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Holomorphic two-spheres in the complex Grassmann manifold $G(k, n)$

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Abstract. In this paper, we study the non-degenerate holomorphic S^2 in the complex Grassmann manifold $G(k, n)$, $2k \leq n$, by the method of moving frame. For a non-degenerate holomorphic one, there exists globally defined positive functions $\lambda_1, \ldots, \lambda_k$ on S^2. We first show that the holomorphic S^2 in $G(k, 2k)$ is totally geodesic if these λ_i are all equal. Conversely, for any totally geodesic immersion f from S^2 into $G(k, n)$, we prove that $f(S^2) \subset G(k, 2k)$ up to $U(n)$-transformation, $\lambda_i = \frac{1}{\sqrt{k}}$, the Gaussian curvature $K = \frac{4}{k}$ and f is given by $(z_0, z_1) \mapsto (z_0 I_k, z_1 I_k, 0)$, in terms of homogeneous coordinate.

Keywords. Moving frame; totally geodesic; Gaussian curvature.

1. Introduction

The complex Grassmann manifold $G(k, n)$ is the set of all k-dimensional complex linear subspace of $\mathbb{C}^n$, particularly, $G(1, n)$ is the complex projective space $\mathbb{C}P^{n-1}$. In 1980's, much attention has been paid to the investigation of minimal surfaces in the complex projective space $\mathbb{C}P^n$. Inspired by the work of Din and Zakrzewski [4], Eells and Wood [5] first gave a rigorous mathematical treatment of the classification of the harmonic two-spheres in $\mathbb{C}P^n$. Later, Ramanathan [9] gave a classification of harmonic two-spheres in complex Grassmann manifold $G(2, 4)$. Wolfson [10] gave a description of harmonic two-spheres in the general complex Grassmann manifold $G(k, n)$. However, all of them paid attention to the construction of minimal (also harmonic) two-spheres in $G(k, n)$, not the geometric properties of submanifolds in $G(k, n)$.

Many perfect geometric properties of minimal two-spheres in $\mathbb{C}P^n$ have been obtained by Bolton *et al.* [1] and Bando and Ohnita [2]. However, the geometric structure of $G(k, n)$ is much more complicated when $k > 1$. For example, when $k > 1$, $G(k, n)$ does not have constant holomorphic sectional curvature and the rigidity of holomorphic curves in $G(k, n)$ fails. For this reason, it is hard to generalize some perfect results of submanifolds in $\mathbb{C}P^n$ to the ones of submanifolds in general complex Grassmannianns. For some special cases (for example, the pseudo-holomorphic two-spheres), there are some results that are similar to the ones in $\mathbb{C}P^n$. For details, one can refer to [8] and [12].

In fact, the present paper is a generalization of our early paper [11] in which we studied the non-degenerate holomorphic immersions from S^2 into the complex Grassmann manifold $G(2,4)$, and proved that the immersion is totally geodesic if $\lambda_1 = \lambda_2$. The study of totally geodesic submanifold is very important in the theory of submanifold. Therefore, we wish to generalize the result in [11] to the one in $G(k, 2k)$, and completely characterize the holomorphic and anti-holomorphic totally geodesic two-spheres in $G(k,n)$. Since the Grassmann manifold $G(k,n)$ and $G(n-k,n)$ are isometric, without loss of generality, we can assume $2k \leq n$.

Throughout this paper we will agree on the following ranges of indices:

$$1 \leq A, B, \ldots \leq n, \quad k+1 \leq \alpha, \beta, \gamma, \ldots \leq n,$$
$$1 \leq i, j, \ldots \leq k, \quad 2k+1 \leq \xi, \eta, \ldots \leq n.$$

And also, we use the summation convention, and the convention $\bar{a}_{i\bar{\alpha}} = a_{\bar{i}\alpha}$, etc.

2. Preliminaries

In this section, we recall some basic formulas of minimal surfaces into complex Grassmannians, and we take the same notations in [3]. The complex Grassmann manifold $G(k,n)$ is isomorphic to the symmetric space $\frac{U(n)}{U(k)\times U(n-k)}$, here $U(n)$ is the unitary group. So, it is convenient to use the method of moving frame to study the minimal surfaces in $G(k,n)$.

Let $Z = (Z_1, Z_2, \ldots, Z_n)^T$ be the elements of $U(n)$, with

$$dZ_A = \omega_{A\bar{B}} Z_B, \tag{2.1}$$

here $\omega_{A\bar{B}}$ are the Maurer–Cartan forms of $U(n)$. They are skew-Hermitian, i.e.

$$\omega_{A\bar{B}} + \omega_{\bar{B}A} = 0. \tag{2.2}$$

Taking the exterior derivative of (2.1), we get the Maurer–Cartan equations of $U(n)$:

$$d\omega_{A\bar{B}} = \sum_C \omega_{A\bar{C}} \wedge \omega_{C\bar{B}}, \tag{2.3}$$

which play an important role in our calculations. The form

$$ds_G^2 = \sum_{i,\alpha} \omega_{i\bar{\alpha}} \omega_{\bar{i}\alpha}, \tag{2.4}$$

defines a positive definite Hermitian metric on $G(k,n)$, which is Kählerian.

Let $f : S^2 \longrightarrow G(k,n)$ be a conformal immersion. Locally, the metric ds^2 on S^2 induced by f can be written as

$$ds^2 = f^* ds_G^2 = \phi\bar{\phi}, \tag{2.5}$$

where ϕ is a local complex-valued one-form of type $(1,0)$ on S^2, which is defined up to a complex factor of absolute value 1.

To express the situation analytically we choose, locally, a field of unitary frame Z_A, such that Z_i span $f(x)$ and Z_α span $f^\perp(x)$ respectively. We set

$$f^*\omega_{A\bar{B}} = a_{A\bar{B}}\phi + b_{A\bar{B}}\bar{\phi}. \tag{2.6}$$

For convenience, we denote by $A := (a_{i\bar{\alpha}})_{k\times(n-k)}$ and $B := (b_{i\bar{\alpha}})_{k\times(n-k)}$. The geometric meanings of A and B are very clear, i.e., $\partial f(x) = [a_{1\bar{\alpha}}Z_\alpha, \ldots, a_{k\bar{\alpha}}Z_\alpha]$, $\bar{\partial} f(x) = [b_{1\bar{\alpha}}Z_\alpha, \ldots, b_{k\bar{\alpha}}Z_\alpha]$, see [3] for details.

Remark.

(1) The rank of the matrices A and B are constants on S^2 except at isolated points, one can refer to [3] for details. We say that f is *non-degenerate* if $\operatorname{rank}(A) = k$ and f is *degenerate* if $\operatorname{rank}(A) < k$.
(2) Note that the matrix A is dependant on the choice of the moving frame Z_A and the complex-valued $(1, 0)$-form ϕ, whose singular values (the square roots if the eigenvalues of AA^*) $\lambda_1, \ldots, \lambda_k$ are globally defined functions on S^2. These λ_i are continuous functions on S^2, but their fundamental polynomials are smooth functions on S^2. Similarly for matrix B.

It is known that f is holomorphic if and only if $b_{i\bar{\alpha}} = 0$ for all i and α. From (2.4), (2.5) and (2.6), one has

$$\sum_{i,\alpha} a_{i\bar{\alpha}} b_{\bar{i}\alpha} = 0, \tag{2.7}$$

$$\sum_{i,\alpha} a_{i\bar{\alpha}} a_{\bar{i}\alpha} + b_{i\bar{\alpha}} b_{\bar{i}\alpha} = 1. \tag{2.8}$$

The structure equations of S^2 with respect to the induced metric are

$$\mathrm{d}\phi = -\rho \wedge \phi, \tag{2.9}$$

$$\mathrm{d}\rho = \frac{K}{2} \phi \wedge \bar{\phi}, \tag{2.10}$$

where the purely imaginary one-form ρ (i.e. $\bar{\rho} = -\rho$) is the connection form with respect to the co-frame ϕ, and K is the Gaussian curvature.

Taking the exterior derivatives of (2.6), here we take $A = i$ and $B = \alpha$, together with (2.3) and (2.10), we get

$$Da_{i\bar{\alpha}} \wedge \phi + Db_{i\bar{\alpha}} \wedge \bar{\phi} = 0, \tag{2.11}$$

where

$$Da_{i\bar{\alpha}} = \mathrm{d}a_{i\bar{\alpha}} - \omega_{i\bar{j}} a_{j\bar{\alpha}} + a_{i\bar{\beta}} \omega_{\beta\bar{\alpha}} - a_{i\bar{\alpha}} \rho, \tag{2.12}$$

$$Db_{i\bar{\alpha}} = \mathrm{d}b_{i\bar{\alpha}} - \omega_{i\bar{j}} b_{j\bar{\alpha}} + b_{i\bar{\beta}} \omega_{\beta\bar{\alpha}} + b_{i\bar{\alpha}} \rho. \tag{2.13}$$

Set

$$Da_{i\bar{\alpha}} = p_{i\bar{\alpha}} \phi + q_{i\bar{\alpha}} \bar{\phi},\ Db_{i\bar{\alpha}} = q_{\bar{i}\alpha} \phi + r_{i\bar{\alpha}} \bar{\phi}. \tag{2.14}$$

Then the immersion f is minimal if and only if $q_{i\bar{\alpha}} = 0$, or we write

$$Da_{i\bar{\alpha}} \equiv 0 \mod \phi, \quad \text{or} \quad Db_{i\bar{\alpha}} \equiv 0 \mod \bar{\phi}. \tag{2.15}$$

for short. The quadratic form

$$\Pi^{\mathbb{C}}_{i\alpha} = Da_{i\bar\alpha}\phi + Db_{i\bar\alpha}\bar\phi = p_{i\bar\alpha}\phi\phi + 2q_{i\bar\alpha}\phi\bar\phi + r_{i\bar\alpha}\bar\phi\,\bar\phi, \tag{2.16}$$

is called *complex second fundamental form* of the immersion f with respect to the co-frames $\omega_{i\bar\alpha}$, and f is totally geodesic if and only if $p_{i\bar\alpha} = q_{i\bar\alpha} = r_{i\bar\alpha} = 0$.

The following lemma will be used in our later proof.

Lemma 2.1. *Let U be an open subset of Riemann surface M, and g be a complex-valued smooth function defined on U, and* $\mathrm{d}s^2 = \phi\bar\phi$ *on U. Suppose that g satisfies*

$$\mathrm{d}g \equiv g\psi, \qquad \mathrm{mod}\ \phi,$$

where ψ is a purely imaginary valued one-form (i.e. $\bar\psi = -\psi$), then

$$\Delta_M \log|g|\ \phi\wedge\bar\phi = 2\mathrm{d}\psi$$

away from its zeros, and Δ_M is the Laplace–Beltrami operator with respect to $\mathrm{d}s^2$.

Proof. The proof can be found in [11]. □

3. Holomorphic two-spheres in $G(k, n)$

Let $f : S^2 \longrightarrow G(k, n)$ be a non-degenerate holomorphic immersion, locally, there exists an open neighborhood U around any point p in S^2 and a suitable moving frame $e = (e_1, \dots, e_n) : S^2 \supset U \longrightarrow U(n)$ along f, i.e., $\pi \circ e = f$, such that the pull-back of the Maurer–Cartan forms $e^*\omega$ as follows:

$$\begin{pmatrix} \Omega_{1\bar 1} & A\phi & 0 \\ -A^*\bar\phi & \Omega_{2\bar 2} & \Phi \\ 0 & -\Phi^* & \Psi \end{pmatrix}, \tag{3.1}$$

where $\Omega_{1\bar 1} = \left(\omega_{i\bar j}\right)$, $A = \left(a_{i\overline{k+j}}\right)$, $\Omega_{2\bar 2} = \left(\omega_{k+i\overline{k+j}}\right)$, $\Phi = \left(\omega_{k+i\bar\xi}\right)$, and $\Psi = \left(\omega_{\xi\bar\eta}\right)$.

As before, we denote the singular values of A by λ_i, then we have

Theorem 3.1. *Let f be a non-degenerate holomorphic immersion from S^2 into $G(k, 2k)$, then f is totally geodesic if $\lambda_1 = \cdots = \lambda_k$.*

Proof. Since $\lambda_1 = \cdots = \lambda_k$, we know that $\lambda_i = \frac{1}{\sqrt{k}}$ by identity (2.8), so we can choose a smooth moving frame such that

$$\omega_{i\overline{k+i}} = \lambda_i\phi \quad \text{and} \quad \omega_{i\overline{k+j}} = 0 \text{ if } i \neq j.$$

Taking the exterior derivative on both sides of the identities above, by (2.3), we obtain

$$\begin{aligned} (\omega_{i\bar i} - \omega_{k+i\overline{k+i}} + \rho)\wedge\phi = 0, \\ \omega_{i\bar j}\wedge\omega_{j\overline{k+j}} + \omega_{i\overline{k+i}}\wedge\omega_{k+i\overline{k+j}} = 0, \end{aligned}$$

which imply

$$\omega_{i\bar i} - \omega_{k+i\overline{k+i}} + \rho = 0, \tag{3.2}$$

by the fact that $\omega_{i\bar{i}}$, $\omega_{k+i\overline{k+i}}$ and ρ are purely imaginary 1-form, and

$$\omega_{i\bar{j}} = \omega_{k+i\overline{k+j}}, \tag{3.3}$$

respectively. Due to (3.2) and (3.3), one can check that $p_{i\bar{\alpha}} = q_{i\bar{\alpha}} = 0$ by (2.12). On the other hand, $r_{i\bar{\alpha}} = 0$ by the fact that $b_{i\bar{\alpha}} = 0$. Therefore, f is a totally geodesic immersion. This completes the proof. □

Remark. We do not know whether this theorem is true when $2k < n$. As far as the author's knowledge is concerned, there is no corresponding example or counterexample.

Example. Through direct calculations, one can show that $f : S^2 \longrightarrow G(k, 2k)$ defined by $(z_0, z_1) \mapsto (z_0 I_k, z_1 I_k)$ is totally geodesic with $\lambda_i = \frac{1}{\sqrt{k}}$ and the Gaussian curvature $K = \frac{4}{k}$.

By the Maurer–Cartan structure equation (2.3), we take the exterior derivative on both sides of the following identity:

$$\omega_{i\bar{\xi}} = 0, \tag{3.4}$$

and get

$$\sum_j \omega_{i\overline{k+j}} \wedge \omega_{k+j\bar{\xi}} = 0, \tag{3.5}$$

or equivalently,

$$\sum_j a_{i\overline{k+j}} b_{k+j\bar{\xi}} = 0, \tag{3.6}$$

by the identities (2.3) and (2.6). Since A is a non-degenerate matrix, equation (3.6) implies

$$b_{k+j\bar{\xi}} = 0. \tag{3.7}$$

Since f is a holomorphic immersion and hence minimal, the identities (2.15) tell us that

$$\mathrm{d}A \equiv \Omega_{1\bar{1}} A - A\Omega_{2\bar{2}} + A\rho \mod \phi, \tag{3.8}$$

from which, we obtain

$$\Delta_M \log|\det(A)| = k\left(K - \frac{4}{k}\right) + \sum_{i,\xi} |a_{k+i\bar{\xi}}|^2, \tag{3.9}$$

by the Proposition 1.1 in [12] directly or by Lemma 2.1 and together with the determinant. Here we have used the identity (3.7).

Therefore, we get the following pinching theorem about the Gaussian curvature K, which will be used in the proof of Proposition 3.6.

PROPOSITION 3.2 [12]

Let $f : S^2 \longrightarrow G(k, n)$ be a non-degenerate holomorphic immersion, K be the Gaussian curvature with respect to the induced metric. If $K \geq \frac{4}{k}$ on S^2, then $K = \frac{4}{k}$.

Proof. Due to Hopf's maximum principle, the identity (3.9) implies $K \equiv \frac{4}{k}$ if $K \geq \frac{4}{k}$ on S^2. □

The following corollary is mentioned in our earlier paper [11] without a proof.

COROLLARY 3.3

Let $f : S^2 \longrightarrow G(k, 2k)$ be a non-degenerate holomorphic immersion, K be the Gaussian curvature with respect to induced metric. If $K \geq \frac{4}{k}$ or $K \leq \frac{4}{k}$ on S^2, then $K = \frac{4}{k}$. Particularly, if f has a constant Gaussian curvature, then $K = \frac{4}{k}$.

Proof. Since $n = 2k$, the identity (3.9) becomes

$$\Delta_M \log(|\det(A)|) = k\left(K - \frac{4}{k}\right), \tag{3.10}$$

which implies that $K = \frac{4}{k}$ by Hopf's maximum principle, under the condition $K \geq \frac{4}{k}$ or $K \leq \frac{4}{k}$. □

In the following, we will pay attention to determine the holomorphic totally geodesic two-spheres in $G(k, n)$, namely, we always assume that f is a non-degenerate holomorphic totally geodesic immersion. Then, for any i and α, we have

$$p_{i\bar{\alpha}} = q_{i\bar{\alpha}} = r_{i\bar{\alpha}} = 0, \tag{3.11}$$

namely,

$$\mathrm{d}a_{i\bar{\alpha}} = \omega_{i\bar{j}}a_{j\bar{\alpha}} - a_{i\bar{\beta}}\omega_{\beta\bar{\alpha}} + a_{i\bar{\alpha}}\rho, \tag{3.12}$$

by equation (2.14). Particularly, taking $\alpha = \xi$ in (3.12), we obtain

$$\sum_j a_{i\overline{k+j}}\omega_{k+j\bar{\xi}} = 0 \tag{3.13}$$

which is equivalent to

$$\omega_{k+i\bar{\xi}} = 0, \tag{3.14}$$

by the fact that the matrix A is non-degenerate. Therefore, the identity (3.9) becomes

$$\Delta_M \log|\det(A)| = k\left(K - \frac{4}{k}\right), \tag{3.15}$$

if the non-degenerate holomorphic immersion f is totally geodesic.

According to the theory of matrix, we can assume the matrix $A = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ by choosing a suitable moving frame along f. These λ_i are positive functions on S^2 with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\sum_i \lambda_i^2 = 1$. We wish to prove that they are constants.

PROPOSITION 3.4

The positive functions $\lambda_1, \lambda_2, \ldots, \lambda_k$ defined above are constants.

Proof. Note that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the singular values of the matrix A. Therefore, it is sufficient to prove that all the eigenvalues of $\tilde{A} = AA^*$ are constants.

Set

$$\tilde{A} = \left(\tilde{a}_{i\bar{j}} \right), \tag{3.16}$$

where $\tilde{a}_{i\bar{j}} = \sum_l a_{i\overline{k+l}} a_{\bar{j}k+l}$. It is easily seen that

$$\mathrm{d}\tilde{a}_{i\bar{j}} = \sum_l (\omega_{i\bar{l}} \tilde{a}_{l\bar{j}} + \tilde{a}_{i\bar{l}} \omega_{\bar{j}l}), \tag{3.17}$$

by equation (3.12).

Let $\tilde{A}\begin{pmatrix} i_1 & \cdots & i_r \\ j_1 & \cdots & j_r \end{pmatrix} = (\tilde{a}_{i_s j_t})_{r\times r}$, $1 \le r \le k$, and $i_s \in \{i_1, \ldots, i_r\}$, $j_t \in \{j_1 \ldots, j_r\}$. Then, the r-th fundamental symmetric polynomial of the eigenvalues of $\tilde{A}$ is given by

$$\sigma_r = \sum_{1 \le i_1 < \ldots < i_r \le k} \det \tilde{A} \begin{pmatrix} i_1 & \cdots & i_r \\ i_1 & \cdots & i_r \end{pmatrix}, \tag{3.18}$$

which are smooth functions defined on S^2. By (3.17) and (2.2), for $r = 1$, we have

$$\begin{aligned} \mathrm{d}\sigma_1 &= \mathrm{d}\Big(\sum_i \tilde{a}_{ii}\Big) \\ &= \sum_{i,l} (\omega_{i\bar{l}} \tilde{a}_{l\bar{i}} + \tilde{a}_{i\bar{l}} \omega_{\bar{i}l}) \\ &= \sum_{i,l} \tilde{a}_{l\bar{i}} (\omega_{i\bar{l}} + \omega_{\bar{l}i}) = 0. \end{aligned}$$

Similarly, through directly calculation, we obtain

$$\mathrm{d}\sigma_r = 0. \tag{3.19}$$

In other words, all the fundamental symmetric polynomials $\sigma_1, \sigma_2, \ldots, \sigma_k$ of $\{\lambda_1, \ldots, \lambda_k\}$ are constants. Hence, we prove that all the eigenvalues of $\tilde{A}$ are constants. This completes the proof. □

By Proposition 3.4, we can choose a moving frame such that $a_{i\bar{\alpha}} = 0$ if $\alpha \ne k + i$ and $a_{i\overline{k+i}} = \lambda_i$ are constants. Therefore, the identities (3.12) imply that

$$\omega_{k+i\overline{k+i}} - \omega_{i\bar{i}} = \rho, \tag{3.20}$$

$$\lambda_j \omega_{i\bar{j}} - \lambda_i \omega_{k+i\overline{k+j}} = 0, \; i \ne j, \tag{3.21}$$

or equivalently,

$$\lambda_j a_{i\bar{j}} = \lambda_i a_{k+i\overline{k+j}}, \quad \lambda_j b_{i\bar{j}} = \lambda_i b_{k+i\overline{k+j}}, \tag{3.22}$$

$$\lambda_j a_{j\bar{i}} = \lambda_i a_{k+i\overline{k+j}}, \quad \lambda_j b_{j\bar{i}} = \lambda_i b_{k+j\overline{k+i}}. \tag{3.23}$$

Now, we can proof the theorem below.

Theorem 3.5. *Let $f : S^2 \longrightarrow G(k,n)$ be a non-degenerate holomorphic totally geodesic immersion. Then f has the constant Gaussian curvature $\frac{4}{k}$, with respect to the induced metric. Moreover, $f(S^2) \subset G(k,2k) \hookrightarrow G(k,n)$, up to $U(n)$-transformation, where the imbedding $G(k,2k) \hookrightarrow G(k,n)$ is given by $[e_1 \wedge \cdots \wedge e_k] \mapsto [\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_k]$, and $\tilde{e}_i = (e_i, 0)$.*

Proof. By Proposition 3.4, we know that $\det(A)$ is a constant which implies that $K = \frac{4}{k}$ by identity (3.15). The identities (3.14) tell us that the pull back of Maurer–Cartan forms $e^*\omega$ are

$$\begin{pmatrix} \Omega_{1\bar{1}} & \Omega_{1\bar{2}} & 0 \\ -\Omega^*_{1\bar{2}} & \Omega_{2\bar{2}} & 0 \\ 0 & 0 & \Psi \end{pmatrix}. \tag{3.24}$$

Furthermore, one can choose a moving frame $e : S^2 \supset U \longrightarrow U(n)$ such that $\Psi = 0$, where U is an open subset of S^2. By reading the pull back of the Maurer–Cartan forms (3.24), we know that $e(U) \subset U(2k) \hookrightarrow U(n)$ up to $U(n)$-transformation, according to Theorems 1.3 and 1.4 in [6]. Here $U(2k) \hookrightarrow U(n)$ is given by $A \mapsto \mathrm{diag}\{A, I_{n-2k}\}$. Since S^2 is connected, $f(S^2) \subset G(k,2k)$ up to $U(n)$-transformation. □

Remark. From Theorem 3.5, we know, essentially, studying the non-degenerate holomorphic totally geodesic two-spheres in $G(k,n)$ is just to study the non-degenerate holomorphic totally geodesic two-spheres in $G(k,2k)$.

Let $f : S^2 \longrightarrow G(k,2k)$ be a non-degenerate holomorphic totally geodesic immersion, $e = (e_1, \ldots, e_{2n}) : S^2 \supset U \longrightarrow U(2k)$ be a moving frame along f, which satisfies the pull back of the Maurer–Cartan forms $e^*\omega$ which are

$$\begin{pmatrix} \Omega_{1\bar{1}} & \Omega_{1\bar{2}} \\ -\Omega^*_{1\bar{2}} & \Omega_{2\bar{2}} \end{pmatrix}, \tag{3.25}$$

where $\Omega_{1\bar{1}} = \left(\omega_{i\bar{j}}\right)$, $\Omega_{1\bar{2}} = \mathrm{diag}\{\lambda_1\phi, \ldots, \lambda_k\phi\}$, $\Omega_{2\bar{2}} = \left(\omega_{k+i\overline{k+j}}\right)$, $\lambda_1 \geq \cdots \geq \lambda_k$ are positive constant functions on S^2.

By identities (3.22) and (3.23), interchanging i and j if necessary, we obtain

$$(\lambda_i^2 - \lambda_j^2) a_{i\bar{j}} = 0, \tag{3.26}$$

$$(\lambda_i^2 - \lambda_j^2) a_{k+i\overline{k+j}} = 0, \tag{3.27}$$

$$(\lambda_i^2 - \lambda_j^2) b_{i\bar{j}} = 0, \tag{3.28}$$

$$(\lambda_i^2 - \lambda_j^2) b_{k+i\overline{k+j}} = 0, \tag{3.29}$$

which gives

$$\omega_{i\bar{j}} = 0, \tag{3.30}$$

$$\omega_{k+i\overline{k+j}} = 0, \tag{3.31}$$

when $\lambda_i \neq \lambda_j$. On the other hand, by (3.22) and (3.23), we have

$$\omega_{i\bar{j}} = \omega_{k+i\overline{k+j}}, \tag{3.32}$$

when $\lambda_i = \lambda_j$ and $i \neq j$. Simultaneously, we have

$$\omega_{k+i\overline{k+i}} - \omega_{i\bar{i}} = \omega_{k+j\overline{k+j}} - \omega_{j\bar{j}} \ \text{ if } \ i \neq j, \tag{3.33}$$

by identity (3.20).

About these constants λ_i, there exist some positive integers $k_1, k_2, \ldots, k_q$ and $\sum_{s=1}^{q} k_s = k$ which satisfy

$$\lambda_{k_1+\cdots+k_{s-1}+1} = \cdots = \lambda_{k_1+\cdots+k_s}, \ \lambda_{k_1+\cdots+k_{s-1}} > \lambda_{k_1+\cdots+k_{s-1}+1}. \tag{3.34}$$

From the identities (3.30) and (3.31), it is clear that the pull back of the Maurer–Cartan forms are

$$\begin{pmatrix} \Omega_{1\bar{1}} & & & \tilde{\lambda}_1 I_{k_1}\phi & & \\ & \ddots & & & \ddots & \\ & & \Omega_{q\bar{q}} & & & \tilde{\lambda}_q I_{k_q}\phi \\ -\tilde{\lambda}_1 I_{k_1}\bar{\phi} & & & \Theta_{1\bar{1}} & & \\ & \ddots & & & \ddots & \\ & & -\tilde{\lambda}_q I_{k_q}\bar{\phi} & & & \Theta_{q\bar{q}} \end{pmatrix}. \tag{3.35}$$

Here $\tilde{\lambda}_s := \lambda_{k_1+\cdots+k_{s-1}+1}$, $\Omega_{s\bar{s}} = (\omega_{h\bar{l}})_{k_s\times k_s}$, $\Theta_{s\bar{s}} = (\omega_{k+h\overline{k+l}})_{k_s\times k_s}$, $\omega_{h\bar{l}} = \omega_{k+h\overline{k+l}}$ when $h \neq l$, the range of indices h, l satisfies $k_1 + \cdots + k_{s-1} < h, l \leq k_1 + \cdots + k_s$ and I_{k_s} is the unit matrix of order k_s.

PROPOSITION 3.6

k_q is defined as above, then $q = 1$.

Proof. By reading the pull back of Maurer–Cartan forms (3.35), for each $1 \leq s \leq q$, it is clear that $g_s := [e_{k_1+\cdots+k_{s-1}+1} \wedge \cdots \wedge e_{k_1+\cdots+k_s}] : S^2 \longrightarrow G(k_s, 2k)$ is a holomorphic map with constant Gaussian curvature $K_q = \frac{4}{kk_q\tilde{\lambda}_q^2}$ with respect to the metric induced by g_s. Since $\sum_{s=1}^{q} k_s \cdot \tilde{\lambda}_s^2 = 1$ and $\lambda_1 \geq \cdots \geq \lambda_k$, we have

$$\tilde{\lambda}_q^2 \leq \frac{1}{k}. \tag{3.36}$$

Therefore, the Gaussian curvature $K_q = \frac{4}{kk_q\tilde{\lambda}_q^2} \geq \frac{4}{k_q}$ implies

$$K_q = \frac{4}{k_q}, \ \text{or} \ \tilde{\lambda}_q^2 = \frac{1}{k}, \tag{3.37}$$

by Proposition 3.2. Due to the facts $\sum_{i=1}^{k} \lambda_i^2 = 1$ and $\lambda_1 \geq \cdots \geq \lambda_k$ again, we get $\lambda_1^2 = \cdots = \lambda_k^2 = \frac{1}{k}$, in other words, we prove that $q = 1$. This completes the proof. □

Now we can characterize the non-degenerate holomorphic totally geodesic two-spheres in $G(k, 2k)$ as follows:

Theorem 3.7. *Any non-degenerate holomorphic totally geodesic two-spheres in $G(k, 2k)$, in terms of homogenous coordinate, is given by $(z_0, z_1) \mapsto (z_0 I_k, z_1 I_k)$ up to $U(2k)$-transformation.*

Proof. A suitable moving frame e along f can be chosen such that the pull back of the Maurer–Cartan forms $e^*\omega$ are

$$\begin{pmatrix} \Omega_{1\bar{1}} & \frac{1}{\sqrt{k}} I_k \phi \\ -\frac{1}{\sqrt{k}} I_k \bar{\phi} & \Omega_{2\bar{2}} \end{pmatrix}, \tag{3.38}$$

by Proposition 3.6, where $\Omega_{1\bar{1}} = (\omega_{i\bar{j}})_{k\times k}$, $\Omega_{2\bar{2}} = (\omega_{k+i\overline{k+j}})_{k\times k}$, $\omega_{i\bar{j}} = \omega_{k+i\overline{k+j}}$ when $i \neq j$, and $\omega_{k+i\overline{k+i}} - \omega_{i\bar{i}} = \omega_{k+j\overline{k+j}} - \omega_{j\bar{j}}$.

Set

$$\Gamma := \{\sigma_{ij}, \overline{\sigma}_{ij}, \tau_{ij}, \varphi_{ij}, \overline{\varphi}_{ij}, \psi_{ij}, \overline{\psi}_{ij}\}, \tag{3.39}$$

where $\sigma_{i\bar{j}} = \omega_{i\bar{j}} - \omega_{k+i\overline{k+j}}$, $\tau_{ij} = \omega_{k+i\overline{k+i}} - \omega_{i\bar{i}} - \omega_{k+j\overline{k+j}} + \omega_{j\bar{j}}$, $\varphi_{i\bar{j}} = \omega_{i\overline{k+i}} - \omega_{j\overline{k+j}}$, $\psi_{ij} = \omega_{i\overline{k+j}}$, $i \neq j$. Through direct calculations, one can find

$$\mathrm{d}\sigma_{ij} \equiv 0, \quad \bmod \Gamma, \tag{3.40}$$

$$\mathrm{d}\tau_{ij} \equiv 0, \quad \bmod \Gamma, \tag{3.41}$$

$$\mathrm{d}\varphi_{ij} \equiv 0, \quad \bmod \Gamma, \tag{3.42}$$

$$\mathrm{d}\psi_{ij} \equiv 0, \quad \bmod \Gamma. \tag{3.43}$$

Hence, according to Frobenius' integrable theorem, $\mathfrak{g} := \{\sigma_{ij} = 0, \tau_{ij} = 0, \varphi_{ij} = 0, \psi_{ij} = 0\}$ determines a Lie subalgebra with dimension $k^2 + 3$ in $\mathfrak{u}(2k)$. It is easily checked that the connected analytic subgroup of $U(2k)$ with Lie algebra $\mathscr{G}$ is

$$G = \left\{ \begin{pmatrix} z_0 A & z_1 A \\ -\bar{z}_1 A & \bar{z}_0 A \end{pmatrix} \;\middle|\; A \in U(k),\ |z_0|^2 + |z_1|^2 = 1 \right\}, \tag{3.44}$$

which is isomorphic to $SU(2) \times U(k)$.

According to Theorems 1.3 and 1.4 in [6], we know $e(U)$ is an integral submanifold of $\mathscr{G}$, and thus $e(U)$ is contained in G up to $U(2k)$-transformation. If we identify S^2 with $\mathbb{C}P^1 = \frac{SU(2)}{S(U(1)\times U(1))}$, then by the same argument in [7] on page 100, we obtain that f is given by $(z_0, z_1) \mapsto (z_0 I_k, z_1 I_k)$ up to $U(2k)$-transformation. □

COROLLARY 3.8

Any non-degenerate holomorphic totally geodesic two-spheres in $G(k, n)$ is given by $(z_0, z_1) \mapsto (z_0 I_k, z_1 I_k, 0)$ up to $U(n)$-transformation.

Proof. By Theorem 3.5 and Theorem 3.7. □

Furthermore, about the degenerate holomorphic totally geodesic two-sphere in $G(k, n)$, we have the following.

Theorem 3.9. *Consider a degenerate holomorphic totally geodesic two-spheres in $G(k, n)$ with* $\operatorname{rank}(A) = k_1 < k$. *Then its Gaussian curvature $K = \frac{4}{k_1}$ and the immersion is given by* $(z_0, z_1) \mapsto \begin{pmatrix} z_0 I_{k_1} & 0 & z_1 I_{k_1} & 0 \\ 0 & I_{k-k_1} & 0 & 0 \end{pmatrix}$ *up to $U(n)$-transformation.*

Proof. The proof is similar as before by reading the pullback of Maurer–Cartan forms under a suitable moving frame. □

Remark. There exist some theorems about the anti-holomorphic totally geodesic two-spheres immersed in $G(k, n)$, which correspond to the above results, except that one needs to change z_0, z_1 to $\bar{z}_0, \bar{z}_1$ respectively.

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