Finally, given a measure $\mu \in \mathcal{P}^+$, we denote by $(\mu_n)_n$ its moment sequence, that is a sequence of numbers

$$\mu_n = \int_{[0,+\infty)} t^n \mu(\mathrm{d}t), \quad n \in \mathbb{N}.$$

The symbol $\mathcal{P}^+_{fm}$ will state for the set of all measures from $\mathcal{P}^+$ which have all moments finite. In general, a sequence $(\mu_n)_n$ is called a *moment sequence* if there exists a (positive Borel) measure μ on $[0, +\infty)$ for which the sequence is a moment sequence. The set of all moment sequences corresponding to measures from $\mathcal{P}^+_{fm}$ will be denoted by $\mathcal{M}^+$.

Let us recall that the moment sequence corresponding to the multiplicative convolution of two measures is

$$(\mu \circ \nu)_n = \mu_n \nu_n.$$

We denote by $\mathbb{N}$ the set of non-negative integers. Then for every $q > 0$, we use the standard notation of the q-calculus (cf. [7,8])

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - q^k a), \quad n = 1, 2, \ldots$$

$$(a; q)_\infty = \lim_{n \to +\infty} (a; q)_n,$$

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad n \in \mathbb{N},$$

$$[0]_q! = 1, \quad [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \quad n = 1, 2, \ldots$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad n, k \in \mathbb{N}.$$

For $q = 1$ these values are identified with the corresponding limits when $q \nearrow 1$ and $q \searrow 1$, thus $[n]_1 = n$, $[n]_1! = n!$ and the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_1$ gives the classical binomial coefficient $\binom{n}{k}$.

2. Generalized convolutions and weak stability

2.1 *Definitions and preliminaries*

The definition of the generalized convolution on $\mathcal{P}^+$ was stated by Urbanik in [18] and studied intensively in a series of papers [18]–[22].

DEFINITION 1

An associative and commutative binary operation $\circledast$ on the set $\mathcal{P}^+$ is called a *generalized convolution on* $\mathcal{P}^+$ if it satisfies the following conditions:

(i) $\delta_0 \circledast \mu = \mu$ for all $\mu \in \mathcal{P}^+$ (δ_0 is the unit element),
(ii) $(c\mu_1 + (1-c)\mu_2) \circledast \nu = c(\mu_1 \circledast \nu) + (1-c)(\mu_2 \circledast \nu)$ whenever $\mu_1, \mu_2, \nu \in \mathcal{P}^+$ and $c \in (0, 1)$,
(iii) $T_a(\mu \circledast \nu) = (T_a\mu) \circledast (T_a\nu)$ for any $\mu, \nu \in \mathcal{P}^+$ and $a \geq 0$,
(iv) if $\mu^{(n)} \xrightarrow{w} \mu$ with $\mu^{(n)}, \mu \in \mathcal{P}^+$ ($n \in \mathbb{N}$), then $\mu^{(n)} \circledast \nu \xrightarrow{w} \mu \circledast \nu$ for all $\nu \in \mathcal{P}^+$, (‘$\xrightarrow{w}$’ denotes the weak convergence of probability measures),
(v) there exists a sequence $(c_n)_n$ of positive numbers such that the sequence $T_{c_n}\delta_1^{\circledast n}$ converges weakly to a measure different from δ_0.

The set $(\mathcal{P}^+, \circledast)$ is called a *generalized convolution algebra*.

Let $(\mathcal{P}^+, \circledast)$ be a generalized convolution algebra. A continuous mapping $h : \mathcal{P}^+ \to \mathbb{R}$ such that

- $h(c\lambda + (1-c)\nu) = ch(\lambda) + (1-c)h(\nu)$,
- $h(\lambda \circledast \nu) = h(\lambda)h(\nu)$

for all $\lambda, \nu \in \mathcal{P}^+$ and $c \in (0, 1)$ is called a *homomorphism of* $(\mathcal{P}^+, \circledast)$. For every probability measure $\lambda \in \mathcal{P}^+$ we have

$$h(\lambda) = \int_{\mathbb{R}_+} h(\delta_x)\lambda(\mathrm{d}x).$$

The homomorphism is non-trivial if it is non-constant. The generalized convolution algebras admitting non-trivial homomorphism, following the paper [18], are called *regular*. For abbreviation, we say that $\circledast$ is *regular* if algebra $(\mathcal{P}^+, \circledast)$ is regular.

It is worth pointing out that every generalized convolution $\circledast$ of two probability measures is uniquely determined by the generalized convolution of the Dirac measures $\rho_{x,y} := \delta_x \circledast \delta_y$, which is called a *probability kernel*. Indeed, for every $\lambda_1, \lambda_2 \in \mathcal{P}_+$ and every Borel set $A \subseteq [0, \infty)$ we have

$$\lambda_1 \circledast \lambda_2(A) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \rho_{x,y}(A)\lambda_1(\mathrm{d}x)\lambda_2(\mathrm{d}y).$$

In parallel to the classical context, we can consider stable measures with respect to a generalized convolution. Notice that there exist also generalized convolutions on the set of all probability measures on the Borel subsets of the real line (cf. [5]).

DEFINITION 2 (cf. [9])

Let $\circledast$ be a generalized convolution. A probability measure $\nu \in \mathcal{P}^+ \backslash \{\delta_0\}$ is *stable with respect to* $(\mathcal{P}^+, \circledast)$ or $\circledast$*-stable* if

$$\forall\, a, b \geqslant 0\ \exists c \geqslant 0 \quad T_a\nu \circledast T_b\nu = T_c\nu. \tag{1}$$

Let $\mathcal{S}(\circledast)$ denote the set of $\circledast$-stable distributions. By Theorem 4 in [18] for any $\circledast$-stable measure ν there exists $\alpha = \alpha(\circledast, \nu) > 0$ such that in (1) we have $c^\alpha = a^\alpha + b^\alpha$. We define the *characteristic exponent* of $\circledast$:

$$\varkappa(\circledast) = \sup\{\alpha > 0 : \exists \nu \in \mathcal{S}(\circledast) \text{ such that } \alpha = \alpha(\nu, \circledast)\}.$$

Notice that by Proposition 2.2 of [21], a measure ν is $\circledast$-stable if and only if there exists a norming sequence $(a_n)_n$ of positive numbers which tends to 0 and a measure $\lambda \in \mathcal{P}^+$ such that

$$T_{a_n} \lambda^{\circledast n} \to \nu.$$

The set of all measures $\lambda \in \mathcal{P}^+$ such that the above limit relation holds is called a *domain of attraction* of measure ν.

In the classical probability theory, if for a given measure μ and for any $\nu_1, \nu_2 \in \mathcal{P}^+$ there exists a measure $\lambda \in \mathcal{P}^+$ such that $(\mu \circ \nu_1) * (\mu \circ \nu_2) = \mu \circ \lambda$ (where $*$ is the convolution in the usual sense), then μ is called *weakly stable*. If moreover, the weakly stable measure μ is non-trivial (by which we mean that $\mu \neq \delta_0$), then a mapping $(\nu_1, \nu_2) \to \lambda$ establishes a new operation, called a *weak generalized convolution* $\otimes_\mu$.

When the classical convolution is replaced by the generalized convolution $\circledast$, we get the notion of weak $\circledast$-stability that has already been studied by Kucharczak and Urbanik in [9].

DEFINITION 3

Let $(\mathcal{P}^+, \circledast)$ be the generalized convolution algebra and let $\mu \in \mathcal{P}^+$. If for any $\nu_1, \nu_2 \in \mathcal{P}^+$ there exists a measure $\lambda \in \mathcal{P}^+$ such that

$$(\mu \circ \nu_1) \circledast (\mu \circ \nu_2) = \mu \circ \lambda,$$

then the measure μ is said to be *weakly stable with respect to* $\circledast$ or *weakly* $\circledast$*-stable*. If, moreover, the measure μ is non-trivial ($\mu \neq \delta_0$), then the operation

$$\otimes_{\mu,\circledast} : \mathcal{P}^+ \times \mathcal{P}^+ \ni (\nu_1, \nu_2) \to \lambda \in \mathcal{P}^+$$

is called a *weak generalized convolution with respect to* $\circledast$ *generated by the measure* μ. We denote this by $\lambda = \nu_1 \otimes_{\mu,\circledast} \nu_2$.

Remark 2.1. In particular, the weak generalized convolution is a weak generalized convolution with respect to the classical convolution $*$.

2.2 *Properties of weak generalized convolution*

In much the same way as in the classical case it can be shown that weakly stable measures with respect to generalized convolution can be cancelled in the following sense:

DEFINITION 4

We say that a weakly $\circledast$-stable measure μ is *cancellable* if for any $\nu_1, \nu_2 \in \mathcal{P}^+$ the equality $\mu \circ \nu_1 = \mu \circ \nu_2$ implies that $\nu_1 = \nu_2$.

Theorem 2.2 (cf. Proposition 1.1 of [9]). *Every nontrivial weakly stable measure μ with respect to a regular generalized convolution $\circledast$ is cancellable.*

It follows that for any generalized convolution $\circledast$ the weak generalized convolution $\otimes_{\mu,\circledast}$ is well defined, since $\nu_1 \otimes_{\mu,\circledast} \nu_2$ is uniquely determined for every $\nu_1, \nu_2 \in \mathcal{P}^+$.

If the operation $\otimes_{\mu,\circledast}$ is a weak generalized convolution with respect to $\circledast$, we can ask whether it inherits the properties (i)–(v) of the induced generalized convolution $\circledast$. The answer is positive for all but the last condition, which in general need not be satisfied. The precise answer to this question appeared in the paper by Kucharczak and Urbanik [9], which we recall below. Since the proof in that paper was only sketched, we present a more detailed proof for the reader's convenience.

Theorem 2.3 (cf. Proposition 1.2 of [9]). *Let μ be a non-trivial, weakly $\circledast$-stable measure in $\mathcal{P}^+$.*

(a) *The weak generalized convolution $\otimes_{\mu,\circledast}$ satisfies the conditions* (i)–(iv) *of the generalized convolution on $\mathcal{P}^+$.*
(b) *The weak generalized convolution $\otimes_{\mu,\circledast}$ is a generalized convolution if and only if there exist a sequence $(a_n)_n$ of positive numbers and a measure $\lambda \in \mathcal{P}^+$ such that $T_{a_n}\mu^{\circledast n} \to \lambda$.*

Proof. In order to prove the first part of the theorem we arrive at the following conditions:

(i) Let $\lambda \in \mathcal{P}^+$. Then $\mu \circ (\delta_0 \otimes_{\mu,\circledast} \lambda) = \delta_0 \circledast \mu \circ \lambda = \mu \circ \lambda$.
(ii) $\mu \circ ((c\lambda_1 + (1-c)\lambda_2) \otimes_{\mu,\circledast} \lambda) = \mu \circ ((c\lambda_1 + (1-c)\lambda_2) \circledast \mu \circ \lambda = \mu \circ (c(\lambda_1 \otimes_{\mu,\circledast} \lambda) + (1-c)(\lambda_2 \otimes_{\mu,\circledast} \lambda))$ whenever $\lambda_1, \lambda_2, \lambda \in \mathcal{P}^+$ and $c \in (0,1)$.
(iii) For any $\lambda_1, \lambda_2 \in \mathcal{P}^+$ and $a \geq 0$ we get $\mu \circ T_a(\lambda_1 \otimes_{\mu,\circledast} \lambda_2) = \mu \circ (T_a\lambda_1 \otimes_{\mu,\circledast} T_a\lambda_2)$.
(iv) If $\lambda^{(n)} \xrightarrow{w} \lambda$ with $\lambda^{(n)}, \lambda \in \mathcal{P}^+$ ($n \in \mathbb{N}$), then $\mu \circ (\lambda^{(n)} \otimes_{\mu,\circledast} \nu) = \mu \circ \lambda^{(n)} \circledast \mu \circ \nu \xrightarrow{w} \mu \circ \lambda \circledast \mu \circ \nu = \mu \circ (\lambda \otimes_{\mu,\circledast} \nu)$ for all $\nu \in \mathcal{P}^+$.

Now it is evident that, since a non-trivial measure μ is cancellable, hence $\otimes_{\mu,\circledast}$ satisfies conditions (i)–(iv).

For the second part, let us observe that by the weak $\circledast$-stability of μ we have

$$T_{a_n}\mu^{\circledast n} = T_{a_n}(\mu \circ \delta_1)^{\circledast n} = \mu \circ T_{a_n}\delta_1^{\otimes_{\mu,\circledast} n}.$$

So if the condition (v) is satisfied for $\otimes_{\mu,\circledast}$, then there exist a sequence $(a_n)_n$ of positive numbers and a measure $\gamma \in \mathcal{P}^+$ such that $T_{a_n}\delta_1^{\otimes_{\mu,\circledast} n} \to \gamma$. In this case, we take $\lambda = \mu \circ \gamma$.

On the other hand, if $T_{a_n}\mu^{\circledast n} \to \lambda$ for some measure λ and a sequence (a_n), then, by Proposition 1.6 of [20] and Theorem 2.2, there exists a measure γ such that $T_{a_n}\delta_1^{\otimes_{\mu,\circledast} n} \to \gamma$ and $\mu \circ \gamma = \lambda$. In particular, the condition (v) holds. □

In the following theorem we show that $\otimes_{\mu,\circledast}$ inherits the regularity of $\circledast$ and that the homomorphism on $(\mathcal{P}^+, \otimes_{\mu,\circledast})$ is given by the explicit formula involving the homomorphism on $(\mathcal{P}^+, \circledast)$ and the weakly stable measure μ.

Theorem 2.4. *Let $(\mathcal{P}^+, \circledast)$ be a regular generalized convolution algebra, μ be a weakly $\circledast$-stable measure and $\otimes_{\mu,\circledast}$ be the weak generalized convolution with respect to $\circledast$. Then $(\mathcal{P}^+, \otimes_{\mu,\circledast})$ is also regular.*

Proof. For any $\nu_1, \nu_2 \in \mathcal{P}^+$ we have

$$(\mu \circ \nu_1) \circledast (\mu \circ \nu_2) = \mu \circ (\nu_1 \otimes_{\mu,\circledast} \nu_2).$$

Since $\circledast$ is regular, there exists a homomorphism $h : \mathcal{P}^+ \to \mathbb{R}$ with $h(\nu_1 \circledast \nu_2) = h(\nu_1)h(\nu_2)$. Applying h to both sides of the relation above, we get

$$h(\mu \circ \nu_1)h(\mu \circ \nu_2) = h[(\mu \circ \nu_1) \circledast (\mu \circ \nu_2)] = h[\mu \circ (\nu_1 \otimes_{\mu,\circledast} \nu_2)].$$

So we can define

$$h_{\otimes_{\mu,\circledast}}(\nu) := h(\mu \circ \nu), \qquad \nu \in \mathcal{P}^+$$

and the condition $h_{\otimes_{\mu,\circledast}}(\nu_1 \otimes_\mu \nu_2) = h_{\otimes_{\mu,\circledast}}(\nu_1)h_{\otimes_{\mu,\circledast}}(\nu_2)$ holds. Note also that such a mapping is continuous (cf. Proposition 1.1 of [20]) and linear with respect to convex combination. Moreover, since $h(\delta_0) = 1$, hence $h_{\otimes_{\mu,\circledast}}(\delta_0) = h(\mu \circ \delta_0) = h(\delta_0) = 1$ and $h_{\otimes_{\mu,\circledast}}$ is non-zero.

On the other hand, if $h_{\otimes_{\mu,\circledast}}(\nu) = 1$ for all $\nu \in \mathcal{P}^+$, then for any $a \geq 0$ we would have $h(T_a\mu) = h(\mu \circ \delta_a) = h_{\otimes_{\mu,\circledast}}(\delta_a) = 1$ and, by Theorem 3 in [18], this would imply that $\mu = \delta_0$ which contradicts the assumption that μ is non-trivial. □

It is to be expected that the weak stability is invariant under rescaling.

Theorem 2.5. *If μ is weakly $\circledast$-stable, then $T_c\mu$ is weakly $\circledast$-stable for any $c > 0$ and $\otimes_{T_c\mu,\circledast} = \otimes_{\mu,\circledast}$.*

Proof. For any $\nu_1, \nu_2 \in \mathcal{P}^+$, we have

$$\begin{aligned} &T_c\mu \circ (\nu_1 \otimes_{T_c\mu,\circledast} \nu_2) \\ &\quad = (T_c\mu \circ \nu_1) \circledast (T_c\mu \circ \nu_2) = T_c(\mu \circ \nu_1) \circledast T_c(\mu \circ \nu_2) \\ &\quad = T_c[(\mu \circ \nu_1) \circledast (\mu \circ \nu_2)] = T_c[\mu \circ (\nu_1 \otimes_{\mu,\circledast} \nu_2)] \\ &\quad = (T_c\mu) \circ (\nu_1 \otimes_{\mu,\circledast} \nu_2). \end{aligned}$$

□

An equivalent condition for the weak stability can be stated in case of regular generalized convolutions. Note that this result has been mentioned in [9] as a remark. Since the proof needs nontrivial arguments, we enclose it in this paper.

Theorem 2.6 [9]. *Let $(\mathcal{P}^+, \circledast)$ be a regular generalized convolution algebra. Then*

$$\forall\, \nu_1, \nu_2 \in \mathcal{P}^+ \;\; \exists\, \lambda \in \mathcal{P}^+ \quad (\mu \circ \nu_1) \circledast (\mu \circ \nu_2) = \mu \circ \lambda, \tag{2}$$

if and only if

$$\forall\, a, b \geqslant 0 \;\; \exists\, \lambda_{a,b} \in \mathcal{P}^+ \quad T_a\mu \circledast T_b\mu = \mu \circ \lambda_{a,b}. \tag{3}$$

Proof. The implication (2) ⇒ (3) holds trivially. The proof of the other implication starts with the observation that if (3) holds then, for any discrete measures $\nu_1 = \sum_i p_i\delta_{a_i}$ and $\nu_2 = \sum_j q_j\delta_{b_j}$, we also have

$$\begin{aligned} (\mu \circ \nu_1) \circledast (\mu \circ \nu_2) &= \sum_i p_i T_{a_i}\mu \circledast \sum_j q_j T_{b_j}\mu = \sum_{i,j} p_i q_j (T_{a_i}\mu \circledast T_{b_j}\mu) \\ &= \mu \circ \sum_{i,j} p_i q_j \lambda_{a_i,b_j}, \end{aligned}$$

so we can take $\lambda = \sum_{i,j} p_i q_j \lambda_{a_i,b_j}$ in (2).

Let $\nu_1, \nu_2 \in \mathcal{P}^+$. We can find two sequences of discrete measures $\nu_1^{(n)}$ and $\nu_2^{(n)}$ such that $\nu_1^{(n)} \xrightarrow{w} \nu_1$ and $\nu_2^{(n)} \xrightarrow{w} \nu_2$. Then the set

$$K_\mu(\nu_1^{(n)}, \nu_2^{(n)}) := \{\lambda : (\mu \circ \nu_1^{(n)}) \circledast (\mu \circ \nu_2^{(n)}) = \mu \circ \lambda\}$$

is not empty for every $n \in \mathbb{N}$ and, by Lemma 2 in [14], is convex and weakly compact. On the other hand,

$$\left\{\mu \circ \lambda : \lambda \in \bigcup_{n=1}^{\infty} K_\mu(\nu_1^{(n)}, \nu_2^{(n)})\right\} = \{(\mu \circ \nu_1^{(n)}) \circledast (\mu \circ \nu_2^{(n)}) : n \in \mathbb{N}\}$$

is tight, which implies that $\bigcup_{n=1}^{\infty} K_\mu(\nu_1^{(n)}, \nu_2^{(n)})$ is also tight.

Let us now choose $\lambda^{(n)} \in K_\mu(\nu_1^{(n)}, \nu_2^{(n)})$ for every $n \in \mathbb{N}$. Then we can find a subsequence $\lambda^{(n_k)}$ such that $\lambda^{(n_k)} \xrightarrow{w} \lambda$. Now, by Theorem 2.1 of [23], we know that $\circledast$ is double continuous (i.e. $\lambda^{(n)} \xrightarrow{w} \lambda$ and $\mu^{(n)} \xrightarrow{w} \mu$ implies that $\lambda^{(n)} \circledast \mu^{(n)} \xrightarrow{w} \lambda \circledast \mu$) and thus $(\mu \circ \nu_1) \circledast (\mu \circ \nu_2) = \mu \circ \lambda$, i.e. $\lambda \in K_\mu(\nu_1, \nu_2)$, which completes the proof. □

2.3 *Construction of weakly stable measures and new generalized convolutions*

We know from the previous part that each weakly stable measure belonging to a domain of attraction of some measure induces the weak generalized convolution, which is again a generalized convolution. An obvious example of such a measure, for any generalized convolution $\circledast$, is δ_1. Indeed, we have

$$T_a\delta_1 \circledast T_b\delta_1 = \delta_1 \circ (\delta_a \circledast \delta_b),$$

but in this case $\otimes_{\delta_1,\circledast} = \circledast$.

Our goal in this section is to present some nontrivial examples of such measures and the related convolutions. That is why we recall the construction of Kucharczak and Urbanik [9], which realizes this purpose, and apply it explicitly to calculate measures weakly stable with respect to the Kendall convolution.

Let $\circledast$ be a fixed regular generalized convolution, h its homomorphism and μ its weakly $\circledast$-stable measure.

Following [9], let us introduce the class of transformations U_s $(s > 0)$ of probability measures with the domain:

$$\mathcal{D}(U_s) = \left\{\mu \in \mathcal{P}^+ : \int_0^\infty x^{s-1}|h(T_x\mu)|\mathrm{d}x < \infty, \quad c_s(\mu) = \int_0^\infty x^{s-1}h(T_x\mu)\mathrm{d}x \neq 0, \quad \exists\, \lambda \in \mathcal{P}^+ \quad h(T_t\lambda) = c_s(\mu)^{-1}\int_t^\infty x^{s-1}h(T_x\mu)\mathrm{d}x\right\}.$$

For $\mu \in \mathcal{D}(U_s)$ we define $U_s\mu := \lambda$.

Theorem 2.7 (Theorem 2.1 of [9]). *Let $s \in (0, \varkappa(\circledast)]$. If $\mu \in \mathcal{D}(U_s)$ is weakly $\circledast$-stable and if there exists $\eta \in \mathcal{P}_+$ such that $U_s\mu = \mu \circ \eta$, then*

(1) *for all $k \in \mathbb{N}$ the probability measures $U_s^k\mu$ are weakly $\circledast$-stable,*
(2) *each $U_s^k\mu$ induces a weak generalized convolution $\otimes_{U_s^k\mu,\circledast}$, which is a generalized convolution,*
(3) $\varkappa(\otimes_{U_s^k\mu,\circledast}) = s$.

Moreover, the probability kernels of $\otimes_{U_s^k\mu,\circledast}$ are given by the following recurrence formula: $\otimes_{U_s^0\mu,\circledast} = \otimes_\mu$ *and*

$$\begin{aligned}\delta_a \otimes_{U_s^k\mu,\circledast} \delta_b &= \frac{a^s}{a^s+b^s} V_s(\delta_a \otimes_{U_s^{k-1}\mu,\circledast} T_b V_{s(k-1)}\eta) \\ &\quad + \frac{b^s}{a^s+b^s} V_s(\delta_b \otimes_{U_s^{k-1}\mu,\circledast} T_a V_{s(k-1)}\eta),\end{aligned}$$

where $a, b \geqslant 0$, $k \geqslant 1$ and $V_s\nu(A) = \upsilon_s(\nu)^{-1}\int_A x^{-s}\nu(\mathrm{d}x)$ provided that $\upsilon_s(\nu) = \int_0^\infty x^{-s}\nu(\mathrm{d}x) < \infty$.

Now we apply this construction to the Kendall convolution case, starting with the weakly stable measure δ_1.

Example 1. Let $\triangle$ be the Kendall convolution, i.e. the generalized convolution with the probability kernel:

$$\delta_1 \triangle \delta_a = (1-a)\delta_1 + a\pi_2$$

for $a \in [0,1]$ and π_2 be the Pareto distribution with the density $\pi_2(\mathrm{d}x) = 2x^{-3}\mathbf{1}_{[1,\infty)}(x)\mathrm{d}x$. The corresponding homomorphism is given by $h(\delta_t) = (1-t)_+$.

We first check that δ_1 belongs to $\mathcal{D}(U_s)$ for a fixed $s \leqslant \varkappa(\triangle) = 1$. By simple computation we arrive at $c_s(\delta_1) = \frac{1}{s(s+1)}$ and

$$h(T_tU_s\delta_1) = s(s+1)\int_t^\infty x^{s-1}(1-x)_+\mathrm{d}x = (1-(s+1)t^s+st^{s+1})\mathbf{1}_{[0,1]}(t). \quad (4)$$

We shall show that there exists $\eta_s = U_s\delta_1$ which satisfies this equation. By definition we have

$$h(T_t\eta_s) = \int_0^\infty h(T_t\delta_x)\eta_s(\mathrm{d}x) = \int_0^{1/t}(1-tx)\eta_s(\mathrm{d}x)$$

and thus η_s is the solution of the following integral equation:

$$\int_0^{1/t}(1-tx)\eta_s(\mathrm{d}x) = (1-(s+1)t^s+st^{s+1})\mathbf{1}_{[0,1]}(t). \quad (5)$$

If $F_s(x)$ denotes the distribution function of the measure η_s, then, after integrating by parts the left-hand side of (5), we get

$$t\int_0^{1/t} F_s(x)\mathrm{d}x = 1-(s+1)t^s + st^{s+1}, \quad \text{for } t \in (0,1].$$

After dividing both sides by t and differentiating the equation with respect to t, we arrive at

$$F_s\left(\frac{1}{t}\right) = 1-(1-s^2)t^s - s^2 t^{s+1},$$

from which it follows that

$$F_s(x) = 1-(1-s^2)x^{-s} - s^2 x^{-s-1}$$

for $x = \frac{1}{t} \geqslant 1$. From this we recover the density function of η_s:

$$\eta_s(\mathrm{d}x) = \frac{s(s+1)}{x^{s+2}}[(1-s)x+s]\mathbf{1}_{[1,\infty)}(x)\mathrm{d}x. \tag{6}$$

By the above reasoning, we showed that $\delta_1 \in \mathcal{D}(U_s)$ and it is obvious that the second assumption of Theorem 2.7 is also satisfied ($U_s\delta_1 = \delta_1 \circ \eta_s$). Thus, by the theorem, the probability measures $U_s^k\delta_1$ are weakly $\triangle$-stable for every $s \leqslant \varkappa(\triangle) = 1$ and $k \geqslant 1$, and induce weak generalized convolutions $\otimes_{U_s^k\delta_1,\triangle}$ which are generalized convolutions.

For $k=1$ the probability kernel of $\otimes_{\eta_s,\triangle}$ is given by

$$\delta_a \otimes_{\eta_s,\triangle} \delta_b = \frac{a^s}{a^s+b^s}V_s(\delta_a \triangle T_b\eta_s) + \frac{b^s}{a^s+b^s}V_s(\delta_b\triangle T_a\eta_s).$$

At the second step of this construction we use the formula (4) to compute $c_s(\eta_s) = \frac{s+1}{2s(2s+1)}$ and

$$h(T_tU_s\eta_s) = \left(1-\frac{2(2s+1)}{s+1}t^s + (2s+1)t^{2s} - \frac{2s^2}{s+1}t^{2s+1}\right)\mathbf{1}_{(0,1)}(t).$$

Solving the integral equation

$$h(T_tU_s\eta_s) = \int_0^\infty h(T_t\delta_x)U_s\eta_s(\mathrm{d}x)$$

in the same manner as above, we prove that $\eta_s \in \mathcal{D}(U_s)$ and we arrive at the next weakly $\triangle$-stable probability measure $U_s^2\delta_1 = U_s\eta_s$ with the density

$$U_s\eta_s(\mathrm{d}x) = \frac{2s(2s+1)}{x^{2s+2}}\left((2s-1)x - \frac{s-1}{s+1}x^s - \frac{2s^2}{(s+1)}\right)\mathbf{1}_{[1,\infty)}(x)\mathrm{d}x.$$

The probability kernel of the generalized convolution $\otimes_{U_s\eta_s,\triangle}$ is described by

$$\delta_a \otimes_{U_s\eta_s,\triangle} \delta_b = \frac{a^s}{a^s+b^s}V_s(\delta_a \otimes_{\eta_s,\triangle} T_b\eta_s) + \frac{b^s}{a^s+b^s}V_s(\delta_b \otimes_{\eta_s,\triangle} T_a\eta_s).$$

Repeating this construction for $k > 2$, we can get a sequence of weakly $\triangle$-stable measures and related generalized convolutions. But this method is really laborious! That is why we present below another way to compute $U_s^k\delta_1$, which is a direct consequence of Lemmas 2.3 and 2.5 in [9].

By Lemma 2.5 in [9] we have

$$U_s^k\delta_1 = \eta_s \circ V_s\eta_s \circ V_{2s}\eta_s \circ V_{3s}\eta_s \circ \cdots \circ V_{(k-1)s}\eta_s$$

for $k = 1, 2, \ldots$ and $s \leqslant 1$. Using (6), we compute directly that

$$\upsilon_r(\eta_s) = \int_0^\infty x^{-r}\eta_s(\mathrm{d}x) = \frac{s(s+1)(r+1)}{(s+r)(s+r+1)}$$

for $s, r \in (0, 1]$. Then we observe that

$$\begin{aligned} V_r\eta_s(A) &= \frac{s}{(s+r)(r+1)} \\ &\quad \times \int_A \left(\frac{(s+r)^2(s+r+1)}{x^{s+r+2}} - \frac{(s+r)\left((s+r)^2-1\right)}{x^{s+r+1}} \right) \mathrm{d}x \\ &\quad + \left(\frac{s}{(s+r)(r+1)} - \frac{s-1}{(s+r-1)(r+1)} \right) \\ &\quad \times \int_A \frac{(s+r)\left((s+r)^2-1\right)}{x^{s+r+1}}\,\mathrm{d}x \end{aligned}$$

for any Borel set $A \subset [0, +\infty)$, from which we get

$$V_r\eta_s = \frac{s}{(s+r)(r+1)}\eta_{s+r} + \left(1 - \frac{s}{(s+r)(r+1)}\right)\pi_{s+r}$$

for $r + s \leqslant 1$, where $\pi_s(\mathrm{d}x) = sx^{-s-1}\mathbf{1}_{[1,\infty)}(x)$.

Now we see that it is enough to know the multiplicative convolutions $\eta_s \circ \eta_r$, $\eta_s \circ \pi_r$ and $\pi_s \circ \pi_r$, for $r, s \leqslant 1$, to describe the weakly $\triangle$-stable probability measures $U_s^k\delta_1$. We compute explicitly

$$\begin{aligned} \eta_s \circ \eta_r &= \frac{sr(r+1)}{(s-r)(r-s-1)}\eta_s + \frac{sr(s+1)}{(s-r)(r-s+1)}\eta_r \\ &\quad + \frac{r(r+1)(r+s-1)(s^2-1)}{(s-r)((s-r)^2-1)}\pi_s - \frac{s(s+1)(s+r-1)(r^2-1)}{(s-r)((s-r)^2-1)}\pi_r, \\ \eta_s \circ \pi_r &= \frac{r}{r-s-1}\eta_s + \frac{r(1-s^2)}{(r-s-1)(s-r)}\pi_s + \frac{s(s+1)(r-1)}{(r-s-1)(s-r)}\pi_r, \\ \pi_s \circ \pi_r &= \frac{s}{s-r}\pi_r - \frac{r}{s-r}\pi_s. \end{aligned}$$

By the above formulas its easy to check that the general form of the k-th measure is

$$U_s^k\delta_1 = U_s^{k-1}\delta_1 \circ V_{(k-1)s}\eta_s = a_k\eta_{ks} + \sum_{j=1}^k b_j^{(k)}\pi_{js},$$

where the coefficients a_k and $b_j^{(k)}$ can be computed recursively. Moreover, we observe that the construction is allowed as long as $k \leqslant \frac{1}{s}$.

The first two steps are the following ($s \leqslant \frac{1}{3}$):

$$\begin{aligned} U_s^2\delta_1 &= U_s\eta_s = \frac{1}{2(s+1)}\eta_s \circ \eta_{2s} + \left(1 - \frac{1}{2(s+1)}\right)\eta_s \circ \pi_{2s} \\ &= -\frac{s}{s+1}\eta_{2s} + \frac{2(1-s)(2s+1)}{s+1}\pi_s + \frac{4s^2-1}{s+1}\pi_{2s}, \\ U_s^3\delta_1 &= \frac{2s^2}{(2s+1)(s+1)}\eta_{3s} + \frac{3(s-1)(3s+1)}{(2s+1)}\pi_s \\ &\quad + \frac{3(2s-1)(3s+1)}{s+1}\pi_{2s} + \frac{(1-9s^2)(3s+1)}{(2s+1)(s+1)}\pi_{3s}. \end{aligned}$$

3. Generalized convolutions on the set of moment sequences

The aim of this part is to give some examples of weak and generalized convolutions related to non-commutative probability. It turns out that they cannot be found among the famous free, boolean or monotone convolution.

As an example, we start by explaining in detail what the boolean convolution is and at which point it fails to be a generalized convolution (this part of the paper is based on [16]). Similar calculations show that the free convolution also satisfies all but one of the Urbanik's conditions. The monotone convolution is not even commutative, and so cannot be a generalized convolution.

In the next part we give the definition of the q-convolution and the $(q, 1)$-convolution and show that they fit better to the Urbanik convolution's family.

3.1 *Boolean convolution is not a generalized convolution*

Given an arbitrary measure μ on $\mathcal{P}$ we define its Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mathrm{d}\mu(x), \quad z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im\mathfrak{m}\, z > 0\}$$

and its *K-transform* (or *self-energy*) by

$$K_\mu(z) = z - \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C}^+.$$

When $z \in C^+$, then the Cauchy transform G_μ takes values in $\mathbb{C}^- = \{z \in \mathbb{C} : \Im\mathfrak{m}\, z < 0\}$, so $K_\mu(z)$ is well defined.

DEFINITION 5

Given $\mu_1, \mu_2 \in \mathcal{P}$ the boolean convolution is the unique measure, denoted by $\mu_1 \uplus \mu_2$, which satisfies the relation

$$K_{\mu_1 \uplus \mu_2}(z) = K_{\mu_1}(z) + K_{\mu_2}(z), \quad z \in \mathbb{C}^+.$$

The operation $\uplus$ is well-defined, since the sum of two K-transforms is again a K-transform of some measure and since K_μ uniquely determines the measure μ. The reader is referred to [16] for more details.

Let us check which conditions of Definition 1 are satisfied by this operation. It follows directly from the definition that $\uplus$ is associative and commutative. Moreover, $G_{\delta_a}(z) = \frac{1}{z-a}$ and so $K_{\delta_a}(z) = a$ and

$$\delta_a \uplus \delta_b = \delta_{a+b}. \tag{7}$$

Since $K_{\delta_0}(z) = 0$, δ_0 must be the neutral element of the convolution.

The weak continuity of $\uplus$ follows from Proposition 3.3 in [16]: if $\mu_n \to \mu$ weakly, then

$$\lim_{n\to\infty} \mathfrak{Im} K_{\mu_n}(x+iy) = \mathfrak{Im} K_\mu(x+iy)$$

for all $x \in \mathbb{R}$ and for some $y > 0$. By adding $K_\nu(x+iy)$ to both sides and applying the opposite implication, we get that $\mu_n \uplus \nu \to \mu \uplus \nu$ weakly.

Thanks to the relation (7) and the limit argument (all measures are weak limit of the linear combination of δ's), we know that the boolean convolution can be restricted to the operation on measures supported only on $[0, +\infty)$.

For the dilation we have the formula

$$K_{T_a\mu}(z) = aK_\mu\left(\frac{z}{a}\right) \quad \text{for} \quad a > 0,$$

from which we see that

$$\begin{aligned} K_{T_a(\mu_1 \uplus \mu_2)}(z) &= aK_{\mu_1 \uplus \mu_2}\left(\frac{z}{a}\right) = aK_{\mu_1}\left(\frac{z}{a}\right) + aK_{\mu_2}\left(\frac{z}{a}\right) \\ &= K_{T_a\mu_1}(z) + K_{T_a\mu_2}(z) = K_{T_a\mu_1 \uplus T_a\mu_2}(z), \end{aligned}$$

so the condition (iii) of Urbanik's definition holds.

The limit theorem, i.e. condition (v), holds for the sequence $a_n = \frac{1}{n}$ and $\lambda = \delta_1$, since $\delta^{\uplus n} = \delta_n$ and

$$K_{T_{a_n}\delta^{\uplus n}}(z) = \frac{1}{n} K_{\delta_n}(nz) = 1 = K_{\delta_1}(z), \quad n \in \mathbb{N}.$$

Unfortunately, the linearity with respect to convex combinations fails for $\uplus$. Indeed, if (ii) holds, then we should have

$$\left(\frac{1}{2}\delta_a + \frac{1}{2}\delta_b\right) \uplus \delta_c \stackrel{?}{=} \frac{1}{2}\delta_{a+c} + \frac{1}{2}\delta_{b+c},$$

but, again in [16] we find that

$$\left(\frac{1}{2}\delta_a + \frac{1}{2}\delta_b\right) \uplus \delta_c = \frac{\Delta + c}{2\Delta}\delta_{(s+\Delta)/2} + \frac{\Delta - c}{2\Delta}\delta_{(s-\Delta)/2}$$

with $\Delta(a,b,c) = \sqrt{(a-b)^2 + c^2}$ and $s = a+b+c$. This means that $\uplus$ does not satisfy (ii).

Remark 3.1. Both the boolean and the free convolution generalize the classical convolution in all but one Urbanik's condition – the linearity does not hold in both cases. It seems that the class of operations on $\mathcal{P}^+$ which are translation-invariant and weakly continuous, has δ_0 as the neutral element, and for which the law of large number exists ('non-linear generalized convolutions') can be an interesting object to study. One more argument is that the observation that if $\diamond$ is a 'non-linear generalized convolution', then the t-transformation of $\diamond$, as defined by Bożejko and Wysoczański in [2], will also share these properties.

3.2 *Definitions of q-convolution and $(q, 1)$-convolution*

We shall study the following operations:

DEFINITION 6

Let q be a positive parameter and let $(\mu_n)_n$ and $(\nu_n)_n$ belong to $\mathcal{M}^+$ (they are moment sequences of measures on $[0, +\infty)$).

- the *q-convolution* is the sequence given by the formula

$$(\mu \star_q \nu)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-k(n-k)} \mu_k \nu_{n-k}, \quad n \in \mathbb{N}. \tag{8}$$

- the $(q, 1)$-*convolution* is the sequence given by the formula

$$(\mu \star_{q,1} \nu)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{-k(n-k)} \mu_k \nu_{n-k}. \tag{9}$$

The original definition of the q-convolution was given by Carnovale and Koornwinder in [3] for some special class of measures (with all moments finite). In that paper the formula for transformation of 'weighted' moments (Lemma 4.5 therein) was proved. This suggested another approach presented in [13], where the q-convolution was considered as an operation on the set of moment sequences (more precisely, the definition was formulated in terms of the q-moment sequences $\mu_n^{(q)} = q^{\binom{n}{2}} \int t^n \mu(\mathrm{d}t)$. The formula presented below follows directly from the fact that a sequence $(\mu_n)_n$ is a q-moment sequence if and only if $(q^{-\binom{n}{2}} \mu_n)_n$ is a moment sequence (see [10] for details)). The $(q, 1)$-convolution was defined in the same paper [13].

If $0 < q < 1$, then $\star_q : \mathcal{M}^+ \times \mathcal{M}^+ \to \mathcal{M}^+$, which means that the resulting sequence belongs to $\mathcal{M}^+$ (see Proposition 3.3 of [13]). For $q > 1$, the q-convolution does not preserve moment sequences (in [13], it was shown that for $a, b > 0$, $\big((\delta_a \star_q \delta_b)_n\big)_{n\in\mathbb{N}}$ cannot be a moment sequence). It is to be checked directly that the operation $\star_q$ is associative and commutative. Finally, let us note that if $q \to 1$, the q-convolution becomes the classical convolution of moment sequences with the formula

$$(\mu \star_1 \nu)_n = \sum_{k=0}^{n} \binom{n}{k} \mu_k \nu_{n-k}.$$

According to Proposition 4.3 of [13], the $(q, 1)$-convolution is also an operation on $\mathcal{M}^+$ (for arbitrary $q > 0$), i.e. $\{(\mu \star_{q,1} \nu)_n\}_n \in \mathcal{M}^+$ if $(\mu_n)_n, (\nu_n)_n \in \mathcal{M}^+$. Moreover, the operation is associative and commutative (here again, we simplified the notion of

(p, q)-convolution presented in [13] to avoid using q-moment sequences. That is why only one parameter appears. In a non-commutative approach, the (p, q)-convolution is related to the $*$-algebras generated by q-normal operators and also has an interpretation in a braided algebra). A special case of this convolution is when $q \to 1$, then the $(1, 1)$-convolution is defined as

$$(\mu \star_{1,1} \nu)_n = \sum_{k=0}^{n} \binom{n}{k}^2 \mu_k \nu_{n-k}.$$

3.3 *Interpretation of the definition*

Unfortunately, both the q-convolution and the $(q, 1)$-convolution are not well defined on the whole set of probability measures $\mathcal{P}^+$, but only on the set of moment sequence $\mathcal{M}^+$, and so they can not satisfy Definition 1 directly. That is, why do we need to reformulate Urbanik's definition so that it fits to the framework of moment sequences?

A way to do it is to notice that the set of moment sequences is isomorphic to a quotient space related to the set of measures:

$$\mathcal{M}^+ \cong \mathcal{P}^+_{fm}/\sim,$$

where $\mathcal{P}^+_{fm}$ is the subset of $\mathcal{P}^+$ containing measures with all moments finite and $\sim$ is the equivalence relation of 'having all moments equal'. More precisely, given two measures $\mu, \nu \in \mathcal{P}^+$, we say that μ is *equivalent* to ν ($\mu \sim \nu$) if $\mu_n = \nu_n$ for all $n \in \mathbb{N}$. This observation suggests the following interpretation of the Urbanik definition on $\mathcal{M}^+$:

(1) all equalities of two measures should be changed into equalities of respective cosets, that is the equalities of all terms of moment sequences,
(2) the weak convergence should be replaced by the convergence of all moments, denoted here by '$\mathtt{m}$-convergence'.

DEFINITION 7

An associative and commutative binary operation $\circledast$ on $\mathcal{M}^+$ is called a *generalized convolution on* $\mathcal{M}^+$ if it satisfies the following conditions:

(i) $\delta_0 \circledast \mu = \mu$ for any $\mu \in \mathcal{M}^+$,
(ii) $(c\mu_1 + (1-c)\mu_2) \circledast \nu = c(\mu_1 \circledast \nu) + (1-c)(\mu_2 \circledast \nu)$ for any $\mu_1, \mu_2, \nu \in \mathcal{M}^+$, $c \in (0, 1)$,
(iii) $T_a(\mu \circledast \nu) = (T_a\mu) \circledast (T_a\nu)$ for any $\mu, \nu \in \mathcal{M}^+$ $a \geq 0$,
(iv) $\forall \mu^{(n)}, \mu, \nu \in \mathcal{M}^+$: $\mu^{(n)} \xrightarrow{\mathtt{m}} \mu \implies \mu^{(n)} \circledast \nu \xrightarrow{\mathtt{m}} \mu \circledast \nu$.
(v) there exists a sequence $(c_n)_n$ of positive numbers such that the sequence $T_{c_n}\delta_1^{\circledast n}$ $\mathtt{m}$-converges to a measure different from δ_0.

Given a generalized convolution on $\mathcal{M}^+$, we can consider, as in the previous section, stable and weakly stable measures, and also the weak generalized convolutions on $\mathcal{M}^+$.

Remark 3.2. If a non-trivial measure μ has all moments finite, then obviously μ is cancellable on $\mathcal{M}^+$, i.e. $(\mu \circ \nu_1)_n = (\mu \circ \nu_2)_n$ implies $(\nu_1)_n = (\nu_2)_n$ for all $n \in \mathbb{N}$. This means that the operation $\otimes_\mu$ satisfies the conditions (i)–(iv) of Definition 7. If, moreover, $T_{a_n}\mu^{\circledast n} \overset{\mathrm{m}}{\to} \lambda$ for some sequence $(a_n)_n$ and measure λ, then $\otimes_\mu$ is a generalized convolution on $\mathcal{M}^+$.

3.4 *Main properties*

We are ready to show that the q-convolution as well as the $(q,1)$-convolution are examples of generalized convolutions on $\mathcal{M}^+$.

Theorem 3.3. *The $(q,1)$-convolution is a generalized convolution on $\mathcal{M}^+$.*

Proof. We need to verify conditions (i)–(v) from Definition 7 for all moments. So let us take $(\mu_n)_n, (\nu_n)_n \in \mathcal{M}^+$.

(i) We have

$$(\mu \star_{q,1} \delta_0)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{-k(n-k)} \mu_k \delta_{n-k} = \mu_n.$$

(ii) Using the elementary fact that $(\alpha\mu + \beta\nu)_k = \alpha\mu_k + \beta\nu_k$ for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
[(c\mu_1+(1-c)\mu_2) \star_{q,1} \nu]_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{-k(n-k)} (c\mu_1+(1-c)\mu_2)_k \nu_{n-k} \\
&= c\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{-k(n-k)} (\mu_1)_k \nu_{n-k} + (1-c) \\
&\quad \times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{-k(n-k)} (\mu_2)_k \nu_{n-k} \\
&= c(\mu_1 \star_{q,1} \nu)_n + (1-c)(\mu_2 \star_{q,1} \nu)_n.
\end{aligned}$$

(iii) Since $(T_a\mu)_n = a^n\mu_n$, we have

$$\begin{aligned}
[T_a(\mu \star_{q,1} \nu)]_n = a^n(\mu \star_{q,1} \nu)_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{-k(n-k)} (a^k\mu_k)(a^{n-k}\nu_{n-k}) \\
&= [(T_a\mu) \star_{q,1} (T_a\nu)]_n.
\end{aligned}$$

(iv) On the set $\mathcal{M}^+$ we identify measures with same moments and thus the weak convergence of measures should be replaced by the convergence of moments. So we need to prove that for $\mu^{(n)}, \mu \in \mathcal{M}^+$ such that $\mu_k^{(n)} \to \mu_k$ for all $k \in \mathbb{N}$ as $n \to +\infty$ we have

$$(\mu^{(n)} \star_{q,1} \nu)_k \to (\mu \star_{q,1} \nu)_k$$

for all $\nu \in \mathcal{M}^+$ and $k \in \mathbb{N}$.

We want to prove that

$$\forall_{\varepsilon>0}\forall_{k\in\mathbb{N}}\ \exists_N\ \forall_{n\geq N} \quad |(\mu^{(n)} \star_{q,1} \nu)_k - (\mu \star_{q,1} \nu)_k| < \varepsilon.$$

Let us fix $k \in \mathbb{N}$ and $\varepsilon > 0$. For each $l = 0, \ldots, k$ there exists N_l such that $|\mu_l^{(n)} - \mu_l| < \varepsilon$ for all $n > N_l$. Set $N := \max\{N_l : 0 \leq l \leq k\}$ and $M := \max\{|\nu_l| : 0 \leq l \leq k\}$. Then

$$\begin{aligned} |(\mu^{(n)} \star_{q,1} \nu)_k - (\mu \star_{q,1} \nu)_k| &\leq \sum_{l=0}^{k} \left[\begin{array}{c} k \\ l \end{array} \right]_q^2 q^{-l(k-l)} |\mu_l^{(n)} - \mu_l|\, |\nu_{k-l}| \\ &\leq \varepsilon \cdot M \sum_{l=0}^{k} \left[\begin{array}{c} k \\ l \end{array} \right]_q^2 q^{-l(k-l)} = \varepsilon \cdot M', \end{aligned}$$

where M' is independent on n.

(v) In Theorem 6.1 of [13] it was shown that the limit theorem holds for $c_n = \frac{1}{\sqrt{n}}$ and the limit measure has moments $\frac{[n]_q![n]_{1/q}!}{n!}$. □

Theorem 3.4. *The $(q, 1)$-convolution is double continuous on $\mathcal{M}^+$, i.e.*

$$\forall_{k\in\mathbb{N}}\ (\mu^{(n)})_k \to \mu_k,\ (\nu^{(n)})_k \to \nu_k \quad \Longrightarrow \quad \forall_{k\in\mathbb{N}}\ (\mu^{(n)} \star_{q,1} \nu^{(n)})_k \to (\mu \star_{q,1} \nu)_k.$$

Proof. The proof is similar to (iv) in the previous theorem.

For a fixed $k \in \mathbb{N}$ and $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that both $|\mu_l^{(n)} - \mu_l| < \varepsilon$ and $|\nu_l^{(n)} - \nu_l| < \varepsilon$ for all $n > N$ and $l = 0, \ldots, k$. Moreover, we set $M \in \mathbb{R}$ such that $|\mu_l| < M$ and $|\nu_l^{(n)}| < M$ for $l = 0, \ldots, k$ and $n > N$ (this is possible since the sequence $(\nu_l^{(n)})_{n\in\mathbb{N}}$ is convergent and thus bounded). Then for $n > N$ we have

$$\begin{aligned} |(\mu^{(n)} \star_{q,1} \nu^{(n)})_k - (\mu \star_{q,1} \nu)_k| &\leq \sum_{l=0}^{k} \left[\begin{array}{c} k \\ l \end{array} \right]_q^2 q^{-l(k-l)} |\mu_l^{(n)} \nu_{k-l}^{(n)} - \mu_l \nu_{k-l}| \\ &\leq \sum_{l=0}^{k} \left[\begin{array}{c} k \\ l \end{array} \right]_q^2 q^{-l(k-l)} \\ &\quad \times (|\mu_l^{(n)} - \mu_l|\, |\nu_{k-l}^{(n)}| + |\mu_l|\, |\nu_{k-l}^{(n)} - \nu_{k-l}|) \\ &\leq \varepsilon \cdot 2M \sum_{l=0}^{k} \left[\begin{array}{c} k \\ l \end{array} \right]_q^2 q^{-l(k-l)} = \varepsilon \cdot M'' \end{aligned}$$

and M'' is independent of n. □

Theorem 3.5. *The q-convolution is a generalized convolution on $\mathcal{M}^+$ and is double continuous on $\mathcal{M}^+$.*

Proof. The proof of (i)–(iv) as well as the double continuity is similar to the previous ones and thus omitted. As to the condition (v), it can be shown that

$$(T_{\frac{1}{k}} \delta_1^{\star_q k})_n \longrightarrow \frac{[n]_{1/q}!}{n!} \quad \text{as} \quad k \to +\infty.$$

The proof of this fact needs purely non-commutative arguments and will be published elsewhere [12]. □

3.5 *Examples of weak stability for deformed convolutions*

Now, we can show that the notion of weak generalized convolution helps to describe the relation between the q- and the $(q,1)$-convolutions. Several examples of the weak-type relations concerning the deformed convolution are also considered.

Example 2. The relation between the q- and the $(q,1)$-convolutions can be described as

$$(N\circ\nu_1)\star_{q,1}(N\circ\nu_2)=N\circ(\nu_1\star_q\nu_2) \tag{10}$$

for any $\nu_1,\nu_2\in\mathcal{M}^+$. This means that the q-convolution is the weak generalized convolution with respect to $(q,1)$-convolution for any $q\in(0,1)$. The generating measure $N=N(q)$ is given by the moment sequence $N_n=[n]_q!$, $n\in\mathbb{N}$.

Let us start by proving the relation (10). We notice that for any $\nu_1,\nu_2\in\mathcal{M}^+$,

$$\begin{aligned}
[(N\circ\nu_1)\star_{q,1}(N\circ\nu_2)]_n &= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q^2 q^{-k(n-k)}(N\circ\nu_1)_k(N\circ\nu_2)_{n-k}\\
&= \sum_{k=0}^{n}\frac{[n]_q!^2}{[k]_q!^2[n-k]_q!^2}q^{-k(n-k)}[k]_q!(\nu_1)_k[n-k]_q!(\nu_2)_{n-k}\\
&= [n]_q!\sum_{k=0}^{n}\frac{[n]_q!}{[k]_q![n-k]_q!}q^{-k(n-k)}(\nu_1)_k(\nu_2)_{n-k}\\
&= [N\circ(\nu_1\star_q\nu_2)]_n.
\end{aligned}$$

We also need to check that the sequence $([n]_q!)_n$ is a moment sequence corresponding to a probability measure on $[0,+\infty)$. This follows from the fact that for $0<q<1$, the probability measure

$$\mu=(q,q)_\infty\sum_{k=0}^{\infty}\frac{q^k}{(q,q)_k}\delta_{q^k},$$

associated to the little q-Laguerre polynomials (cf. Subsection 3.20 of [7]), is supported on $\{q^k;\,k\in\mathbb{N}\}$ and has the moments $\mu_n=(q,q)_n$. Since the sequence $\left(\frac{1}{(1-q)^n}\right)_n$ is a moment sequence corresponding to the measure $\delta_{\frac{1}{1-q}}$ on $[0,+\infty)$, $[n]_q!=\frac{(q,q)_n}{(1-q)^n}$ is such too.

Finally, let us note that N belongs to a domain of attraction of a $\star_{q,1}$-stable measure but it is not $\star_{q,1}$ -stable measure. Indeed, the sequence $([n]_q!)_n$ has zeros and the first moment is equal to one, and thus satisfies the condition of the limit theorem for the $(q,1)$-convolution. Therefore, it belongs to a domain of attraction of the limit sequence $L=\left(\frac{[n]_q![n]_{1/q}!}{n!}\right)_n$. On the other hand if N was $\star_{q,1}$-stable, i.e. for all $a,b\geq 0$ there exists $c\geq 0$ such that

$$T_aN\star_{q,1}T_bN=T_cN,$$

we would get

$$\begin{aligned}
[n]_q!(\delta_c)_n &= (N\circ\delta_c)_n=(T_cN)_n=(T_aN\star_{q,1}T_bN)_n\\
&= [(N\circ\delta_a)\star_{q,1}(N\circ\delta_b)]_n=[n]_q!\cdot(\delta_a\star_q\delta_b)_n.
\end{aligned}$$

This would imply that $(\delta_a \star_q \delta_b)_n = (\delta_c)_n$. But for $b = 1$ the sequence $\{(\delta_a \star_q \delta_1)_n\}_n$ is not compactly supported (see [11] for details) and thus can not correspond to a delta measure. In the next example we show that L is $(q, 1)$-stable.

Combining this example with Theorem 2.3 and Remark 3.2, we get Theorem 3.5 for free: the q-convolution is the Urbanik's generalized convolution since it is a weak generalized convolution with respect to $\star_{q,1}$, generated by a weakly $\star_{q,1}$-stable measure which belongs to a domain of attraction of a $\star_{q,1}$-stable measure.

Example 3. The classical convolution (restricted to $\mathcal{M}^+$) is a weak generalized convolution with respect to $\star_{q,1}$. The generating measure corresponds to the sequence $L = \left(\frac{[n]_q![n]_{1/q}!}{n!}\right)_n$ (see [11] for the description of the measure).

First note that $[n]!_{1/q} = q^{-\binom{n}{2}}[n]_q!$, and so $L_n = q^{-\binom{n}{2}}\frac{[n]_q!^2}{n!}$. Then

$$\begin{aligned}[(L\circ\nu_1)\star_{q,1}(L\circ\nu_2)]_n &= \sum_{k=0}^{n}\frac{[n]_q!^2}{[k]_q!^2[n-k]_q!^2}q^{-k(n-k)}\\ &\quad\times q^{-\binom{k}{2}}\frac{[k]_q!^2}{k!}(\nu_1)_k q^{-\binom{n-k}{2}}\frac{[n-k]_q!^2}{(n-k)!}(\nu_2)_{n-k}\\ &= q^{-\binom{n}{2}}\frac{[n]_q!^2}{n!}\sum_{k=0}^{n}\binom{n}{k}(\nu_1)_k(\nu_2)_{n-k} = [L\circ(\nu_1 * \nu_2)]_n,\end{aligned}$$

where $*$ is the usual convolution. The sequence L is $\star_{q,1}$-stable (we see from the previous calculations that $c = a + b$ in the relation $T_a L \star_{q,1} T_b L = T_c L$).

Example 4. As the limit case of the two previous examples, we get that the classical convolution is the weak generalized convolution with respect to the $(1, 1)$-convolution, i.e.

$$[(N(1)\circ\nu_1)\star_{1,1}(N(1)\circ\nu_2)]_n = [N(1)\circ(\nu_1 * \nu_2)]_n.$$

The generating measure $N(1)$ is the exponential distribution with parameter $\lambda = 1$ and the density function $f(x) = \mathrm{e}^{-x}$ for $x \geq 0$, which corresponds to the moment sequence $(n!)_n$.

Example 5. One shows in the same manner as in Example 3, that the classical convolution (restricted to $\mathcal{M}^+$) is a weak generalized convolution with respect to $\star_q$, generated by the ($\star_q$-stable) limit measure with the moment sequence $M = M(q) = \left(q^{-\binom{n}{2}}\frac{[n]_q!}{n!}\right)_n$:

$$[(M\circ\nu_1)\star_q(M\circ\nu_2)]_n = [M\circ(\nu_1 * \nu_2)]_n.$$

The measure corresponding to the sequence M is described in [11].

Example 6. In the general case of (p, q)-convolution, another relation of this weak-type can be shown. Indeed, the $(p, 1)$-convolution is the weak generalized convolution with respect to (p, q)-convolution for any $q > 1$ with the generating measure $Q = Q(q)$ corresponding to the sequence $Q_n = q^{\frac{n(n-1)}{2}}$.

3.6 *Remark on regularity*

Unfortunately, it is still unknown whether $(\mathcal{M}^+, \star_q)$ or $(\mathcal{M}^+, \star_{q,1})$ are regular or not. Below we present two partial results on this topic.

In case of the q-convolution $(0 < q < 1)$ we have a natural candidate for the generalized characteristic function. Namely, given a measure μ on $[0, +\infty)$, its q-Laplace transform (cf. the q-Fourier transform in [3]) is defined as

$$\mathcal{L}^q(\mu)(x) = \int E_q(-xt)\mu(\mathrm{d}t), \quad \text{with } E_q(x) := \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{(q,q)_k}. \tag{11}$$

One can check that the formal power series

$$\mathcal{L}^q(\mu)(x) = \int \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}(-tx)^k}{(q,q)_k}\mu(\mathrm{d}t) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}(-1)^k \mu_k x^k}{(q,q)_k} \tag{12}$$

satisfies

$$\mathcal{L}^q(\mu \star_q \nu)(x) = \mathcal{L}^q(\mu)(x) \cdot \mathcal{L}^q(\nu)(x).$$

Notice that for the limit sequence $\lambda = \left(q^{-\binom{n}{2}} \frac{[n]_q!}{n!}\right)_n$, we have

$$\mathcal{L}^q(\lambda)(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}(-1)^k x^k}{(q,q)_k}\lambda_k = \sum_{k=0}^{\infty} \frac{(-1)^k(1-q)^k x^k}{k!} = \mathrm{e}^{-(1-q)x}.$$

Unfortunately, it remains unclear whether the function $\mathcal{L}^q(\mu)$ is well-defined for every sequence in $\mathcal{M}^+$ – at least, the formal power series become divergent for some measures.

In case of the (1, 1)-convolution we can consider the transformation defined by

$$\mathcal{L}^{1,1}(\mu)(x) := \int_0^{+\infty} \mathcal{J}_0(2\sqrt{xt})\mu(\mathrm{d}t),$$

where $\mathcal{J}_0(z)$ is the Bessel integral

$$\mathcal{J}_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2k} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{iz\cos\theta}\mathrm{d}\theta.$$

Note that the value of $\mathcal{L}^{1,1}(\mu)(x)$ is finite and real for any $x \in \mathbb{R}$ and any measure in $\mathcal{P}^+$. This is due to the fact that $\mathcal{J}_0(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ and to the estimation

$$|\mathcal{L}^{1,1}(\mu)(x)| \le \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi} |\mathrm{e}^{i2\sqrt{xt}\cos\theta}|\mathrm{d}\theta\, \mu(\mathrm{d}t) = 1.$$

It is also easy to check that the transformation $\mu \to \mathcal{L}^{1,1}(\mu)$ is linear and that $\mathcal{L}^{1,1}(T_a\mu)(x) = \mathcal{L}^{1,1}(\mu)(ax)$ for $a \ge 0$.

Formally, $\mathcal{L}^{1,1}$ turns the (1, 1)-convolution of moment sequences into the multiplication of functions; indeed it can be rewritten as a power series

$$\mathcal{L}^{1,1}(\mu)(x) = \int \sum_{k=0}^{\infty} \frac{(-tx)^k}{(k!)^2}\mu(\mathrm{d}t) = \sum_{k=0}^{\infty} \frac{(-x)^k}{(k!)^2}\mu_k,$$

from which it follows (formally) that

$$\mathcal{L}^{1,1}(\mu \star_{1,1} \nu)(x) = \mathcal{L}^{1,1}(\mu)(x) \cdot \mathcal{L}^{1,1}(\nu)(x). \tag{13}$$

To extend this property beyond the formal power series (to the whole set $\mathcal{M}^+$, and also to $\mathcal{P}^+$) we would need to know either the explicit formula for the (1, 1)-convolution of measures, or the inverse transformation $(\mathcal{L}^{1,1})^{-1}$. In the first case we would be able to calculate $\mathcal{L}^{1,1}(\mu \star_{1,1} \nu)$ on the level of integrals, whereas in the second case, the extended definition would mimic one of the boolean convolution (cf. § 3.1) and involve the desired relation.

We hope to consider this problem in the future.

Acknowledgements

The second author (AK) was partially supported by the MNiSW research grant N201 36 44 36.

References

[1] Berg Ch., On a generalized Gamma convolution related to the q-calculus, in: Theory and Applications of Special Functions, *Dev. Math.* **13** (2005) 61–76

[2] Bożejko M and Wysoczański, Remarks on t-transformations of measures and convolutions, *Ann. Inst. H. Poincaré Probab. Statist.* **37(6)** (2001) 737–761

[3] Carnovale G and Koornwinder T H, A q-analogue of convolution on the line, *Methods Appl. Anal.* **7** (2000) 705–726

[4] Franz U and Schott R, Stochastic Processes and Operator Calculus on Quantum Groups, Mathematics and its Applications (1999) vol. 490

[5] Jasiulis B, Limit property for regular and weak generalized convolutions, *J. Theor. Probab.* **23(1)** (2010) 315–327

[6] Kempf A and Majid S, Algebraic q-integration and Fourier theory on quantum and braided spaces, *J. Math. Phys.* **35(12)** (1994) 6802–6837

[7] Koekoek R and Swarttouw R F, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 94-05 (Delft University of Technology) (1994)

[8] Koornwinder T H, Special functions and q-commuting variables, *Fields Inst. Commun.* **14** (1997) 127–166

[9] Kucharczak J and Urbanik K, Transformations Preserving Weak Stability, *Bull. Pol. Acad. Sci. Math.* **34** (1986) 475–486

[10] Kula A, A q-analogue of complete monotonicity, *Colloq. Math.* **111** (2008) 169–181

[11] Kula A, The q-deformed convolutions: Examples and applications to moment problem, *Operators and Matrices* **4(4)** (2010) 593–603

[12] Kula A, Limit theorem for the q-convolution, to appear in Noncommutative Harmonic Analysis with Applications to Probability III, Banach Center Publications 96 (2012)

[13] Kula A and Ricard E, On a convolution for q-normal operators, *Inf. Dim. Anal. Quantum Prob. Rel. Topics* **11** (2008) 565–588

[14] Misiewicz J K, Oleszkiewicz K and Urbanik K, Classes of measures closed under mixing and convolution, Weak stability, *Studia Math.* **167(3)** (2005) 195–213

[15] Muraki N, Monotonic Lévy-Khintchine formula, preprint (2000)

[16] Speicher R and Woroudi R, Boolean convolution, in: Free probability theory, Fields Inst. Commun. 12, Amer. Math. Soc., Providence, RI (1997) pp. 267–279

[17] Urbanik K, A counterexample on generalized convolutions, *Colloq. Math.* **54** (1987) 143–147

[18] Urbanik K, Generalized convolutions, *Studia Math.* **23** (1964) 217–245
[19] Urbanik K, Generalized convolutions II, *Studia Math.* **45** (1973) 57–70
[20] Urbanik K, Generalized convolutions III, *Studia Math.* **80** (1984) 167–189
[21] Urbanik K, Generalized convolutions IV, *Studia Math.* **83** (1986) 57–95
[22] Urbanik K, Generalized convolutions V, *Studia Math.* **91** (1988) 153–178
[23] Urbanik K, Quasi-regular generalized convolutions, *Coll. Math.* **55(1)** (1988) 147–162
[24] Voiculescu D V, Addition of certain non-commuting random variables, *J. Funct. Anal.* **66** (1986) 323–346