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Finsler metrics with constant (or scalar) flag curvature

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Abstract. By finding Killing vector fields of general Bryant's metric we give a lot of new Finsler metrics of constant (or scalar) flag curvature and determine their scalar curvature.

Keywords. Finsler metric; constant flag curvature; navigation representation; general (α, β)-metric.

1. Introduction

Killing fields on a Finsler manifold M are vector fields induced by local 1-parameter group of isometric transformations of M. Such vector fields are solutions of Killing equations on M. They are the natural extension of Killing fields on a Riemannian manifold and thereby are important in both mathematics and physics.

For instance, Li, Chang and Mo related some Killing fields of Finsler metrics to the symmetry of very special relativity (VSR for short). They find that the isometric group of a class of (α, β)-manifold is the same with the symmetry of VSR [6]. The very special relativity is an interesting theory of investigating the violation of Lorentz invariance which is developed by Cohen and Glashow [5]. In [9], the authors showed that the Killing navigation representation has the flag curvature preserving property. In particular, it preserves scalar (or constant) flag curvature.

The flag curvature is the most important Riemannian quantity in Finsler geometry because it is an analogue of sectional curvature in Riemannian geometry. Furthermore, Finsler metrics of constant flag curvature (or scalar curvature and dim $n \geq 3$) are the natural extension of Riemannian metrics of constant sectional curvature.

One of the fundamental problems in Finsler geometry is to study and classify Finsler metrics of constant (or scalar) curvature. Many Finslerian geometers have made effort to study Finsler metrics of constant (or scalar) curvature [1,3,7,8,11]. For instance, in [8], by using projectively flat (α, β)-metrics due to Mo and Yu [10], the author obtained a lot of new Finsler metric of scalar (or constant) flag curvature in terms of Killing shortest time problem. A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesics are straight in $\mathcal{U}$.

The aim of this paper is to construct Killing fields of general Bryant's metrics (see Proposition 4.1) and then manufacture new Finsler metrics of scalar curvature (see Theorem

4.2). In particular, we show that there is at least an $\frac{n(n+1)}{2}$-dimensional family of new Finsler metrics of positive constant flag curvature. Precisely we prove the following:

Theorem 1.1. *Let* $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and*

$$\Phi = \mathrm{Im}\frac{-\langle \lambda x + a, y\rangle + i\sqrt{(\mathrm{e}^{i\varphi} + |\lambda x + a|^2)|y|^2 - \langle \lambda x + a, y\rangle^2}}{\mathrm{e}^{i\varphi} + |\lambda x + a|^2}$$

be a general (α, β)*-metric on an the open subset* $\mathcal{U}$ *at origin in* $\mathbb{R}^n$ *where* $a \in \mathbb{R}^n$ *is an arbitrary constant vector and* λ *an arbitrary non-zero constant. Assume that* V *is a vector field on* $\mathcal{U}$ *defined by* (4.3) *where* Q *is skew-symmetric and satisfies* (4.4) *and* $\Phi(x, V_x) < 1$. *Then Finsler metric* F *is given by*

$$\Phi\left(x, \frac{y}{F(x, y)} + V_x\right) = 1, \quad \forall x \in \mathcal{U}, \quad y \in T_x\mathcal{U}$$

which is of positive constant flag curvature λ^2.

Finsler metrics in the form $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ are called *general* (α, β)-metrics where α is a Riemannian metric and β is a 1-form (for definition, see §2).

One of our main approach is to determine the flag curvature of the projectively flat general (α, β)-metrics due to Yu and Zhu which include Bryant's metrics [14].

It is well-known that Bryant's metrics on $\mathbb{S}^n$ have constant flag curvature $K = 1$ [2]. However, to prove directly the fact that $K = 1$, one may need a faster computer [12].

By exploring some interesting properties of elementary functions introduced by Chern and Shen [4] we verify directly that Bryant's metrics have constant flag curvature 1. After that we extend Chern–Shen's elementary functions which preserve their nice properties and determine thereby the scalar flag curvature of Yu–Zhu's projectively flat general (α, β)-metrics (see Theorem 3.1). On the basis of such formula of scalar flag curvature, we determine finally the flag curvature in Theorem 4.2 and Theorem 1.1.

2. Some lemmas

Let $|\ ,\ |$ and $\langle\ ,\ \rangle$ be the standard Euclidean norm and inner product in $\mathbb{R}^n$. Let $\Omega = B^n(r)$, where $r = 1/\sqrt{-\mu}$ if $\mu < 0$ and $r = +\infty$ if $\mu \geq 0$. A function f defined on $T\Omega$ can be expressed as $f(x^1, \ldots, x^n; y^1, \ldots, y^n)$. We use the following notation:

$$f_0 = \frac{\partial f}{\partial x^i} y^i.$$

Lemma 2.1. *On* $T\Omega\backslash\{0\}$, *define*

$$\alpha := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2},$$

$$\beta := \frac{\langle a, y\rangle}{\sqrt{1 + \mu|x|^2}} + \frac{\lambda - \mu\langle a, x\rangle}{(\sqrt{1 + \mu|x|})^3}\langle x, y\rangle, \quad b := \|\beta\|_\alpha$$

where $a \in \mathbb{R}^n$ is a constant vector and λ is a constant. Then

(i) $(\alpha^2)_0 = -\frac{4\mu\langle x,y\rangle}{\omega^2}\alpha^2$;
(ii) $\beta_0 = \frac{\zeta}{\omega}\alpha^2 - \frac{2\mu\langle x,y\rangle}{\omega^2}\beta$;
(iii) $(b^2)_0 = \frac{2\zeta}{\omega}\beta$;

where $\omega := \sqrt{1+\mu|x|^2}$ and $\zeta := \lambda - \mu\langle a, x\rangle$.

Proof. By direct calculations one obtains

$$\left(\frac{|y|^2}{\omega^2}\right)_0 = -\frac{2\mu|y|^2\langle x, y\rangle}{\omega^4}, \tag{2.1}$$

$$\left(\frac{\langle x, y\rangle^2}{\omega^4}\right)_0 = \frac{2\langle x, y\rangle}{\omega^4}\left(|y|^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}\right), \tag{2.2}$$

$$\left(\frac{\langle a, y\rangle}{\omega}\right)_0 = -\frac{\mu\langle a, y\rangle\langle x, y\rangle}{\omega^3}, \tag{2.3}$$

$$\left(\frac{\zeta\langle x, y\rangle}{\omega^3}\right)_0 = \frac{\zeta|y|^2 - \mu\langle a, y\rangle\langle x, y\rangle}{\omega^3} - \frac{3\mu\zeta\langle x, y\rangle^2}{\omega^5}, \tag{2.4}$$

$$\left(\frac{|x|^2}{\omega^2}\right)_0 = \frac{2\langle x, y\rangle}{\omega^4}, \tag{2.5}$$

$$\left(\frac{\langle a, x\rangle}{\omega^2}\right)_0 = \frac{\langle a, y\rangle}{\omega^2} - \frac{2\mu\langle a, x\rangle\langle x, y\rangle}{\omega^4}, \tag{2.6}$$

$$\left(\frac{\langle a, x\rangle^2}{\omega^2}\right)_0 = 2\langle a, x\rangle\left(\frac{\langle a, y\rangle}{\omega^2} - \frac{\mu\langle a, x\rangle\langle x, y\rangle}{\omega^4}\right). \tag{2.7}$$

(i) By using (2.1) and (2.2) we obtain

$$\begin{aligned}
\left(\alpha^2\right)_0 &= \left(\frac{|y|^2}{\omega^2} - \frac{\mu\langle x, y\rangle^2}{\omega^4}\right)_0 \\
&= \left(\frac{|y|^2}{\omega^2}\right)_0 - \mu\left(\frac{\langle x, y\rangle^2}{\omega^4}\right)_0 \\
&= -\frac{2\mu|y|^2\langle x, y\rangle}{\omega^4} - \mu\frac{2\langle x, y\rangle}{\omega^4}\left(|y|^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}\right) \\
&= \frac{4\mu}{\omega^4}\langle x, y\rangle\left(\frac{\mu\langle x, y\rangle^2}{\omega^2 - |y|^2}\right) = -4\mu\langle x, y\rangle\frac{\alpha^2}{\omega^2}.
\end{aligned}$$

(ii) Similar to the proof of (i) where we use (2.3) and (2.4) instead of (2.1) and (2.2).
(iii) Similar to the proof of (i) where we use (2.6) and (2.7) instead of (2.1) and (2.2).

In the following we explore some interesting properties of elementary functions A, B, U, V and E in Lemma 2.2.

Lemma 2.2. *Let* $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *On* $T\Omega \setminus \{0\}$, *define*

$$A := (\alpha^2\cos\varphi + b^2\alpha^2 - \beta^2)^2 + (\alpha^2\sin\varphi)^2,$$
$$B := \alpha^2\cos\varphi + b^2\alpha^2 - \beta^2, \quad U := \beta\sin\varphi,$$
$$V := \beta(\cos\varphi + b^2), \quad E := b^4 + 2b^2\cos\varphi + 1.$$

Then

(i) $A_0 = -\frac{8\mu\langle x,y\rangle}{\omega}A$;
(ii) $B_0 = -\frac{4\mu\langle x,y\rangle}{\omega^2}B$;
(iii) $U_0 = \sin\varphi\frac{\zeta\alpha^2}{\omega} - \frac{2\mu\langle x,y\rangle}{\omega^2}U$;
(iv) $V_0 = \frac{\zeta}{\omega}B + \frac{3\zeta}{\omega}\beta^2 - \frac{2\mu\langle x,y\rangle}{\omega^2}V$;
(v) $E_0 = \frac{4\zeta}{\omega}V$;

where $\alpha,\ \beta,\ b,\ \omega$ *and* ζ *are defined in Lemma* 2.1.

Proof. First of all, we show (ii).

$$\begin{aligned} B_0 &= [\alpha^2(\cos\varphi + b^2) - \beta^2]_0 \\ &= (\alpha^2)_0(\cos\varphi + b^2) + \alpha^2(b^2)_0 - 2\beta\beta_0 \\ &= -\frac{4\mu\langle x, y\rangle}{\omega^2}\alpha^2(\cos\varphi + b^2) + \alpha^2\frac{2\zeta}{\omega}\beta - 2\beta\left(\frac{\zeta}{\omega}\alpha^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}\beta\right) \\ &= -\frac{4\mu\langle x, y\rangle}{\omega^2}(\alpha^2\cos\varphi + b^2\alpha^2 - \beta^2) = -\frac{4\mu\langle x, y\rangle}{\omega^2}B, \end{aligned}$$

from Lemma 2.1.

To show (i), we use (ii) and Lemma 2.1 and we obtain

$$\begin{aligned} A_0 &= [B^2 + (\alpha^2\sin\varphi)^2]_0 \\ &= 2BB_0 + 2\alpha^2(\alpha^2)_0\sin^2\varphi \\ &= 2B\left(-\frac{4\mu\langle x, y\rangle}{\omega^2}B\right) + 2\alpha^2\left(-\frac{4\mu\langle x, y\rangle}{\omega^2}\alpha^2\right)\sin^2\varphi \\ &= -\frac{8\mu\langle x, y\rangle}{\omega^2}(B^2 + \alpha^4\sin^2\varphi) = -\frac{8\mu\langle x, y\rangle}{\omega^2}A. \end{aligned}$$

To show (iii), from Lemma 2.1(ii) we have

$$U_0 = (\beta\sin\varphi)_0$$

$$
\begin{aligned}
&= \beta_0 \sin\varphi \\
&= \left(\frac{\zeta}{\omega}\alpha^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}\beta\right)\sin\varphi \\
&= \sin\varphi\frac{\zeta\alpha^2}{\omega} - \frac{2\mu\langle x, y\rangle}{\omega^2}U.
\end{aligned}
$$

We show (iv) by using Lemma 2.1 and we get

$$
\begin{aligned}
V_0 &= [\beta(\cos\varphi + b^2)]_0 \\
&= \beta_0(\cos\varphi + b^2) + \beta(b^2)_0 \\
&= \left(\frac{\zeta}{\omega}\alpha^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}\beta\right)(\cos\varphi + b^2) + \frac{2\zeta}{\omega^2}\beta^2 \\
&= \frac{\zeta}{\omega}\left(\alpha^2\cos\varphi + b^2\alpha^2\right) + \frac{2\zeta}{\omega^2}\beta^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}\beta(\cos\varphi + b^2) \\
&= \frac{\zeta}{\omega}\left(\alpha^2\cos\varphi + b^2\alpha^2 - \beta^2\right) + \frac{3\zeta}{\omega}\beta^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}V \\
&= \frac{\zeta}{\omega}B + \frac{3\zeta}{\omega}\beta^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}V.
\end{aligned}
$$

Then we show (v) using Lemma 2.1(iii) and we have

$$
\begin{aligned}
E_0 &= (b^4 + 2b^2\cos\varphi + 1)_0 \\
&= 2b^2(b^2)_0 + 2(b^2)_0\cos\varphi \\
&= 2(b^2 + \cos\varphi)\frac{2\zeta}{\omega}\beta \\
&= \frac{4\zeta}{\omega}\beta(b^2 + \cos\varphi) = \frac{4\zeta}{\omega}V.
\end{aligned}
$$

Remark. When $\mu = 0$, $\lambda = 1$ and $a = 0$, elementary functions A, B, U, V and E are defined in [4] and they satisfy

$$A_0 = B_0 = 0, \quad U_0 = \sin\varphi|y|^2, \quad V_0 = B + 3\langle x, y\rangle^2, \quad E_0 = 4V.$$

The following Lemma 2.3 gives some closed relations among these elementary functions.

Lemma 2.3. *Suppose that α, β, b, A, B, U, V and E are defined in Lemma* 2.1 *and Lemma* 2.2. *Then*

(i) $U^2 + V^2 = E\beta^2$;
(ii) $U : V = \sin\varphi : (\cos\varphi + B^2)$;
(iii) $(\frac{B}{2} + \frac{U^2}{E})^2 + (\frac{UV}{E} - \frac{\alpha^2\sin\varphi}{2})^2 = \frac{A}{4}$.

Proof. (i) By the definitions of U, V and E we have

$$\begin{aligned} U^2 + V^2 &= \beta^2 \sin^2\varphi + \beta^2(\cos\varphi + b^2)^2 \\ &= \beta^2(\sin^2\varphi + \cos^2\psi + 2b^2\cos\varphi + b^4) \\ &= \beta^2(1 + 2b^2\cos\varphi + b^4) = E\beta^2. \end{aligned}$$

(ii) Similar to the proof of (i).
(iii) From (i) we have

$$\begin{aligned} &\left(\frac{B}{2} + \frac{U^2}{E}\right)^2 + \left(\frac{UV}{E} - \frac{\alpha^2\sin\varphi}{2}\right)^2 \\ &\quad= \frac{B^2 + (\alpha^2\sin\varphi)^2}{4} + \frac{U^4 + U^2V^2}{E^2} + \frac{BU^2}{E} - \frac{UV\alpha^2\sin\varphi}{E}, \\ &\quad= \frac{A}{4} + (I), \end{aligned}$$

where

$$\begin{aligned} (I) &= \frac{U^2}{E}\beta^2 + \frac{BU^2}{E} - \frac{UV\alpha^2\sin\varphi}{E} \\ &= \frac{U}{E}[(\beta^2 + B)U - V\alpha^2\sin\varphi] \\ &= \frac{U}{E}[\alpha^2(\cos\varphi + b^2)U - V\alpha^2\sin\varphi] \\ &= \frac{U\alpha^2}{E}[U(\cos\varphi + b^2) - V\sin\varphi] = 0 \end{aligned}$$

for which we have used (ii).

COROLLARY 2.4

Define

$$\epsilon = sgn\left(\frac{\alpha^2\sin\varphi}{2} - \frac{UV}{E}\right).$$

Then

$$\frac{\alpha^2\sin\varphi}{2} - \frac{UV}{E} = \epsilon DQE,$$

where

$$D = \sqrt{\frac{\sqrt{A} + B}{2E} + \left(\frac{U}{E}\right)^2}, \qquad Q = \sqrt{\frac{\sqrt{A} - B}{2E} - \left(\frac{U}{E}\right)^2}.$$

Proof. By using the definition of ϵ and Lemma 2.3(iii) we obtain

$$\begin{aligned}\epsilon DQ &= \epsilon\sqrt{\left[\frac{\sqrt{A}}{2E}+\left(\frac{B}{2E}+\frac{U^2}{E^2}\right)\right]\left[\frac{\sqrt{A}}{2E}-\left(\frac{B}{2E}-\frac{U^2}{E^2}\right)\right]}\\ &= \epsilon\sqrt{\frac{A}{4E^2}-\left(\frac{B}{2E}+\frac{U^2}{E^2}\right)}\\ &= \frac{\epsilon}{E}\sqrt{\frac{A}{4}-\left(\frac{B}{2}+\frac{U}{E}\right)^2}=\frac{1}{E}\left(\frac{\alpha^2\sin\varphi}{2}-\frac{UV}{E}\right).\end{aligned}$$

Lemma 2.5. *Suppose D and Q are given in Corollary* 2.4. *Then*

(i) $(\frac{U}{E})_0 = \frac{2\epsilon\zeta}{\omega}DQ - \frac{2\zeta UV}{\omega E^2} - \frac{2\mu\langle x,y\rangle}{\omega^2}\frac{U}{E}$;

(ii) $(\frac{V}{E})_0 = \frac{\zeta(B+3\beta^2)}{\omega E} - \frac{4\zeta}{\omega}(\frac{V}{E})^2 - \frac{2\mu\langle x,y\rangle}{\omega^2}\frac{V}{E}$;

(iii) $(\sqrt{A}\pm B)_0 = -\frac{4\mu\langle x,y\rangle}{\omega^2}(\sqrt{A}\pm B)$;

(iv) $D_0 = \frac{2\epsilon\zeta}{\omega E}UQ - \frac{2\zeta}{\omega E}VD - \frac{2\mu\langle x,y\rangle}{\omega^2}D$;

(v) $Q_0 = -\frac{2\zeta}{\omega E}VQ - \frac{2\epsilon\zeta}{\omega E}UD - \frac{2\mu\langle x,y\rangle}{\omega^2}Q$.

Proof.

(i) By using Lemma 2.2 and Corollary 2.4 we obtain

$$\begin{aligned}\left(\frac{U}{E}\right)_0 &= \frac{U_0E-UE_0}{E^2}\\ &= \left(\frac{\zeta\alpha^2}{\omega}\sin\varphi-\frac{2\mu\langle x,y\rangle}{\omega^2}U\right)\frac{1}{E}-\frac{4\zeta UV}{\omega E^2}\\ &= (I)-\frac{2\zeta UV}{\omega E^2}-\frac{2\mu\langle x,y\rangle}{\omega^2}\frac{U}{E},\end{aligned} \tag{2.8}$$

where

$$(I)=\frac{2\zeta}{\omega E}\left(\frac{\alpha^2\sin\varphi}{2}-\frac{UV}{E}\right)=\frac{2\zeta}{\omega E}\epsilon DQE=\frac{2\epsilon\zeta}{\omega}DQ.$$

Plugging this into (2.8) yields (i).

(ii) From Lemma 2.2, we have

$$\begin{aligned}\left(\frac{V}{E}\right)_0 &= \frac{V_0E-VE_0}{E^2}\\ &= \left(\frac{\zeta}{\omega}B+\frac{3\zeta}{\omega}\beta^2-\frac{2\mu\langle x,y\rangle}{\omega^2}V\right)\frac{1}{E}-\frac{4\zeta}{\omega}V\frac{V}{E^2}\\ &= \frac{\zeta(B+3\beta^2)}{\omega E}-\frac{4\zeta}{\omega}\left(\frac{V}{E}\right)^2-\frac{2\mu\langle x,y\rangle}{\omega^2}\frac{V}{E}.\end{aligned}$$

(iii) By using Lemma 2.2, we have

$$2\sqrt{A}(\sqrt{A})_0 = A_0 = -\frac{8\mu\langle x, y\rangle}{\omega^2}A.$$

It follows that

$$(\sqrt{A})_0 = -\frac{4\mu\langle x, y\rangle}{\omega^2}\sqrt{A}$$

from which together with Lemma 2.2(ii) we obtain

$$\begin{aligned}(\sqrt{A} \pm B)_0 &= (\sqrt{A})_0 + B_0\\ &= -\frac{4\mu\langle x, y\rangle}{\omega^2}\sqrt{A} \pm \left(-\frac{4\mu\langle x, y\rangle}{\omega}B\right)\\ &= -\frac{4\mu\langle x, y\rangle}{\omega^2}(\sqrt{A} \pm B).\end{aligned}$$

(iv) By (i), (ii) and Lemma 2.2(v), we see that

$$\begin{aligned}2DD_0 &= (D^2)_0\\ &= \left[\frac{\sqrt{A}+B}{2E} + \left(\frac{U}{E}\right)^2\right]_0\\ &= \frac{(\sqrt{A}+B)_0E - (\sqrt{A}+B)E_0}{2E^2} + 2\frac{U}{E}\left(\frac{U}{E}\right)_0,\\ DD_0 &= \frac{1}{4E}\left[-\frac{4\mu\langle x, y\rangle}{\omega^2}(\sqrt{A}+B)\right] - \frac{1}{4E^2}(\sqrt{A}+B)\frac{4\zeta}{\omega}V\\ &\quad + \frac{U}{E}\left(\frac{2\epsilon\zeta}{\omega}DQ - \frac{2\zeta UV}{\omega E^2} - \frac{2\mu\langle x, y\rangle}{\omega^2}\frac{U}{E}\right)\\ &= -\frac{2\mu\langle x, y\rangle}{\omega^2}\left(\frac{\sqrt{A}+B}{2E} + \frac{U^2}{E^2}\right)\\ &\quad - \frac{2\zeta V}{\omega E}\left(\frac{\sqrt{A}+B}{2E} + \frac{U^2}{E^2}\right) + \frac{2\epsilon\zeta}{\omega E}UQD\\ &= \frac{2\epsilon\zeta}{\omega E}UQD - \frac{2\zeta}{\omega E}VD^2 - \frac{2\mu\langle x, y\rangle}{\omega^2}D,\end{aligned}$$

thus we obtain (iv).

(v) Similar to the proof of (iii).

A Finsler metric on a manifold M in the following form is said to be *general* (α, β) *type*

$$F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right),$$

where α is a Riemannian metric, β is a 1-form on M, $b = \|\beta\|_\alpha$ and $\phi(\rho, s)$ is a C^∞ function (see [14]). The following lemma will be used in §4.

Lemma 2.6. *Let V be a vector field on a manifold M and $\Phi := \alpha\phi(b^2, \frac{\beta}{\alpha})$ a general (α, β)-metric on M. Then*

$$X_V(\Phi) = \left(\phi - \frac{\beta}{\alpha}\phi_2\right) X_V(\alpha) + \phi_2 X_V(\beta) + \alpha\phi_1 X_V(b^2), \tag{2.9}$$

where

$$\phi = \phi\left(b^2, \frac{\beta}{\alpha}\right), \ \ \phi_1 = \frac{\partial}{\partial\rho}\phi\left(b^2, \frac{\beta}{\alpha}\right), \ \ \phi_2 = \frac{\partial}{\partial s}\phi\left(b^2, \frac{\beta}{\alpha}\right)$$

and

$$X_V = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i} = V + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i}. \tag{2.10}$$

Proof. By straightforward computation one obtains

$$\begin{aligned}
\frac{\partial\Phi}{\partial x^i} &= \frac{\partial}{\partial x^i}\left[\alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)\right] \\
&= \phi\frac{\partial\alpha}{\partial x^i} + \alpha\left[\phi_1\frac{\partial b^2}{\partial x^i} + \phi_2\frac{\frac{\partial\beta}{\partial x^i}\alpha - \frac{\partial\alpha}{\partial x^i}\beta}{\alpha^2}\right] \\
&= \left(\phi - \frac{\beta}{\alpha}\phi_2\right)\frac{\partial\alpha}{\partial x^i} + \phi_2\frac{\partial\beta}{\partial x^i} + \alpha\phi_1\frac{\partial b^2}{\partial x^i}.
\end{aligned} \tag{2.11}$$

Similarly, we have

$$\frac{\partial\Phi}{\partial y^i} = \left(\phi - \frac{\beta}{\alpha}\phi_2\right)\frac{\partial\alpha}{\partial y^i} + \phi_2\frac{\partial\beta}{\partial y^i} + \alpha\phi_1\frac{\partial b^2}{\partial y^i}. \tag{2.12}$$

By using (2.10), (2.11) and (2.12) we have

$$\begin{aligned}
X_V(\Phi) &= V^i\left[\left(\phi - \frac{\beta}{\alpha}\phi_2\right)\frac{\partial\alpha}{\partial x^i} + \phi_2\frac{\partial\beta}{\partial x^i} + \alpha\phi_1\frac{\partial b^2}{\partial x^i}\right] \\
&\quad + y^j\frac{\partial V^i}{\partial x^j}\left[\left(\phi - \frac{\beta}{\alpha}\phi_2\right)\frac{\partial\alpha}{\partial y^i} + \phi_2\frac{\partial\beta}{\partial y^i} + \alpha\phi_1\frac{\partial b^2}{\partial y^i}\right] \\
&= \left(\phi - \frac{\beta}{\alpha}\phi_2\right)\left(V^i\frac{\partial\alpha}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial\alpha}{\partial y^i}\right) \\
&\quad + \phi_2\left(V^i\frac{\partial\beta}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial\beta}{\partial y^i}\right) + \alpha\phi_1\left(V^i\frac{\partial b^2}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial b^2}{\partial y^i}\right) \\
&= \left(\phi - \frac{\beta}{\alpha}\phi_2\right)X_V(\alpha) + \phi_2 X_V(\beta) + \alpha\phi_1 X_V(b^2).
\end{aligned}$$

3. Scalar curvature

In this section we are going to determine the scalar flag curvature of Yu–Zhu's projectively flat general (α, β)-metrics [14]. A Finsler metric F on a manifold M is said to be of *scalar curvature* if the flag curvature $\mathbf{K}(P,\ y) = \mathbf{K}(x,\ y)$ is a scalar function on the slit tangent bundle $TM\backslash\{0\}$. In particular, F is said to have constant (flag) curvature if the flag curvature $\mathbf{K}(P,\ y) =$ constant. In general, the flag curvature is a function $\mathbf{K}(P,\ y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$.

PROPOSITION 3.1

Define

$$F = \sqrt{\frac{\sqrt{A}+B}{2E} + \left(\frac{U}{E}\right)^2} + \frac{U}{E},$$

where $A,\ B,\ U$ and E are defined in Lemma 2.2. *Then F has a projective factor*

$$P = -\frac{\mu\langle x,\ y\rangle}{\omega^2} + \frac{\zeta}{\omega}\left(\epsilon Q - \frac{V}{E}\right) \tag{3.1}$$

and a scalar flag curvature

$$K = \frac{\zeta^2}{\omega^2} + \frac{\mu}{F^2}\left[\alpha^2 + \beta\left(\epsilon Q - \frac{V}{E}\right)\right]. \tag{3.2}$$

Proof. Recently Yu and Zhu has proved that F is projectively flat [14], equivalently, its geodesic coefficients G^i satisfies the following [13]:

$$G^i = Py^i,$$

where P is its projective factor. Hence P and its scalar flag curvature K are given by

$$P = \frac{F_0}{2F}, \qquad K = \frac{P^2 - P_0}{F^2}. \tag{3.3}$$

By Lemma 2.5, we see that

$$\begin{aligned} F_0 &= \left(D + \frac{U}{E}\right)_0 \\ &= D_0 + \left(\frac{U}{E}\right)_0 \\ &= \frac{2\epsilon\zeta}{\omega E}UQ - \frac{2\zeta}{\omega E}VD - \frac{2\mu\langle x,\ y\rangle}{\omega^2}D + \frac{2\epsilon\zeta}{\omega}DQ - \frac{2\zeta UV}{\omega E^2} - \frac{2\mu\langle x,\ y\rangle}{\omega^2}\frac{U}{E} \\ &= -\frac{2\mu\langle x,\ y\rangle}{\omega^2}\left(D + \frac{U}{E}\right) + \frac{2\zeta}{\omega}\left(\epsilon Q - \frac{V}{E}\right)\left(D + \frac{U}{E}\right) \\ &= -\frac{2\mu\langle x,\ y\rangle}{\omega^2}F + \frac{2\zeta}{\omega}\left(\epsilon Q - \frac{V}{E}\right)F. \end{aligned}$$

Together with the first equation of (3.3) we obtain (3.1). By direct calculation, one obtains

$$\left(\frac{\langle x,\ y\rangle}{\omega^2}\right)_0 = \frac{|y|^2\omega^2 - 2\mu\langle x,\ y\rangle^2}{\omega^4}, \qquad \left(\frac{\zeta}{\omega}\right)_0 = -\mu\frac{\langle a,\ y\rangle\omega^2 + \zeta\langle x,\ y\rangle}{\omega^3}.$$

Together with (3.1) and Lemma 2.5 yields

$$
\begin{aligned}
P^2 - P_0 = & \left[\frac{\zeta}{\omega}\left(\epsilon Q - \frac{V}{E}\right) - \frac{\mu\langle x, y\rangle}{\omega^2}\right]^2 - \left(\frac{\zeta}{\omega}\right)_0\left(\epsilon Q - \frac{V}{E}\right) \\
& - \frac{\zeta}{\omega}\left[\epsilon Q_0 - \left(\frac{V}{E}\right)_0\right] + \mu\left(\frac{\langle x, y\rangle}{\omega^2}\right)_0 \\
= & \frac{\zeta^2}{\omega^2}\left(Q^2 - 2\epsilon Q\frac{V}{E} + \frac{V^2}{E^2}\right) - 2\frac{\mu\zeta\langle x, y\rangle}{\omega^3}\left(\epsilon Q - \frac{V}{E}\right) \\
& + \frac{\mu^2\langle x, y\rangle^2}{\omega^4} + \mu\frac{\langle a, y\rangle\omega^2 + \zeta\langle x, y\rangle}{\omega^3}\left(\epsilon Q - \frac{V}{E}\right) \\
& + \mu\frac{|y|^2\omega^2 - 2\mu\langle x, y\rangle^2}{\omega^4} \\
& - \frac{\zeta}{\omega}\left[\epsilon\left(-\frac{2\zeta}{\omega E}VQ - \frac{2\epsilon\zeta}{\omega E}UD - \frac{2\mu\langle x, y\rangle}{\omega^2}Q\right)\right. \\
& \qquad \left. - \frac{\zeta(B + 3\beta^2)}{\omega E} + \frac{4\zeta}{\omega}\left(\frac{V}{E}\right)^2 + \frac{2\mu\langle x, y\rangle}{\omega^2}\frac{V}{E}\right] \\
= & \frac{\zeta^2}{\omega^2}\left(\frac{\sqrt{A} - B}{2E} - \frac{U^2}{E^2}\right) - 3\frac{\zeta^2}{\omega^2}\frac{V^2}{E^2} - \frac{\mu\zeta\langle x, y\rangle}{\omega^3}\left(\epsilon Q - \frac{V}{E}\right) \\
& + \frac{\mu\langle a, y\rangle}{\omega}\left(\epsilon Q - \frac{V}{E}\right) + \mu\alpha^2 + 2\frac{\zeta^2 UD}{\omega^2 E} \\
& + \frac{\zeta^2(B + 3\beta^2)}{\omega^2 E} + \frac{2\mu\zeta\langle x, y\rangle}{\omega^3}\left(\epsilon Q - \frac{V}{E}\right) \\
= & \text{(I)} + \text{(II)},
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
\text{(I)} = & \frac{\zeta^2}{\omega^2}\left(\frac{\sqrt{A} - B}{2E} - \frac{U^2}{E^2}\right) - 3\frac{\zeta^2}{\omega^2}\frac{V^2}{E^2} + 2\frac{\zeta^2 UD}{\omega^2 E} + \frac{\zeta^2(B + 3\beta^2)}{\omega^2 E} \\
= & \frac{\zeta^2}{\omega^2}\left[\frac{\sqrt{A} + B}{2E} + \frac{U^2}{E^2} + \frac{3}{E}\left(\beta^2 - \frac{U^2}{E} - \frac{V^2}{E}\right) + 2\frac{U}{E}D + \left(\frac{U}{E}\right)^2\right] \\
= & \frac{\zeta^2}{\omega^2}\left[D^2 + 2\frac{U}{E}D + \left(\frac{U}{E}\right)^2\right] \\
= & \frac{\zeta^2}{\omega^2}\left(D + \frac{U}{E}\right)^2 = \frac{\zeta^2}{\omega^2}F^2
\end{aligned}
\tag{3.5}
$$

and

$$\begin{aligned}(\mathrm{II}) &= \mu\left[\frac{\langle a, y\rangle}{\omega}\left(\epsilon Q - \frac{V}{E}\right) + \frac{\zeta\langle x, y\rangle}{\omega^3}\left(\epsilon Q - \frac{V}{E}\right) + \alpha^2\right] \\ &= \mu\left[\alpha^2 + \beta\left(\epsilon Q - \frac{V}{E}\right)\right], \end{aligned} \tag{3.6}$$

where we have used Lemma 2.3. Plugging (3.5) and (3.6) into (3.4) yields

$$P^2 - P_0 = \frac{\zeta^2}{\omega^2}F^2 + \mu\left[\alpha^2 + \beta\left(\epsilon Q - \frac{V}{E}\right)\right].$$

Together with the second equation of (3.3) yields (3.2).

Recall that all projectively flat Finsler metrics are of scalar curvature (Proposition 6.1.3 of [4]). Thus we obtain the following:

Theorem 3.1. *Let $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\Omega = B^n(r)$, where $r = 1/\sqrt{-\mu}$ if $\mu < 0$ and $r = +\infty$ if $\mu \geq 0$. Define*

$$F = \sqrt{\frac{\sqrt{A} + B}{2E} + \left(\frac{U}{E}\right)^2} + \frac{U}{E}, \tag{3.7}$$

where A, B, U and E are defined in Lemma 2.2. F is a Finsler metric on Ω with scalar flag curvature

$$K = \frac{(\lambda - \mu\langle a, x\rangle)^2}{1 + \mu|x|^2} + \frac{\mu}{F^2}\left[\alpha^2 + \beta\left(\epsilon\sqrt{\frac{\sqrt{A} - B}{2E} - \left(\frac{U}{E}\right)^2} - \frac{V}{E}\right)\right],$$

where V and ϵ are defined in Lemma 2.2 and Corollary 2.4.

Let us take a look at the special case: when $\mu = 0$,

$$\alpha = |y|, \quad \beta = \langle \lambda x + a, y\rangle$$

Then we have the following:

COROLLARY 3.2

Let $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $a \in \mathbb{R}^n$ be an arbitrary constant vector and λ an arbitrary constant. The following Finsler metric is projectively flat with non-negative constant flag curvature $\mathbf{K} = \lambda^2$,

$$F = \mathrm{Im}\frac{-\langle \lambda x + a, y\rangle + i\sqrt{(\mathrm{e}^{i\varphi} + |\lambda x + a|^2)|y|^2 - \langle \lambda x + a, y\rangle^2}}{\mathrm{e}^{i\varphi} + |\lambda x + a|^2}.$$

Remark. When $\lambda = 1$ and $a = 0$, then F was constructed by Bryant with constant flag curvature $K = 1$ [2].

4. Killing fields and new Finsler metrics of scalar curvature

In this section we are going to prove our main results. A vector field V on a Finsler manifold (M, Φ) is said to be a *Killing field* if its flow is isometric, equivalently, $X_V(\Phi) = 0$ where X_V is given in (2.10) [8,9].

PROPOSITION 4.1

Let $\Phi = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Define

$$\alpha := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}, \tag{4.1}$$

$$\beta := \frac{\langle a, y\rangle}{\sqrt{1 + \mu|x|^2}} + \frac{\lambda - \mu\langle a, x\rangle}{(\sqrt{1 + \mu|x|})^3}\langle x, y\rangle, \tag{4.2}$$

where λ, μ are constants and $a \in \mathbb{R}^n$ is a constant vector. Let V denote a vector field on $\mathcal{U}$ defined by

$$V_x = xQ \qquad \text{at } x \in \mathscr{U}, \tag{4.3}$$

where Q is skew-symmetric and satisfies that

$$Qa^T = 0. \tag{4.4}$$

Then V is of Killing type with respect to Φ.

Proof. It is easy to see that

$$a^{ij} = \omega^2(\delta^{ij} + \mu x^i x^j), \tag{4.5}$$

where $(a^{ij}) = (a_{ij})^{-1}$, $\alpha^2 = a_{ij}y^i y^j$, ω is defined in Lemma 2.1.

By (4.2), we obtain

$$b_i = \frac{(\lambda - \mu\langle a, x\rangle)x^i + \omega^2 a^i}{\omega^3},$$

where $\beta = b_i y^i$. It follows that

$$\begin{aligned}\omega^6 b_i b_j &= \lambda^2 x^i x^j + \lambda\omega^2(a^i x^j + a^j x^i) - 2\lambda\mu\langle a, x\rangle x^i x^j \\ &\quad + \omega^4 a^i a^j - \mu\omega^2\langle a, x\rangle(a^i x^j + a^j x^i) + \mu^2\langle a, x\rangle^2 x^i x^j.\end{aligned}$$

Together with (4.5) we get

$$\begin{aligned}\omega^4 b^2 &= \omega^4 a^{ij} b_i b_j \\ &= \omega^6(\delta^{ij} + \mu x^i x^j)b_i b_j \\ &= \lambda^2\omega^2|x|^2 + 2\lambda\omega^2\langle a, x\rangle + |a|^2\omega^4 - \mu\omega^2\langle a, x\rangle^2.\end{aligned}$$

We obtain

$$b^2 = \frac{\lambda^2}{\omega^2}|x|^2 + \frac{2\lambda}{\omega^2}\langle a, x\rangle + |a|^2 - \frac{\mu}{\omega^2}\langle a, x\rangle^2. \tag{4.6}$$

By Lemma 3.2 in [8], we have

$$\frac{\partial\alpha}{\partial x^i} = \mu\frac{(2\mu\langle x, y\rangle^2 - \omega^2|y|^2)x^i - \omega^2\langle x, y\rangle y^i}{\omega^4\sqrt{\omega^2|y|^2 - \mu\langle x, y\rangle^2}}, \tag{4.7}$$

$$\frac{\partial\alpha}{\partial y^i} = \frac{\omega^2 y^i - \mu\langle x, y\rangle x^i}{\omega^2\sqrt{\omega^2|y|^2 - \mu\langle x, y\rangle^2}}. \tag{4.8}$$

Let $Q = (q_{ij})$. Since Q is anti-symmetric, we have

$$(x^j y^i + x^i y^j)q_{ji} = x^j y^i(q_{ji} + q_{ij}) = 0 \tag{4.9}$$

and

$$x^i x^j q_{ij} = y^i y^j q_{ij} = 0. \tag{4.10}$$

Together with (2.10), (4.7), (4.8) and (4.9) we get

$$\begin{aligned} X_V(\alpha) &= x^j q_{ji}\frac{\partial\alpha}{\partial x^i} + y^j q_{ji}\frac{\partial\alpha}{\partial y^i} \\ &= x^j q_{ji}\mu\frac{(2\mu\langle x, y\rangle^2 - \omega^2|y|^2)x^i - \omega^2\langle x, y\rangle y^i}{\omega^4\sqrt{\omega^2|y|^2 - \mu\langle x, y\rangle^2}} \\ &\quad + y^j q_{ji}\frac{\omega^2 y^i - \mu\langle x, y\rangle x^i}{\omega^2\sqrt{\omega^2|y|^2 - \mu\langle x, y\rangle^2}} \\ &= -(x^j y^i + x^i y^j)q_{ji}\frac{\mu\langle x, y\rangle}{\omega^2\sqrt{\omega^2|y|^2 - \mu\langle x, y\rangle^2}} = 0. \end{aligned} \tag{4.11}$$

By simple calculations, we have

$$\frac{\partial\omega^2}{\partial x^i} = 2\mu x^i, \quad \frac{\partial\omega^2}{\partial y^i} = 0, \tag{4.12}$$

$$\frac{\partial}{\partial x^i}\frac{\langle a, y\rangle}{\omega} = -\frac{\mu\langle a, y\rangle x^i}{\omega^3}, \quad \frac{\partial}{\partial y^i}\frac{\langle a, y\rangle}{\omega} = \frac{a^i}{\omega}. \tag{4.13}$$

By using the first formula of (4.12) we obtain

$$\frac{\partial\omega^3}{\partial x^i} = 3\mu\omega x^i. \tag{4.14}$$

Together with (4.13) yields

$$\begin{aligned} \frac{\partial}{\partial x^i}\frac{\zeta\langle x, y\rangle}{\omega^3} &= \frac{1}{\omega^6}\{[\langle x, y\rangle(-\mu a^i) + \zeta y^i]\omega^3 - \zeta\langle x, y\rangle 3\mu\omega x^i\} \\ &= \frac{-\mu\langle x, y\rangle a^i + \zeta y^i}{\omega^3} - \mu\frac{3\zeta\langle x, y\rangle x^i}{\omega^5}, \end{aligned} \tag{4.15}$$

$$\frac{\partial}{\partial y^i}\frac{\zeta\langle x,y\rangle}{\omega^3}=\frac{\zeta}{\omega^3}\frac{\partial}{\partial y^i}\langle x,y\rangle=\frac{\zeta x^i}{\omega^3}. \tag{4.16}$$

The condition (4.4) implies that

$$\sum a^i q_{ji}=0. \tag{4.17}$$

Combining this with (4.2), (4.13), (4.15) and (4.16) we have

$$\begin{aligned}
X_V(\beta) &= \left(x^j q_{ji}\frac{\partial}{\partial x^i}+y^j q_{ji}\frac{\partial}{\partial y^i}\right)\left(\frac{\langle a,y\rangle}{\omega}+\frac{\zeta\langle x,y\rangle}{\omega^3}\right)\\
&= x^j q_{ji}\frac{\partial}{\partial x^i}\frac{\langle a,y\rangle}{\omega}+y^j q_{ji}\frac{\partial}{\partial y^i}\frac{\langle a,y\rangle}{\omega}\\
&\quad +x^j q_{ji}\frac{\partial}{\partial x^i}\frac{\zeta\langle x,y\rangle}{\omega^3}+y^j q_{ji}\frac{\partial}{\partial y^i}\frac{\zeta\langle x,y\rangle}{\omega^3}\\
&= x^j q_{ji}\left(-\frac{\mu\langle a,y\rangle x^i}{\omega^3}\right)+y^j q_{ji}\frac{a^i}{\omega}\\
&\quad +x^j q_{ji}\left(-\frac{\mu\langle a,y\rangle a^i+\zeta y^j}{\omega^3}-\mu\frac{3\zeta\langle x,y\rangle x^i}{\omega^5}\right)+y^j q_{ji}\frac{\zeta x^i}{\omega^3}\\
&= (x^j y^i+x^i y^j)\frac{\zeta}{\omega^3}=0.
\end{aligned} \tag{4.18}$$

By simple calculations, we have

$$\frac{\partial}{\partial x^i}\left(\frac{|x|^2}{\omega^2}\right)=\frac{2x^i(\omega^2-\mu|x|^2)}{\omega^4}=\frac{2x^i}{\omega^4}, \tag{4.19}$$

$$\frac{\partial}{\partial x^i}\left(\frac{\langle a,x\rangle}{\omega^2}\right)=\frac{a^i}{\omega^2}-\frac{2\mu\langle a,x\rangle x^i}{\omega^4} \tag{4.20}$$

and

$$\frac{\partial}{\partial x^i}\left(\frac{\langle a,x\rangle^2}{\omega^2}\right)=2\langle a,x\rangle\left(\frac{a^i}{\omega^2}-\frac{\mu\langle a,x\rangle x^i}{\omega^4}\right). \tag{4.21}$$

Note that b^2 is a scalar function on $\mathcal{U}$. From (2.10), (4.6), (4.19), (4.20) and (4.21), we obtain

$$\begin{aligned}
X_V(b^2) &= x^j q_{ji}\frac{\partial}{\partial x^i}(b^2)\\
&= x^j q_{ji}\frac{\partial}{\partial x^i}\left(\frac{\lambda^2}{\omega^2}|x|^2+\frac{2\lambda}{\omega^2}\langle a,x\rangle+|a|^2-\frac{\mu}{\omega^2}\langle a,x\rangle^2\right)\\
&= \lambda^2 x^j q_{ji}\frac{\partial}{\partial x^i}\left(\frac{|x|^2}{\omega^2}\right)+2\lambda x^j q_{ji}\frac{\partial}{\partial x^i}\left(\frac{\langle a,x\rangle}{\omega^2}\right)\\
&\quad -\mu x^j q_{ji}\frac{\partial}{\partial x^i}\left(\frac{\langle a,x\rangle^2}{\omega^2}\right)
\end{aligned}$$

$$= \lambda^2 x^j q_{ji} \frac{2x^i}{\omega^4} + 2\lambda x^j q_{ji} \left(\frac{a^j}{\omega^2} - \frac{2\mu\langle a, x\rangle x^i}{\omega^4} \right)$$
$$- \mu x^j q_{ji} \left[2\langle a, x\rangle \left(\frac{a^i}{\omega^2} - \frac{\mu\langle a, x\rangle x^i}{\omega^4} \right) \right] = 0.$$

Together with (2.9), (4.11) and (4.18) we have $X_V(\Phi) = 0$. Therefore V is a Killing field of Φ.

Let $\Phi = \alpha\phi(b^2, \beta/\alpha)$ be a general (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Assume that α and β satisfies (4.1) and (4.2). Proposition 4.1 tells us $V := xQ$ is a Killing field of Φ, where Q satisfies $Q^T = -Q$ and $Qa^T = 0$. Define

$$\phi(\rho, s) = \text{Im} \frac{-s + i\sqrt{e^{i\varphi} + \rho - s^2}}{e^{i\varphi} + \rho}. \tag{4.22}$$

Then the elementary function expression of Φ is given in (3.7) and Theorem 3.1 implies that Φ is of scalar curvature. Let V be a Killing field of Φ on $\mathcal{U}$ with $\Phi(x, V_x) < 1$. Define a new Finsler metric F by [1,8,9]

$$\Phi\left(x, \frac{y}{F(x, y)} + V_x\right) = 1, \quad \forall x \in \mathcal{U}, \quad y \in T_x\mathcal{U}. \tag{4.23}$$

By using Mo–Huang's Theorem 1.1 and Corollary 7.1 [9], we obtain F is also of scalar curvature. Moreover, its scalar flag curvature K_{F} is given by

$$K_{\text{F}}(x, y) = K_{\Phi}(x, y - \Phi(x, y)V_x). \tag{4.24}$$

Combining this with Theorem 3.1 we have the following:

Theorem 4.2. *Let*

$$\Phi = \text{Im} \frac{-\beta + i\sqrt{\alpha^2 e^{i\varphi} + b^2\alpha^2 - \beta^2}}{e^{i\varphi} + b^2}$$

be a general-(α, β) metric on an open subset $\mathcal{U}$ at origin in $\mathbb{R}^n$, where α and β are defined by (4.1) *and* (4.2)*. Assume that V is a vector field on $\mathcal{U}$ defined by* (4.3)*, where Q is skew-symmetric and satisfies that* (4.4) *and $\Phi(x, V_x) < 1$. Then Finsler metric F given by* (4.23) *is of scalar curvature with the flag curvature*

$$K_{\text{F}} = \frac{(\lambda - \mu\langle a, x\rangle)^2}{1 + \mu|x|^2} + \frac{\mu}{F^2} \left[\alpha^2 + \beta \left(\epsilon\sqrt{\frac{\sqrt{A} - B}{2E} - \left(\frac{U}{E}\right)^2} - \frac{V}{E} \right) \right]$$

where A, B, U, V, E and ϵ are defined in Lemma 2.2 *and Corollary* 2.4, *where*

$$\alpha = \alpha(x, y - \Phi(x, y)V_x), \qquad \beta = \beta(x, y - \Phi(x, y)V_x).$$

Proof of Theorem 1.1. We take $\mu = 0$ in Theorem 4.2. Then Φ is the Finsler metric defined in Corollary 3.2. Together with (4.24) yields Theorem 1.1.

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