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Uncertainty principles for the Cherednik transform

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Abstract. We shall investigate two uncertainty principles for the Cherednik transform on the Euclidean space $\mathfrak{a}$; Miyachi's theorem and Beurling's theorem. We give an analogue of Miyachi' theorem for the Cherednik transform and under the assumption that $\mathfrak{a}$ has a hypergroup structure, an analogue of Beurling's theorem for the Cherednik transform.

Keywords. Cherednik operators; Cherednik transform; Miyachi's theorem; Beurling's theorem.

1. Introduction

Uncertainty principle for the Fourier transform on $\mathbb{R}$ was first formulated as Heisenberg's inequality and Hardy's theorem in 1930's. Then various generalizations have been studied. Many of the variants of uncertainty principles follow from the following three theorems, which we call master theorems for uncertainty principles.

Theorem 1.1 **($L^p - L^q$ Morgan's theorem).** *Let $a, b > 0$, $p, q \in [1, \infty]$, $\alpha \geq 2$ and $\beta > 0$ with $1/\alpha + 1/\beta = 1$ and $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\frac{\pi}{2}(\beta - 1)))^{1/\beta}$. If a measurable function f on $\mathbb{R}$ satisfies $\mathrm{e}^{a|x|^\alpha} f \in L^p(\mathbb{R})$ and $\mathrm{e}^{b|y|^\beta} \hat{f} \in L^q(\mathbb{R})$, then $f = 0$ almost everywhere.*

Theorem 1.2 **(Miyachi's theorem).** *Let $a, b > 0$ and $ab = 1/4$. If a measurable function f on $\mathbb{R}$ satisfies the conditions $f(x)\mathrm{e}^{ax^2} \in L^\infty(\mathbb{R}) + L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} \log^+ \frac{|\hat{f}(\lambda)|\mathrm{e}^{b\lambda^2}}{C(1+|\lambda|)^N} \mathrm{d}\lambda < \infty$ for some $0 < C, N < \infty$, then f is a constant multiple of e^{-ax^2}.*

Theorem 1.3 **(Generalized Beurling's theorem).** *Let $f \in L^2(\mathbb{R})$ and $N \geq 0$. Then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(x)||\hat{f}(y)|}{(1+|x|+|y|)^N} \mathrm{e}^{2\pi|xy|}\mathrm{d}x\mathrm{d}y < \infty$ if and only if f may be written as $f(x) = P(x)\mathrm{e}^{-ax^2}$, where $a > 0$ and P is a polynomial of degree $< (N-1)/2$.*

The $L^p - L^q$ Morgan's theorem on $\mathbb{R}$ was obtained in [9] for $p = q = \infty$ and generalized on $\mathbb{R}^d$ in [1] for general $p, q \in [1, \infty]$. Miyachi's theorem on $\mathbb{R}$ was obtained in [8] and generalized on $\mathbb{R}^d$ in [5]. Beurling's theorem, which is the case of $N = 0$ in Theorem 1.3, was found by Beurling and a proof was given in [7]. A generalization of Beurling's theorem for the Fourier transform on $\mathbb{R}^d$ was obtained in [2]. These theorems are also extended to generalized Fourier transforms $\mathcal{F}$ on non-Euclidean spaces such as Heisenberg groups, semisimple Lie groups, Jacobi transforms, Dunkl transforms, etc. We refer to [1,5,12] and references therein. The common key to obtain extensions of uncertainty principles for $\mathcal{F}$ is a slice formula, that is, the generalized Fourier transform $\mathcal{F}$ is decomposed as a composition of the Euclidean Fourier transform and the so-called Radon transform. By this formula these extensions resolve into the Euclidean cases. For example, see §3 in [5] for a generalization of Miyachi's theorem.

In this paper we shall discuss analogous results for the Cherednik transform on the Euclidean space $\mathfrak{a}$, which has no slice formula at present. However, we have a sharp estimate of the heat kernel obtained in [13]. Hence, by using the heat kernel to express the decay of functions, we can reconstruct uncertainty principles for the Cherednik transform. In the previous paper [3], the authors obtained $L^p - L^q$ Morgan's theorem for the Cherednik transform. Here, we shall obtain Miyachi's theorem for the Cherednik transform (see §3). As for generalized Beurling's theorem, a substantial amount of difficulty is anticipated. In §4 we shall obtain a partial result on Beurling's theorem for the Cherednik transform.

2. Harmonic analysis associated with the Cherednik operator

In this section, we review basic results on harmonic analysis associated with the Cherednik operator, which are due to Opdam [10,11].

Let $\mathfrak{a}$ be a Euclidean space of dimension d equipped with an inner product $(.,.)$ and $\mathcal{R}$ be a root system in $\mathfrak{a}$. Let W denote the associated Weyl group, which is generated by orthogonal reflections $s_\alpha \in O(\mathfrak{a})$ along hyperplanes $\ker\alpha$ for $\alpha \in \mathcal{R}$. Let $k = (k_\alpha)_{\alpha\in\mathcal{R}}$ be a non-negative multiplicity function, that is, $k_\alpha \geqq 0$ and $k_\alpha = k_\beta$ if $\alpha \in W\beta$. Choose a set $\mathcal{R}^+$ of positive roots in $\mathcal{R}$ and put $\mathfrak{a}_+ = \{x \mid \forall\alpha \in \mathcal{R}_+, (\alpha, x) > 0\}$. We denote by $\overline{\mathfrak{a}_+}$ its closure and put

$$\rho = \frac{1}{2} \sum_{\alpha\in\mathcal{R}^+} k_\alpha \alpha.$$

Let $\xi \in \mathfrak{a}_\mathbf{C}$. The Cherednik operator T_ξ is the differential-difference operator on $\mathfrak{a}$ defined by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{(\alpha, \xi)}{1 - \mathrm{e}^{-(\alpha,x)}} \{f(x) - f(r_\alpha x)\} - (\rho, \xi) f(x),$$

where ∂_ξ is the directional derivative along ξ. For any $\xi, \eta \in \mathfrak{a}_\mathbb{C}$, $[T_\xi, T_\eta] = 0$. Then there exists a neighborhood U of 0 in $\mathfrak{a}$ and a unique holomorphic function $(\lambda, x) \mapsto G_\lambda(x)$ on $\mathfrak{a}_\mathbb{C} \times (\mathfrak{a} + iU)$ such that

$$\begin{cases} T_\xi G_\lambda(x) = \langle\lambda, \xi\rangle\, G_\lambda(x) \quad (\forall \xi \in \mathfrak{a}) \\ G_\lambda(0) = 1. \end{cases}$$

For $\lambda \in \mathfrak{a}$, G_λ is real and strictly positive. Moreover, the following estimate holds on $a_{\mathbb{C}} \times \mathfrak{a}$:

(a) $|G_\lambda(x)| \leq G_{\Re(\lambda)}(x)$,

(b) $|G_\lambda(x)| \leq G_0(x)\mathrm{e}^{\max_w(\Re(w\lambda),x)}$,

(c) $$G_0(x) \sim \prod_{\alpha \in \mathcal{R}_0^+,(\alpha,x)\geq 0} (1+(\alpha,x))\mathrm{e}^{(-\rho,x^+)}, \tag{1}$$

where $\mathcal{R}_0^+$ is the set of positive indivisible roots and x^+ is the unique conjugate of x in $\overline{\mathfrak{a}_+}$.

Let μ denote the measure on $\mathfrak{a}$ defined by

$$\mathrm{d}\mu(x) = \prod_{\alpha \in \mathcal{R}^+} \left| 2\sinh \frac{\langle \alpha, x\rangle}{2} \right|^{2k_\alpha} \mathrm{d}x. \tag{2}$$

Let f be a nice function on $\mathfrak{a}$, say f belongs to the space $C_c^\infty(\mathfrak{a})$. The Cherednik transform of f is defined by

$$\mathcal{F}f(\lambda) = \int_{\mathfrak{a}} f(x) G_{-i\lambda}(-x)\mathrm{d}\mu(x).$$

When f is W-invariant, it can be rewritten as

$$\mathcal{F}f(\lambda) = \int_{\mathfrak{a}_+} f(x) F_{-i\lambda}(-x)\mathrm{d}\mu(x),$$

where

$$F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(w \cdot x),$$

which satisfies the same estimates (1)(b), (c) of G_λ on $\mathfrak{a}_{\mathbb{C}} \times \mathfrak{a}_+$.

Let $C_R^\infty(\mathfrak{a})$, $R > 0$, denote the space of C^∞ functions on $\mathfrak{a}$ vanishing outside the ball $B_R = \{x \in \mathfrak{a} : ||x|| \leq R\}$ and let $H_R^\infty(\mathfrak{a})$ denote the space of holomorphic functions h on $\mathfrak{a}_{\mathbb{C}}$ such that, for every integer $N > 0$,

$$\sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}} (1+\|\lambda\|)^N \mathrm{e}^{-R\|\Im(\lambda)\|} \|h(\lambda)\| < +\infty.$$

The Paley–Wiener theorem and the Plancherel formula hold for the Cherednik transform $\mathcal{F}$ on $\mathfrak{a}$.

Theorem 2.1. *$\mathcal{F}$ is an isomorphism of $C_R^\infty(\mathfrak{a})$ onto $H_R^\infty(\mathfrak{a})$ for every $R > 0$ and there exists a measure $\mathrm{d}\nu$ on $\mathfrak{a}$ such that*

$$\int_{\mathfrak{a}} f(x) g(-x)\mathrm{d}\mu(x) = \int_{\mathfrak{a}} \mathcal{F}f(\lambda)\mathcal{F}g(\lambda)\mathrm{d}\nu(\lambda).$$

Let $p_t^W(x,y)$, $t > 0$, denote the W-invariant heat kernel on $\mathfrak{a} \times \mathfrak{a}$ and put $h_t(x) = |W|^{-1} p_{2t}^W(0,x)$. Then h_t is the W-invariant function on $\mathfrak{a}$ such that

$$\mathcal{F}h_t(\lambda) = \mathrm{e}^{-t(\|\lambda\|^2+\|\rho\|^2)}.$$

Moreover, h_t is strictly positive and satisfies for all $t > 0$ and $x \in \mathfrak{a}$,

$$h_t(x) \sim t^{-\gamma-\frac{n}{2}} \prod_{\alpha\in\mathcal{R}_0^+} (1+|(\alpha,x)|)(1+t+|(\alpha,x)|)^{k_\alpha+k_{2\alpha}-1}$$

$$\times\, \mathrm{e}^{-t\|\rho\|^2-(\rho,x^+)-\frac{\|x\|^2}{4t}}, \tag{3}$$

where $\gamma = \sum_{\alpha\in\mathcal{R}_+} k_\alpha$.

3. Miyachi's theorem

We shall obtain an extension of Miyachi's theorem for the Cherednik transform. We define the measure $d_k x$ on $\mathfrak{a}$ by

$$d_k x = \Pi_{\alpha\in\mathcal{R}^+} |\tanh(\alpha,x)|^{2k_\alpha} \prod_{\alpha\in\mathcal{R}_0^+} (1+|(\alpha,x)|)^{k_\alpha+k_{2\alpha}+1} \mathrm{d}x. \tag{4}$$

Here $\Pi_{\alpha\in\mathcal{R}^+}|\tanh(\alpha,x)|^{2k_\alpha}$ corresponds to the asymptotic behavior of $\mathrm{d}\mu(x)$ around $x = 0$ (see (2)), similarly, $\prod_{\alpha\in\mathcal{R}_0^+}(1+|(\alpha,x)|)$ and $\prod_{\alpha\in\mathcal{R}_0^+}(1+|(\alpha,x)|)^{k_\alpha+k_{2\alpha}}$ correspond to the polynomial parts of $G_0(x)$ and $h_t(x)$ with fixed t respectively (see (1), (3)). We put

$$L^\infty(\mathfrak{a}) + L^1(\mathfrak{a}, d_k x) = \{f_1 + f_2\,;\ f_1 \in L^\infty(\mathfrak{a}),\, f_2 \in L^1(\mathfrak{a}, d_k x)\}.$$

Theorem 3.1. *Let $a, b > 0$ and $ab = 1/4$. Suppose f is a measurable function on $\mathfrak{a}_+$ satisfying*

(A) $f(x)h_{1/4a}^{-1}(x) \in L^\infty(\mathfrak{a}) + L^1(\mathfrak{a}, \mathrm{d}_k x)$,

(B) $\displaystyle\int_{\mathfrak{a}} \log^+ \frac{|\mathcal{F}(f)(\lambda)\mathrm{e}^{b\|\lambda\|^2}|}{C(1+\|\lambda\|)^N} \mathrm{d}\lambda < \infty$ *for some* $0 < C, N < \infty$.

Then f is a constant multiple of $h_{1/4a}$.

Proof. The first condition (A) implies that $fh_{1/4a}^{-1} = u + v$, where $u \in L^\infty(\mathfrak{a})$ and $v \in L^1(\mathfrak{a}, d_k x)$ and hence, $f = h_{1/4a}u + h_{1/4a}v$. For the first term, it follows from (1)(a) that for all $\lambda = \xi + i\eta \in \mathfrak{a}_C$,

$$|\mathcal{F}(h_{1/4a}u)(\lambda)| \le \|u\|_\infty \int_{\mathfrak{a}} h_{1/4a}(x) G_\eta(x) \mu(x)\mathrm{d}x$$

$$= c\mathcal{F}h_{1/4a}(i\eta) = c\mathrm{e}^{b\|\eta\|^2}.$$

For the second term, first we suppose that $\max_{w\in W}(w\eta, x) = (\eta, x^+) > (\rho, x^+)$ for $x \in \mathfrak{a}$. Then it follows from (3) and (1)(b), (c) that

$$|\mathcal{F}(h_{1/4a}v)(\lambda)| \le c \int_{\mathfrak{a}} |v(x)|\mathrm{e}^{-(\rho,x^+)-a\|x\|^2} \prod_{\alpha\in\mathcal{R}_0^+} (1+|(\alpha,x)|)^{k_\alpha+k_{2\alpha}}$$

$$\times\, (1+|(\alpha,x)|)\mathrm{e}^{(\eta-\rho,x^+)} \mu(x)\mathrm{d}x$$

$$\leq c \int_{\mathfrak{a}} |v(x)| \prod_{\alpha \in \mathcal{R}_0^+} (\tanh |(\alpha, x)|)^{2k_\alpha} (1 + |(\alpha, x)|)^{k_\alpha + k_{2\alpha} + 1}$$

$$\times\, \mathrm{e}^{-a\|x^+ - \eta/2a\|^2} \mathrm{d}x \cdot \mathrm{e}^{\|\eta\|^2/4a}$$

$$\leq c\|v\|_{L^1(\mathbf{R}, d_k x)} \mathrm{e}^{b\|\eta\|^2}.$$

On the other hand, if $(\eta, x^+) \leq (\rho, x^+)$ for $x \in \mathfrak{a}$, since $G_\lambda(x)$ is bounded and $\mathrm{e}^{-a\|x\|^2} \leq c\mathrm{e}^{-(\rho, x^+)}$ for $x \in \mathfrak{a}$, it easily follows that

$$|\mathcal{F}(h_{1/4a} v)(\lambda)| \leq c\|v\|_{L^1(\mathbf{R}_+, d_k x)} \leq c\mathrm{e}^{b\|\eta\|^2}.$$

Hence, $\mathcal{F}(f)(\lambda)$ is entire and it satisfies $|\mathcal{F}(f)(\lambda)| \leq c\mathrm{e}^{b\|\eta\|^2}$ for all $\lambda \in \mathfrak{a}_C$ and (B). Here we consider $F(z) = \mathcal{F}(f)(z)\mathrm{e}^{b\|z\|^2}/C$. Then it is easy to see that $F(z)$ satisfies the assumption of the following lemma.

Lemma 3.2. *Suppose $F(z)$ is an entire function on $\mathbb{C}^d$ and there exist constants $A, B > 0$ and $N \geq 0$ such that*

$$|F(z)| \leq A\mathrm{e}^{B\|\Re(z)\|^2} \quad \textit{and} \quad \int_{\mathbb{R}^n} \log^+ \frac{|F(x)|}{(1 + \|x\|)^N} \mathrm{d}x < \infty.$$

Then F is a constant function.

Proof. When $d = 1$, this lemma was obtained by [8]. Let us suppose that $d > 1$ and put $H(z_1) = F(z_1, z_2, \ldots, z_d)$. Then there exists a $E \subset \mathbb{R}^{d-1}$ with positive measure such that for each $(x_2, x_3, \ldots, x_d) \in E$,

$$\int_{-\infty}^{\infty} \log^+ \frac{|H(x_1)|}{C(1 + |x_1|)^N} \mathrm{d}x_1 < \infty.$$

Hence $H(z_1)$ is constant. Since F is entire, F is also constant. Therefore, $\mathcal{F}(f)(\lambda)\mathrm{e}^{b\|\lambda\|^2}/C$ is constant and $\mathcal{F}(f)(\lambda) = c\mathrm{e}^{-b\|\lambda\|^2}$. This implies that $f(x) = ch_{1/4a}(x)$. □

Remark 3.3. The process of the above proof is the same as in [4] for the Jacobi transform on $[0, \infty)$. Since the Jacobi function $\phi_\lambda^{\alpha,\beta}(x)$ satisfies the Harish-Chandra integral formula (see §3 of [6]), for example, $\phi_\lambda(x) = \int_K \mathrm{e}^{(i\lambda - \rho)(H(a_x k)}\mathrm{d}k$ in the case of symmetric spaces, it follows that $|\phi_\lambda^{\alpha,\beta}(x)| \leq c\mathrm{e}^{(\Im\lambda - \rho)x}$ if $x > 0$ and $\Im\lambda - \rho \geq 0$ (see Lemma 11 of [6]). Hence we can remove the term $\prod_{\alpha \in \mathcal{R}_0^+}(1 + |(\alpha, x)|)$ from (4) (see §3 of [4]). However, in our setting, it is not known that $F_\lambda(x)$ satisfies the Harish-Chandra integral formula. Hence the estimate (1)(c) is used even if $(\Im\lambda - \rho, x^+) \geq 0$.

4. Beurling's theorem

In this section we shall consider a version of Beurling's theorem for the Cherednik transform under some assumptions. As in §1, several generalized Beurling's theorems are known for non-Euclidean Fourier transforms $\mathcal{F}$, and in each case, to prove the generalized

Beurling's theorem, one uses the fact that $\mathcal{F}$ is compatible with a convolution structure such that $\mathcal{F}(f*g) = \mathcal{F}f\cdot\mathcal{F}g$. Hence, in what follows, we suppose a hypergroup structure on $\mathfrak{a}_+$, that is, there exists a positive function $K(x, y, z)$ on $\mathfrak{a}_+^3$ for which

$$\text{(A)} \qquad F_\lambda(x)F_\lambda(y) = \int_{\mathfrak{a}_+} F_\lambda(z)K(x, y, z)\mathrm{d}\mu(z).$$

As is well-known, the Jacobi functions on $[0, \infty)$ and the spherical functions on semisimple Lie groups satisfy this property. However, for the Dunkl kernels, K are not positive in the generic case. By using the kernel $K(x, y, z)$, we can define a translation $f_y(x)$ by

$$f_y(x) = \int_{\mathfrak{a}_+} f(z)K(x, y, z)\mathrm{d}\mu(z) = \int_{\mathfrak{a}_+} \mathcal{F}f(\lambda)F_\lambda(y)F_\lambda(x)\mathrm{d}\nu(\lambda)$$

and a convolution $f * g(x)$ by

$$f * g(x) = \int_{\mathfrak{a}_+} f(y)g_x(y)\mathrm{d}\mu(y) = \int_{\mathfrak{a}_+} \mathcal{F}f(\lambda)\mathcal{F}g(\lambda)F_\lambda(x)\mathrm{d}\nu(\lambda)$$

for suitable functions f, g on $\mathfrak{a}_+$. Especially it follows that

$$\mathcal{F}(f * g)(\lambda) = \mathcal{F}f(\lambda)\mathcal{F}g(\lambda)$$

for $\lambda \in \mathfrak{a}$. Then the following lemma easily follows.

Lemma 4.1. *For $t > 0$ and $z, y \in \mathfrak{a}$,*

$$\mathrm{e}^{-t\|y\|^2}\int_{\mathfrak{a}_+} (h_t)_z(x)F_y(x)\mathrm{d}\mu(x) = c_t F_y(z).$$

Under the assumption (A) we shall prove the following.

Theorem 4.2. *Suppose that $f \in L^2(\mathfrak{a})$ is W-invariant and satisfies*

$$\int\int_{\mathfrak{a}_+\times\mathfrak{a}_+} |f(x)||\mathcal{F}f(y)|F_y(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y) < \infty. \tag{5}$$

Then $f = 0$.

Proof. We use the same arguments used in the proof of Proposition 2.2 of [2]. As in their first step, we see that $f \in L^1(\mathfrak{a}_+, \mathrm{d}\mu)$ and $\mathcal{F}f \in L^1(\mathfrak{a}_+, \mathrm{d}\nu)$. We put $g = f * h_{\frac{1}{2}}$ and show that $g = 0$. Since $\mathcal{F}f \in L^1(\mathfrak{a}_+, \mathrm{d}\nu)$, it follows that

$$\int_{\mathfrak{a}_+} \mathcal{F}g(y)\mathrm{e}^{\frac{1}{2}\|y\|^2}\mathrm{d}\nu(y) < \infty.$$

and since $f \in L^1(\mathfrak{a}_+, \mathrm{d}\mu)$, it follows that $\|\mathcal{F}f\|_\infty < \infty$ and thus,

$$|\mathcal{F}g(y)| \le c\mathrm{e}^{-\frac{1}{2}\|y\|^2}. \tag{6}$$

Lemma 4.1 and the fact that $\mathcal{F}(h_t)(iy)\mathcal{F}(h_t)(y) = 1$ yield that

$$\begin{aligned}
&\iint_{\mathfrak{a}_+\times\mathfrak{a}_+} |g(x)||\mathcal{F}g(y)|F_y(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\
&\quad \le \iint_{\mathfrak{a}_+\times\mathfrak{a}_+} |f|*h_t(x)|\mathcal{F}f(y)|\mathcal{F}(h_t)(y)F_y(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\
&\quad = c_t\iint_{\mathfrak{a}_+\times\mathfrak{a}_+} \mathcal{F}(|f|)(iy)\mathcal{F}(h_t)(iy)|\mathcal{F}f(y)|\mathcal{F}(h_t)(y)\mathrm{d}\nu(y)\\
&\quad = c_t\iint_{\mathfrak{a}_+\times\mathfrak{a}_+} \mathcal{F}(|f|)(iy)|\mathcal{F}f(y)|\mathrm{d}\mu(y)\\
&\quad = c_t\iint_{\mathfrak{a}_+\times\mathfrak{a}_+} |f(x)||\mathcal{F}f(y)|F_y(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y) < \infty.
\end{aligned}$$

Moreover, it follows from (6) and $F_y(x) \le c\mathrm{e}^{\|x\|\|y\|}$ that for each $c > 2$,

$$\begin{aligned}
&\int_{\|x\|\le R}\int_{\mathfrak{a}_+} |g(x)||\mathcal{F}g(y)|F_y(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\
&\quad = \int_{\|x\|\le R}|g(x)|\left(\int_{\|y\|\ge cR}+\int_{\|y\|<cR}\right)|\mathcal{F}g(y)|F_y(x)\mathrm{d}\nu(y)\mathrm{d}\mu(x)\\
&\quad \le \int_{\|x\|\le R}|g(x)|\Bigg(\int_{\|y\|\ge cR} c\mathrm{e}^{(\frac{1}{c}-\frac{1}{2})\|y\|^2}\mathrm{d}\nu(y)\\
&\qquad + \int_{\|y\|<cR}|\mathcal{F}g(y)|F_y(x)\mathrm{d}\nu(y)\Bigg)\mathrm{d}\mu(x)\\
&\quad \le c\|g\|_{L^1(\mathfrak{a}_+,\mathrm{d}\mu)} + \int_{\|x\|\le R}\int_{\|y\|<cR}|g(x)||\mathcal{F}g(y)|F_y(x)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\\
&\quad < \infty.
\end{aligned}$$

We note that (6) implies that g admits an holomorphic extension to $\mathfrak{a}_{\mathbb{C}}$ and $|g(z)| \le c\mathrm{e}^{\frac{1}{2}\|x\|^2}$. Moreover, for all $x \in \mathfrak{a}_+$ and $\mathrm{e}^{i\theta}$ of modulus 1,

$$\begin{aligned}
|g(\mathrm{e}^{i\theta}x)| &\le \int_{\mathfrak{a}_+}|\mathcal{F}g(y)||F_{-i\mathrm{e}^{i\theta}x}(y)|\mathrm{d}\nu(y)\\
&\le \int_{\mathfrak{a}_+}|\mathcal{F}g(y)|F_{\sin\theta\cdot x}(y)\mathrm{d}\nu(y)\\
&\le c\int_{\mathfrak{a}_+}|\mathcal{F}g(y)|F_x(y)\mathrm{d}\nu(y),
\end{aligned}$$

where, in the last step we use the asymptotic behavior of $F_y(x)$, $x, y \in \mathfrak{a}_+$ obtained Remark 3.1 of [13]. Since g and $\mathcal{F}g$ are W-invariant, all integrals over $\mathfrak{a}_+$ that appeared in the previous arguments can be replaced by the ones over $\mathfrak{a}$. Then, by defining a function G on $\mathfrak{a}$ as

$$G(z) = \int_0^{z_1}\cdots\int_0^{z_d} g(u)(g)(iu)\mathrm{d}\mu(u),$$

we can deduce that $g = 0$ as in the proof of Proposition 2.2 of [2]. □

Remark 4.3 In order to obtain a generalized Beurling's theorem for the Cherednik transform as in Theorem 1.1 of [2], we need to estimate the following integral:

$$e^{-t\|y\|^2}\int_{\mathfrak{a}_+}\frac{(h_t)_z(x)}{(1+\|x\|)^N}F_y(x)\mathrm{d}\mu(x).$$

At present we have no idea to estimate this integral.

References

[1] Ben Farah S and Mokni K, Uncertainty principle and the $L^p - L^q$-version of Morgan's theorem on some groups, *Russ. J. Math. Phys.* **10** (2003) 245–260

[2] Bonami A, Demange B and Jaming P, Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms, *Rev. Mat. Iberoamericana* **19** (2003) 23–55

[3] Daher R and Hamad S L, An $L^p - L^q$-version of Morgan's theorem for the Dunkl-Cherednik transform, *Math. Sci. Res. J.* **13** (2009) 208–212

[4] Daher R and Kawazoe T, Generalized of Hardy's theorem for Jacobi transform, *Hiroshima J. Math.* **36** (2006) 331–337

[5] Daher R, Kawazoe T and Mejjaoli H, A generalization of Miyachi's theorem, *J. Math. Soc. Japan* **61** (2009) 551–558

[6] Flensted-Jensen M, Paley-Wiener type theorems for a differential operator connected with symmetric spaces, *Ark. Mat.* **10** (1972) 143–162

[7] Hörmander L A, A uniqueness theorem of Beurling for Fourier transform pairs, *Ark. Mat.* **29** (1991) 237–240

[8] Miyachi A, A generalization of theorem of Hardy, Harmonic Analysis Seminar held at Izunagaoka, Shizuoka-Ken Japan (1997) 44–51

[9] Morgan G W, A note on Fourier transforms, *J. London Math. Soc.* **9** (1934) 187–192

[10] Opdam E, Harmonic analysis for certain representations of graded Hecke algebras, *Acta. Math.* **175** (1995) 75–121

[11] Opdam E, Lectures on Dunkl Operators for Real and Complex Reflection Groups, MSJ Memoirs, Vol. 8, Mathematical Society of Japan (2000)

[12] Ray S K and Sarkar R P, A theorem of Beurling and Hörmander on Damek–Ricci spaces, *Adv. Pure Appl. Math.* **1** (2010) 65–79

[13] Schapira B, Contributions to the hypergeometric function theory of Heckman and Opdam: Sharp estimates, Schwartz space, heat kernel, *Geom. Funct. Anal.* **18** (2008) 222–250