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On the mean curvature of semi-Riemannian graphs in semi-Riemannian warped products

ZONGLAO ZHANG

College of Mathematics and Information Science, Wenzhou University, Wenzhou, Zhejiang 325035, People's Republic of China
E-mail: zonglao@163.com; zonglao@sohu.com

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Abstract. We investigate the mean curvature of semi-Riemannian graphs in the semi-Riemannian warped product $M \times_f \mathbb{R}_\varepsilon$, where M is a semi-Riemannian manifold, $\mathbb{R}_\varepsilon$ is the real line $\mathbb{R}$ with metric $\varepsilon \mathrm{d}t^2$ ($\varepsilon = \pm 1$), and $f : M \to \mathbb{R}^+$ is the warping function. We obtain an integral formula for mean curvature and some results dealing with estimates of mean curvature, among these results is a Heinz–Chern type inequality.

Keywords. Mean curvature; semi-Riemannian graphs; semi-Riemannian warped product; Heinz–Chern type inequality.

1. Introduction

The study of the mean curvature of graph-like surfaces has a long history, but it is still a hot research area in differential geometry. In 1955, Heinz [8] investigated the graph in $\mathbb{R}^3$ defined by the function $\varphi(x, y)$, $x^2 + y^2 < R^2$, and proved that if the mean curvature of the graph satisfies $|H| \geq c > 0$, then $c \leq \frac{1}{R}$. Thus if H is constant, then $|H| \leq \frac{1}{R}$. In 1965 and 1966, Chern [6] and Flanders [7] independently extended Heinz inequality to graphs defined by function $z = z(x_1, \ldots, x_n)$, respectively. These inequalities are known as the Heinz–Chern inequalities for graphs.

In 1989, Salavessa [14] considered the graph Γ_h in the Riemannian product $M \times N$ defined by a smooth function $h : M \to N$ (where M, N are Riemannian manifolds), and obtained the following result:

Theorem 1.1 [14]. *If Γ_h has parallel mean curvature with $c = \|H\|$, then for each oriented compact domain $D \subset M$, we have $c \leq \frac{1}{m}\frac{A(\partial D)}{V(D)}$, thus $c \leq \frac{1}{m}\mathfrak{h}(M)$, where $m = \dim M$, $\mathfrak{h}(M)$ denotes the Cheeger constant of M, $A(\partial D)$ the area of ∂D, and $V(D)$ the volume of D.*

In 2008, Salavessa [15] considered space-like graphs Γ_h of simple products $M \times N$ with semi-Riemannian metric $g_M \times -g_N$, where M and N are Riemannian manifolds, with metric g_M and g_N, respectively. A Heinz–Chern type inequality was obtained for the case $\dim N = 1$.

Some other works relevant to this study can be found in [2–4, 13, 16].

In this paper, we extend this research to graphs in the semi-Riemannian warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$ ($\varepsilon = \pm 1$), where M is a semi-Riemannian manifold, $\mathbb{R}_\varepsilon = \varepsilon\mathbb{R}$, and

f is the warping function. A graph Σ will be defined by a smooth function φ on M: $\Sigma = \{(x, \varphi(x)) \mid x \in M\} \subset \bar{M}$. The graph Σ is a hypersurface of $\bar{M}$. Unlike the usual case, we generally do not make the assumption that Σ is space-like. Some new phenomena occur in this case.

The organization of this paper is as follows: In §2 we recall some notations and results about semi-Riemannian warped product. In §3 we compute the mean curvature of a semi-Riemannian graph in semi-Riemannian warped product, and obtain the mean curvature equation. In §4, as a preparation for §5, we extend the well-known Gauss' theorem on Riemannian manifolds to the case of semi-Riemannian manifolds. In the final section, we give an integral formula for the mean curvature of semi-Riemannian graphs and its applications. The main results of this paper are from Theorem 5.1 through Theorem 5.5. Among these results, Theorem 5.4 is a Heinz–Chern type inequality. In Theorems 5.1–5.5, the base manifold M is assumed, for simplicity, to be oriented, but the results are also valid for nonorientable manifolds with a little revision of the statement (see Remark 4.2).

2. Preliminaries

Let M be an n-dimensional semi-Riemannian manifold with metric tensor g, f be a positive smooth function defined on M. Let $\mathbb{R}_\varepsilon$ denote the real line $\mathbb{R}$ endowed with the metric $\tilde{g} = \varepsilon \mathrm{d}t^2$ ($\varepsilon = +1$ or -1). The (semi-Riemannian) warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$ is the product manifold $M \times \mathbb{R}$ furnished with the semi-Riemannian metric

$$\bar{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(\tilde{g}),$$

where π and σ denote the projections of $M \times \mathbb{R}$ onto M and $\mathbb{R}$, respectively.

We recall some notations and results about warped product in [12].

We only consider the warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. For $x \in M, t \in \mathbb{R}$, we call $x \times \mathbb{R} = \pi^{-1}(x)$ the fibers and $M \times t = \sigma^{-1}(t)$ the leaves of the warped product. Vectors of $\bar{M}$ tangent to leaves are said to be horizontal and vectors of $\bar{M}$ tangent to fibers are said to be vertical. Denote by $\mathscr{H}$ the orthogonal projection of $T_{(x,t)}(\bar{M})$, where $T_{(x,t)}(\bar{M})$ denotes the tangent space to $\bar{M}$ at (x, t), onto its horizontal subspace $T_{(x,t)}(M \times t)$, and by $\mathscr{V}$ the orthogonal projection of $T_{(x,t)}(\bar{M})$ onto its vertical subspace $T_{(x,t)}(x \times \mathbb{R})$.

Throughout this paper, we use the following conventions. For a manifold N, let $\mathfrak{F}(N)$ be the set of all smooth functions on N, and $\mathfrak{X}(N)$ the set of all smooth vector fields on N. For the warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$, if $h \in \mathfrak{F}(M)$, $v \in \mathfrak{F}(\mathbb{R})$, $X \in \mathfrak{X}(M)$ and $V \in \mathfrak{X}(\mathbb{R})$, let $\bar{h} = h \circ \pi$ be the lift of h to $\bar{M}$, $\bar{v} = v \circ \sigma$ the lift of v, $\bar{X}$ the (horizontal) lift of X to $\bar{M}$, and $\bar{V}$ the (vertical) lift of V. For the sake of simplification, sometimes, if there is no confusion, we omit the bars over the letters (that denote lift), for instance, we write h for $\bar{h}$, whenever we know the exact meaning of that from the context.

The following two results proved in Chapter 7 of [12] will be needed in the sequel.

PROPOSITION 2.1

Consider the warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. If $h \in \mathfrak{F}(M)$, then

$$\bar{\nabla}(\bar{h}) = \overline{(\nabla h)}, \tag{2.1}$$

where, here and in the sequel, ∇h *stands for the gradient of* h *on* M, *and* $\bar{\nabla}(\bar{h})$ *the gradient of* $\bar{h}$ *on* $\bar{M}$.

PROPOSITION 2.2

Let $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. *Here and in the sequel, we denote by* ∇, $\bar{\nabla}$ *and* $\tilde{\nabla}$ *the Levi-Civita connections of* M, $\bar{M}$ *and* $\mathbb{R}_\varepsilon$ *respectively. Let* $II^\star(\cdot,\cdot)$ *be the second fundamental form of all fibers. If* $X, Y \in \mathfrak{X}(M)$ *and* $V, W \in \mathfrak{X}(\mathbb{R})$, *then*

(1) $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{(\nabla_X Y)}$;
(2) $\bar{\nabla}_{\bar{X}}\bar{V} = \bar{\nabla}_{\bar{V}}\bar{X} = (X\bar{f}/f)\,\bar{V}$;
(3) $\mathscr{H}(\bar{\nabla}_{\bar{V}}\bar{W}) = II^\star(\bar{V},\bar{W}) = -[\bar{g}(\bar{V},\bar{W})/\bar{f}\,]\bar{\nabla}\bar{f}$;
(4) $\mathscr{V}(\bar{\nabla}_{\bar{V}}\bar{W}) = \overline{(\tilde{\nabla}_V W)}$.

3. Mean curvature equation of semi-Riemannian graphs

In this section we will compute the mean curvature of a semi-Riemannian graph in the semi-Riemannian warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$.

Let Ω be an open subset of M, $\varphi \in \mathfrak{F}(\Omega)$. We mean by the graph of the function φ the subset

$$\Sigma = \{(x, \varphi(x)) \mid x \in \Omega\}$$

of $\bar{M}$. The graph Σ is a submanifold of $\bar{M}$, but need not be semi-Riemannian (this is different from the case of Riemannian products). When the graph Σ is a semi-Riemannian submanifold of $\bar{M}$, we call Σ a semi-Riemannian graph. The graph Σ is said to be entire if $\Omega = M$.

The graph Σ can be described as a level set of some function on $\bar{M}$. In fact it is easy to see that $\Sigma = \omega^{-1}(0)$ where ω is the function given by

$$w(x,t) = t - \varphi(x), \qquad ((x,t) \in \Omega \times \mathbb{R} \subset \bar{M}). \tag{3.1}$$

In order to decide if the graph Σ is semi-Riemannian or not, we compute the gradient of w on $\bar{M}$. We first prove the following fact.

PROPOSITION 3.1

Consider the warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. *If* $v \in \mathfrak{F}(\mathbb{R})$, *then*

$$\bar{\nabla}\,\bar{v} = \frac{1}{(\bar{f})^2}\overline{(\tilde{\nabla} v)}, \tag{3.2}$$

where $\tilde{\nabla} v$ *stands for the gradient of* v *on* $\mathbb{R}_\varepsilon$.

Proof. It suffices to show that $\bar{f}^{\,2}\bar{\nabla}\,\bar{v}$ is vertical (that is, tangent to the fibers) and σ-related to $\tilde{\nabla}\,v$. For any horizontal vector field Z on $\bar{M}$ (that is, tangent to the leaves),

$$\bar{g}(\bar{f}^{\,2}\bar{\nabla}\,\bar{v}, Z) = \bar{f}^{\,2} Z(v \circ \sigma) = 0,$$

thus $\bar{f}^{\,2}\bar{\nabla}\,\bar{v}$ is vertical. On the other hand, for any vertical vector field Q on $\bar{M}$, and at any point of $\bar{M}$,

$$\begin{aligned}\tilde{g}(\mathrm{d}\sigma(\bar{f}^{\,2}\bar{\nabla}\,\bar{v}), \mathrm{d}\sigma(Q)) &= \bar{g}(\bar{\nabla}\,\bar{v}, Q)) = Q(v\circ\sigma)\\ &= \mathrm{d}v(\mathrm{d}\sigma(Q)) = \mathrm{d}\sigma(Q)(v) = \tilde{g}(\tilde{\nabla}v, \mathrm{d}\sigma(Q)),\end{aligned}$$

so that

$$\mathrm{d}\sigma(\bar{f}^{\,2}\bar{\nabla}\,\bar{v}) = \tilde{\nabla}v,$$

that is, $\bar{f}^{\,2}\bar{\nabla}\,\bar{v}$ is σ-related to $\tilde{\nabla}\,v$. □

Now we compute the gradient of the function w on $\bar{M}$. From (3.1), by using Propositions 2.1 and 3.1, we have

$$\bar{\nabla}\,w = \bar{\nabla}\,\bar{t} - \bar{\nabla}\,\bar{\varphi} = \frac{1}{(\bar{f})^2}\overline{\tilde{\nabla}t} - \overline{(\nabla\varphi)} = \frac{\varepsilon}{(\bar{f})^2}\overline{\partial_t} - \overline{(\nabla\varphi)}, \tag{3.3}$$

where $\overline{\partial_t}$ denotes the lift of the coordinate vector field ∂_t on $\mathbb{R}$ to $\bar{M}$. For simplicity, we omit some of the lift symbols. Then (3.3) can be written as

$$\bar{\nabla}\,w = \bar{\nabla}\,t - \bar{\nabla}\,\varphi = \frac{\varepsilon}{f^2}\partial_t - \nabla\varphi. \tag{3.4}$$

PROPOSITION 3.2

Consider the graph Σ of $\varphi \in \mathfrak{F}(\Omega)$ in $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. Let $w(x,t) = t - \varphi(x)$. Then,

(1) *The graph Σ is a semi-Riemannian graph if and only if $\frac{1}{f^2} + \varepsilon\, g(\nabla\varphi, \nabla\varphi)$ is > 0 or < 0 on $\Omega \subset M$. In this case, when restricted to Σ,*

$$\xi \equiv \frac{\bar{\nabla}\,w}{\|\bar{\nabla}\,w\|_{\bar{M}}} = \frac{\frac{\varepsilon}{f^2}\partial_t - \nabla\varphi}{\psi} \tag{3.5}$$

is a unit normal vector field on Σ, where by $\|\cdot\|_{\bar{M}}$ we mean taking the norm on $\bar{M}$, $\psi \equiv \|\bar{\nabla}\,w\|_{\bar{M}} = \|\frac{\varepsilon}{f^2}\partial_t - \nabla\varphi\|_{\bar{M}}$.

(2) *If $\varepsilon\, g(\nabla\varphi, \nabla\varphi) \geq 0$ on Ω, then Σ is a semi-Riemannian graph.*

Proof.

(1) By (3.4), we have

$$\bar{g}(\bar{\nabla}\,w, \bar{\nabla}\,w) = \varepsilon\left(\frac{1}{f^2} + \varepsilon\, g(\nabla\varphi, \nabla\varphi)\right).$$

Then (1) follows by applying Proposition 17 of Chapter 4 in [12].

(2) This is an immediate consequence of (1). □

DEFINITION 3.3

Let M be a semi-Riemannian manifold with metric tensor g, and let Ω be an open subset of M, $h \in \mathfrak{F}(\Omega)$, $\varepsilon = \pm 1$. If $\varepsilon g(\nabla h, \nabla h) \geq 0$ on Ω, we call h an ε-proper function on Ω.

From now on, we suppose that φ is an ε-proper function defined on an open subset of M. Hence the graph Σ of function φ in $\bar{M} = M \times_f \mathbb{R}_\varepsilon$ is a semi-Riemannian graph.

Before computing the mean curvature of a semi-Riemannian graph, we show some basic results dealing with the divergence, Laplacian of a lift to the product space $\bar{M} = M \times_f \mathbb{R}_\varepsilon$.

PROPOSITION 3.4

Let $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. Denote by div_M, $\operatorname{div}_{\bar{M}}$ and $\operatorname{div}_{\mathbb{R}_\varepsilon}$ the divergence on M, $\bar{M}$ and $\mathbb{R}_\varepsilon$ respectively. If $X \in \mathfrak{X}(M)$ and $V \in \mathfrak{X}(\mathbb{R})$, then,

(1) $\operatorname{div}_{\bar{M}} \bar{X} = \operatorname{div}_M X + \frac{Xf}{f}$.

(2) $\operatorname{div}_{\bar{M}} \bar{V} = \operatorname{div}_{\mathbb{R}_\varepsilon} V$.

Note that we have omitted the lift symbols on the right sides of the above equalities.

Proof.

(1) Let $n = \dim M$. We choose a local field of orthonormal frames $E_1, \ldots, E_n$ in M. Let $\overline{E_{n+1}} \equiv \frac{\bar{\partial}_t}{\|\bar{\partial}_t\|_{\bar{M}}}$ (note here that $\overline{E_{n+1}}$ can not be seen as the lift of E_{n+1} because E_{n+1} has no definition). Then $\overline{E_1}, \ldots, \overline{E_n}, \overline{E_{n+1}}$ is a local field of orthonormal frames in $\bar{M}$. Let $\varepsilon_i = \bar{g}(\overline{E_i}, \overline{E_i})$ $(i = 1, \ldots, n+1)$. Then $\varepsilon_j = g(E_j, E_j)$ $(j = 1, \ldots, n)$ and $\varepsilon_{n+1} = \varepsilon$. By using Proposition 2.2 we have

$$\begin{aligned}
\operatorname{div}_{\bar{M}} \bar{X} &= \sum_{i=1}^{n+1} \varepsilon_i \bar{g}(\bar{\nabla}_{\overline{E_i}} \bar{X}, \overline{E_i}) \\
&= \sum_{i=1}^{n} \varepsilon_i g(\nabla_{E_i} X, E_i) + \varepsilon \bar{g}\left(\frac{Xf}{f}\overline{E_{n+1}}, \overline{E_{n+1}}\right) \\
&= \operatorname{div}_M X + \frac{Xf}{f}.
\end{aligned}$$

Thus (1) is proved.

(2) Using the same notations as above, we have

$$\begin{aligned}
\operatorname{div}_{\bar{M}} \bar{V} &= \sum_{i=1}^{n+1} \varepsilon_i \bar{g}(\bar{\nabla}_{\overline{E_i}} \bar{V}, \overline{E_i}) = \varepsilon_{n+1} \bar{g}(\bar{\nabla}_{\overline{E_{n+1}}} \bar{V}, \overline{E_{n+1}}) \\
&= \varepsilon \bar{g}(\mathscr{V}(\bar{\nabla}_{\overline{E_{n+1}}} \bar{V}), \overline{E_{n+1}}) = \frac{\varepsilon}{(\|\bar{\partial}_t\|_{\bar{M}})^2} \bar{g}(\mathscr{V}(\bar{\nabla}_{\bar{\partial}_t} \bar{V}), \bar{\partial}_t) \\
&= \frac{\varepsilon}{f^2} \bar{g}(\mathscr{V}(\bar{\nabla}_{\bar{\partial}_t} \bar{V}), \bar{\partial}_t) = \frac{\varepsilon}{f^2} \bar{g}(\overline{(\tilde{\nabla}_{\partial_t} V)}, \bar{\partial}_t) \\
&= \varepsilon\, \tilde{g}(\tilde{\nabla}_{\partial_t} V, \partial_t) = \operatorname{div}_{\mathbb{R}_\varepsilon} V.
\end{aligned}$$

□

PROPOSITION 3.5

Let $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. Denote by Δ, $\bar{\Delta}$ and $\tilde{\Delta}$ the Laplace operators on M, $\bar{M}$ and $\mathbb{R}_\varepsilon$ respectively. If $h \in \mathfrak{F}(M)$ and $v \in \mathfrak{F}(\mathbb{R})$, then

(1) $\bar{\Delta}\,(\bar{h}) = \Delta h + \frac{(\nabla h) f}{f}$.
(2) $\bar{\Delta}(\bar{v}) = \frac{1}{f^2}\tilde{\Delta} v$.

Proof.

(1) By definition of Laplacian, and using Propositions 2.1 and 3.4, we have

$$\begin{aligned}\bar{\Delta}\,(\bar{h}) &= \operatorname{div}_{\bar{M}}\,(\bar{\nabla}(\bar{h})) = \operatorname{div}_{\bar{M}}\,(\bar{\nabla} h)\\ &= \operatorname{div}_{M}\,(\nabla h) + \frac{(\nabla h) f}{f} = \Delta h + \frac{(\nabla h) f}{f}.\end{aligned}$$

(2) By Proposition 3.1,

$$\begin{aligned}\bar{\Delta}(\bar{v}) &= \operatorname{div}_{\bar{M}}\,\left(\bar{\nabla}(\bar{v})\right) = \operatorname{div}_{\bar{M}}\left[\frac{1}{\bar{f}^2}\overline{(\tilde{\nabla} v)}\right]\\ &= \overline{(\tilde{\nabla} v)}(\frac{1}{\bar{f}^2}) + \frac{1}{\bar{f}^2}\operatorname{div}_{\bar{M}}\,\overline{(\tilde{\nabla} v)} = \frac{1}{\bar{f}^2}\operatorname{div}_{\bar{M}}\,\overline{(\tilde{\nabla} v)}\\ &= \frac{1}{\bar{f}^2}\operatorname{div}_{\mathbb{R}_\varepsilon}\,(\tilde{\nabla} v) = \frac{1}{f^2}\tilde{\Delta} v.\end{aligned}$$

□

Now we are in position to compute the mean curvature of a semi-Riemannian graph $\Sigma = \{(x, \varphi(x)) \mid x \in \Omega\} \subset \bar{M} = M \times_f \mathbb{R}_\varepsilon$, where φ is an ε-proper function on Ω so that Σ is a semi-Riemannian graph, and hence Σ with the restricted metric $g^* = \bar{g}|_\Sigma$ is a semi-Riemannian manifold. Denote by ∇^* the connection of (Σ, g^*).

Let $\xi = \frac{\bar{\nabla} w}{\|\bar{\nabla} w\|_{\bar{M}}}$ (a unit normal vector field of Σ when restricted to Σ). We choose a local field of orthonormal frames $e_1, \ldots, e_n, e_{n+1}$ ($n = \dim M$) in $\bar{M}$ such that, restricted to Σ, the vectors $e_1, \ldots, e_n$ are tangent to Σ and $e_{n+1} = \xi$. Let II be the second fundamental form of Σ, that is, for any $X, Y \in \mathfrak{X}(\Sigma)$, $II(X, Y) = \bar{\nabla}_X Y - \nabla^*_X Y$. By definition, the mean curvature vector $\vec{H}$ of Σ is

$$\vec{H} = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i II(e_i, e_i).$$

Define

$$H \equiv \bar{g}(\vec{H}, \xi).$$

We call H the mean curvature of Σ with respect to ξ.

The shape operator of Σ derived from ξ is the (1, 1) tensor field S on Σ such that

$$\bar{g}\,(S(V), W) = \bar{g}\,(II(V, W), \xi) \quad \text{for all } V, W \in \mathfrak{X}(\Sigma).$$

It is a well-known fact that $S(V) = -\bar{\nabla}_V \xi$.

Theorem 3.6. *Let M be an n-dimensional semi-Riemannian manifold, $\Omega \subset M$ be an open subset of M, and let φ be an ε-proper function on Ω ($\varepsilon = \pm 1$). Consider the graph*

Σ of function φ in the semi-Riemannian warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. Let H be the mean curvature of Σ (with respect to ξ defined in Proposition 3.2). Then, on Ω, we have that

$$nH = q\Delta\varphi + (\nabla\varphi)q + \frac{q}{f}(\nabla\varphi)f, \tag{3.6}$$

where ∇ and Δ denote the gradient and the Laplacian in the metric of M respectively, and

$$q = \frac{1}{\sqrt{\frac{1}{f^2} + \|\nabla\varphi\|_M^2}}.$$

We can also write (3.6) as

$$nH = \mathrm{div}_M\,(q\nabla\varphi) + \frac{q}{f}(\nabla\varphi)f. \tag{3.7}$$

Proof. By the definition of H, we get

$$\begin{aligned} -nH &= -\bar{g}\left(\sum_{i=1}^{n} \varepsilon_i II(e_i, e_i), \xi\right) \\ &= -\sum_{i=1}^{n} \varepsilon_i \bar{g}\,(S(e_i), e_i) = \sum_{i=1}^{n} \varepsilon_i \bar{g}(\bar{\nabla}_{e_i}\xi, e_i) \\ &= \sum_{i=1}^{n} \varepsilon_i \bar{g}(\bar{\nabla}_{e_i}\xi, e_i) + \varepsilon\bar{g}(\bar{\nabla}_{\xi}\xi, \xi) = \mathrm{div}_{\bar{M}}\,\xi, \end{aligned}$$

where $\mathrm{div}_{\bar{M}}\,\xi$ means $\{\mathrm{div}_{\bar{M}}\,\xi\}|_\Sigma$. That is, we obtain

$$H = -\frac{1}{n}\mathrm{div}_{\bar{M}}\,\xi. \tag{3.8}$$

Applying (3.5) to (3.8), we have

$$-nH = \mathrm{div}_{\bar{M}}\left[\frac{\frac{\varepsilon}{f^2}\partial_t - \nabla\varphi}{\psi}\right],$$

where

$$\psi \equiv \|\bar{\nabla}\,w\|_{\bar{M}} = \sqrt{\frac{1}{f^2} + \varepsilon g(\nabla\varphi, \nabla\varphi)} = \sqrt{\frac{1}{f^2} + \|\nabla\varphi\|_M^2}, \tag{3.9}$$

a function independent of the variable t. Let

$$p = \frac{\varepsilon}{f^2\psi}, \qquad q = \frac{1}{\psi}.$$

Then, using Propositions 3.4 and 3.5, we have

$$\begin{aligned} -nH &= \mathrm{div}_{\bar{M}}\,(p\,\partial_t) - \mathrm{div}_{\bar{M}}\,(q\,\nabla\varphi) \\ &= \partial_t(p) + p\,\mathrm{div}_{\bar{M}}\;\partial_t - (\nabla\varphi)q - q\,\mathrm{div}_{\bar{M}}\,(\nabla\varphi) \end{aligned}$$

$$= p\,\mathrm{div}_{\mathbb{R}_\varepsilon}\partial_t - (\nabla\varphi)q - q\left[\mathrm{div}_M(\nabla\varphi) + \frac{(\nabla\varphi)f}{f}\right] \qquad (\partial_t(p)=0)$$
$$= -(\nabla\varphi)q - q\Delta\varphi - \frac{q}{f}(\nabla\varphi)f \qquad \left(\mathrm{div}_{\mathbb{R}_\varepsilon}\partial_t = 0\right).$$

Then we get (3.6) and (3.7). □

4. Gauss' theorem on semi-Riemannian manifolds

In the next section, we will discuss the integral of the mean curvature H of a semi-Riemannian graph Σ. Before doing it, as a preparation, we extend the well-known Gauss' theorem on Riemannian manifolds to the case of semi-Riemannian manifolds, which is interesting in itself.

Throughout this section, let N be an n-dimensional oriented semi-Riemannian manifold, possibly with boundary. Denote by g_N the metric tensor of N, and by ∂N the boundary of N. Then ∂N is also orientable and we take the induced orientation of ∂N (see [1]). Let ω be the (natural) semi-Riemannian volume element of N, that is, the unique smooth n-form ω such that $\omega(e_1, \ldots, e_n) = 1$ for any positively oriented local orthonormal frame $e_1, \ldots, e_n$ on N. Denote by μ the (natural) semi-Riemannian volume element of ∂N.

Let h be a continuous function defined on a domain $\Omega \subset N$. If $\int_\Omega h\,\omega$ exists, we call it the integral of h over Ω, denoted by $\int_\Omega h$. We call $\int_\Omega \omega$ the volume of Ω (see [10,11]). If $\bar{\Omega}$ (the closure of Ω) is compact, then the volume of Ω is finite. We refer to [10] for details about the properties of the measure derived from the volume element ω.

Recall the two operators i_X and L_X (see [1] for details). For every $X \in \mathfrak{X}(N)$, we define operators i_X and L_X on differential forms as follows: for smooth r-form θ on N, $X_1, \ldots, X_{r-1} \in \mathfrak{X}(N)$,

$$(i_X\theta)(X_1, \ldots, X_{r-1}) = \theta(X, X_1, \ldots, X_{r-1})$$

and

$$L_X\theta = i_X \mathrm{d}\theta + \mathrm{d}(i_X\theta).$$

Then we know that

$$L_X(\omega) = (\mathrm{div}_N X)\omega.$$

If X has compact support, then

$$\int_N (\mathrm{div}_N X)\omega = \int_{\partial N} i_X\omega. \tag{4.1}$$

PROPOSITION 4.1 (Gauss' theorem on semi-Riemannian manifolds)

Let N be an oriented semi-Riemannian manifold, possibly with boundary. Let ω be the semi-Riemannian volume elements of N.

If $\partial N \neq \emptyset$ (where $\emptyset$ denotes empty set), we suppose that ∂N is a semi-Riemannian submanifold of N with the orientation induced by N. Let μ be the semi-Riemannian volume element of ∂N, and ν the outward pointing unit normal vector field of ∂N. If X is a smooth vector field on N with compact support, then we have that

$$\int_N (\mathrm{div}_N X)\omega = \int_{\partial N} \eta\, g_N(X, \nu)\mu, \tag{4.2}$$

where g_N denotes the metric tensor of N, and $\eta = g_N(\nu, \nu)$.

If $\partial N = \emptyset$, the integral over N on the left side of (4.2) vanishes as long as X has compact support, no matter N is compact or not.

Proof. If $\partial N \neq \emptyset$, since ∂N is a semi-Riemannian submanifold of N, then for $\forall x \in \partial N$, the tangent space of ∂N at x, $T_x(\partial N)$ is a nondegenerate subspace of $T_x N$, and hence $T_x(\partial N)^{\perp}$ is also a nondegenerate subspace of $T_x N$. This means that the unit normal vector field ν along ∂N does exist. Thus we have the direct sum decomposition

$$T_x N = T_x(\partial N) \oplus T_x(\partial N)^{\perp}.$$

By (4.1), in order to show (4.2), it is enough to prove that

$$i_X \omega = \eta\, g_N(X, \nu)\mu$$

along the boundary of ∂N. In fact, for any $x \in \partial N$, let $v_1, v_2, \ldots, v_{n-1}$ be a positively oriented orthonormal basis of $T_x(\partial N)$. Then, by the definition of induced orientation, $\nu(x), v_1, v_2, \ldots, v_{n-1}$ is a positively oriented orthonormal basis of $T_x N$. Let

$$X(x) = X^n \nu(x) + \sum_{i=1}^{n-1} X^i v_i$$

be the expression for the vector $X(x)$ in this basis. At any $x \in \partial N$, we get

$$\begin{aligned}
(i_X\omega)(v_1, v_2, \ldots, v_{n-1}) &= \omega\,(X(x), v_1, v_2, \ldots, v_{n-1}) \\
&= X^n \omega\,(\nu(x), v_1, v_2, \ldots, v_{n-1}) \\
&= \eta\, g_N(X(x), \nu(x))\omega\,(\nu(x), v_1, v_2, \ldots, v_{n-1}) \\
&= \eta\, g_N(X(x), \nu(x)) \\
&= \eta\, g_N(X(x), \nu(x))\mu(v_1, v_2, \ldots, v_{n-1}).
\end{aligned}$$

Thus we have

$$i_X \omega = \eta\, g_N(X, \nu)\mu.$$

Hence (4.2) is true.

When ∂N is empty, it is a well-known fact that the integral over M on the left of (4.2) vanishes (see p. 281 of [9]).

The proof of Proposition 4.1 is complete. □

Remark 4.7. The Gauss' theorem on semi-Riemannian manifolds is also valid for nonorientable semi-Riemannian manifolds by replacing the volume elements in the theorem with the associated densities (like for the Riemannian case in [1]).

5. An integral formula for the mean curvature of semi-Riemannian graphs and its applications

Based on the results of §§4 and 5, we are now ready to give the following integral formula for the mean curvature of semi-Riemannian graphs in the semi-Riemannian warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$:

Theorem 5.1. *Let M be an n-dimensional oriented semi-Riemannian manifold, possibly with boundary. Let φ be an ε-proper function on $M(\varepsilon = \pm 1)$ such that φ is constant outside a compact set. Consider the graph Σ of function φ in the semi-Riemannian warped product $\bar{M} = M \times_f \mathbb{R}_\varepsilon$. Let H be the mean curvature of Σ (with respect to ξ defined in Proposition* 3.2*).*

If $\partial M \neq \emptyset$, we let ∂M have the orientation induced by M and suppose ∂M to be a semi-Riemannian hypersurface of M. Then we have

$$\int_M nH\omega = \int_{\partial M} \eta\, g(q\nabla\varphi, \nu)\mu + \int_M \frac{q}{f}\, g(\nabla\varphi, \nabla f)\,\omega, \tag{5.1}$$

where ω and μ denote the semi-Riemannian volume elements of M and ∂M respectively, ∇ is the gradient on M, g is the metric of M, ν stands for the outward pointing unit normal vector field of ∂M, and

$$q = \frac{1}{\sqrt{\frac{1}{f^2} + \|\nabla\varphi\|_M^2}}, \qquad \eta = g(\nu, \nu).$$

When ∂M is empty, we have

$$\int_M nH\omega = \int_M \frac{q}{f}\, g(\nabla\varphi, \nabla f)\,\omega. \tag{5.2}$$

Proof. First note that the integrals in (5.1) and (5.2) are all well defined because $\nabla\varphi$ has compact support. Integrating (3.7), we get

$$\int_M nH\omega = \int_M \mathrm{div}_M\,(q\nabla\varphi)\omega + \int_M \frac{q}{f}\, g(\nabla\varphi, \nabla f)\,\omega.$$

If $\partial M \neq \emptyset$, by using Proposition 4.1, we get

$$\int_M \mathrm{div}_M\,(q\nabla\varphi)\omega = \int_{\partial M} \eta\, g(q\nabla\varphi, \nu)\mu,$$

and thus we obtain (5.1). If $\partial M = \emptyset$, applying Proposition 4.1 again, we have

$$\int_M \mathrm{div}_M\,(q\nabla\varphi)\omega = 0,$$

and hence (5.2) is true. □

In the following we will give some applications of the above theorem.

Theorem 5.2. *In the notations and hypotheses of Theorem* 5.1*, if $\partial M = \emptyset$, $f \equiv 1$ (that is, $\bar{M} = M \times \mathbb{R}_\varepsilon$), and either M is compact or φ is constant outside a compact set, then we get*

$$\int_M H\omega = 0. \tag{5.3}$$

That is to say, the total mean curvature of Σ is zero. Thus if H does not change sign, then $H \equiv 0$.

Proof. It immediately follows from (5.2).

Let $\mathbb{V}$ be a real vector space, $v_1, \ldots, v_k \in \mathbb{V}$. Denote by $\operatorname{span}\{v_1, \ldots, v_k\}$ the span of the vectors $v_1, \ldots, v_k$.

Theorem 5.3. *Let M be an n-dimensional compact oriented semi-Riemannian manifold (with or without boundary), and φ a smooth function on M such that $g(\nabla\varphi, \nabla\varphi) \geq 0$. Consider the graph Σ of φ in $\bar{M} = M \times_f \mathbb{R}$.*

When $\partial M \neq \emptyset$, we suppose ∂M to be a semi-Riemannian hypersurface of M and let ∂M have the orientation induced by M. If $\operatorname{span}\{\nabla\varphi, \nu\}$ is space-like at any point of ∂M, and $\operatorname{span}\{\nabla\varphi, \nabla f\}$ is space-like at any point of M, then we have

$$\left|\int_M nH\omega\right| \leq A(\partial M) + \int_M \frac{\|\nabla f\|}{f}\omega, \tag{5.4}$$

where $A(\partial M)$ stands for the $(n-1)$-dimensional volume of ∂M.

When $\partial M = \emptyset$, inequality (5.4) still holds if we agree that $A(\partial M) = 0$ and suppose that $\operatorname{span}\{\nabla\varphi, \nabla f\}$ is space-like at any point of M.

Proof. We only give the proof of the case that $\partial M \neq \emptyset$.

By (5.1), we get

$$\begin{aligned}\left|\int_M nH\omega\right| &\leq \left|\int_{\partial M} \eta\, g(q\nabla\varphi, \nu)\mu\right| + \left|\int_M \frac{q}{f}\, g(\nabla\varphi, \nabla f)\,\omega\right| \\ &\leq \int_{\partial M} |g(q\nabla\varphi, \nu)|\mu + \int_M \frac{q}{f}\, |g(\nabla\varphi, \nabla f)|\,\omega.\end{aligned}$$

Since $\operatorname{span}\{\nabla\varphi, \nu\}$ and $\operatorname{span}\{\nabla\varphi, \nabla f\}$ are space-like, the Schwarz inequality holds. Thus we have

$$|g(q\nabla\varphi, \nu)| \leq \|q\nabla\varphi\| \cdot \|\nu\| \leq \frac{\|\nabla\varphi\|}{\sqrt{\frac{1}{f^2} + \|\nabla\varphi\|^2}} \leq 1$$

and

$$\frac{q}{f}\,|g(\nabla\varphi, \nabla f)| \leq \frac{q}{f}\|\nabla\varphi\| \cdot \|\nabla f\| \leq \frac{\|\nabla f\|}{f}.$$

Hence we obtain

$$\left|\int_M nH\omega\right| \leq \int_{\partial M} \mu + \int_M \frac{\|\nabla f\|}{f}\omega = A(\partial M) + \int_M \frac{\|\nabla f\|}{f}\omega.$$

□

Based on the above theorem, we now give an estimate of the mean curvatures for constant mean curvature graphs. First we recall that the Cheeger constant of a noncompact Riemannian manifold M (possibly with boundary) is defined by

$$\mathfrak{h}(M) = \inf_{\Omega} \frac{A(\partial\Omega)}{V(\Omega)},$$

where Ω ranges over all open submanifolds of M with compact closure in M and smooth boundary (see p. 95 of [5]), and $V(\Omega)$ denotes the volume of Ω.

Theorem 5.4. *Let M be an n-dimensional noncompact oriented Riemannian manifold (possibly with boundary), $f, \varphi \in \mathfrak{F}(M)$ and $f > 0$. Consider the graph Σ of φ in $\bar{M} = M \times_f \mathbb{R}$. We suppose that Σ has constant mean curvature H, and $L \equiv \sup_M\{\frac{\|\nabla f\|}{f}\} < +\infty$. Then we have*

$$n|H| \leq \mathfrak{h}(M) + L. \tag{5.5}$$

Proof. Let Ω be an open submanifold of M with compact closure and smooth boundary. Applying Theorem 5.3 to the graph $\Sigma|_{\bar{\Omega}} = \{(x, \varphi(x)) \mid x \in \bar{\Omega}\}$ in $\bar{\Omega} \times_f \mathbb{R}$ ($\bar{\Omega}$ denotes the closure of Ω in M). Then we get

$$n|H|V(\Omega) = \left|\int_{\bar{\Omega}} nH\omega\right| \leq A(\partial\Omega) + \int_{\Omega} \frac{\|\nabla f\|}{f}\omega \leq A(\partial\Omega) + L \cdot V(\Omega),$$

and hence

$$n|H| \leq \frac{A(\partial\Omega)}{V(\Omega)} + L.$$

Taking $\inf_{\Omega}$ with Ω rangers over all open submanifolds of M with compact closure and smooth boundary, we obtain (5.5). □

Theorem 5.5. *Let M be an n-dimensional oriented Riemannian manifold. Assume that there exists an increasing family of subsets of M:*

$$\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_i \subset \Omega_{i+1} \subset \ldots$$

such that

(1) *for $\forall i \in \mathbb{Z}^+$, Ω_i is a compact n-dimensional submanifold of M with smooth boundary;*
(2) *$A(\partial\Omega_i) < C$, for all $i \in \mathbb{Z}^+$, where C is a positive constant, and $A(\partial\Omega_i)$ stands for the $(n-1)$-dimensional volume of $\partial\Omega_i$;*
(3) *$M = \bigcup_{i=1}^{\infty} \Omega_i$.*

Let f be a positive smooth function on M such that f is constant outside a compact set. Then, for any $\varphi \in \mathfrak{F}(M)$, the total mean curvature of the graph Σ of φ in $\bar{M} = M \times_f \mathbb{R}$ is bounded, that is,

$$\left|\int_M H\omega\right| < +\infty. \tag{5.6}$$

Hence if the volume of M is infinity and $H = constant$, then $H = 0$. That is to say, all the graphs in $\bar{M} = M \times_f \mathbb{R}$ with constant mean curvature is necessarily minimal.

Proof. For each $i \in \mathbb{Z}^+$, applying Theorem 5.3 to Ω_i, we get

$$\left|\int_{\Omega_i} nH\omega\right| \leq A(\partial\Omega_i) + \int_{\Omega_i} \frac{\|\nabla f\|}{f}\omega.$$

By the hypotheses of the theorem, $\mathrm{supp}(\nabla f)$ (the support of ∇f) is compact, so that $\exists\, j \in \mathbb{Z}^+$ such that $\mathrm{supp}(\nabla f) \subset \Omega_j$. Then for any $k \geq j$, we have

$$\left|\int_{\Omega_k} nH\omega\right| \leq A(\partial\Omega_k) + \int_{\Omega_k} \frac{\|\nabla f\|}{f}\omega \leq C + \int_{\Omega_j} \frac{\|\nabla f\|}{f}\omega.$$

Letting $k \to +\infty$, it follows that

$$\left|\int_{M} nH\omega\right| \leq C + \int_{\Omega_j} \frac{\|\nabla f\|}{f}\omega < +\infty.$$

Thus we get (5.6).

The rest of the theorem is obvious. □

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References

[1] Abraham R, Marsden J E and Ratiu T, Manifolds, tensor analysis, and applications (London: Addison-Wesley) (1983)

[2] Albujer A L, Camargo F E C and de Lima H F, Complete space-like hypersurfaces with constant mean curvature in $-\mathbb{R} \times \mathbb{H}^n$, *J. Math. Anal. Appl.* **368** (2010) 650–657

[3] Alías L J and Dajczer M, Constant mean curvature hypersurfaces in warped product spaces, *Proc. Edinb. Math. Soc.* **50** (2007) 511–526

[4] Alías L J, Romero A and Sánchez M, Space-like hypersurfaces of constant mean curvature in certain spacetimes, *Nonlinear Analysis TMA* **30** (1997) 655–661

[5] Chavel I, Eigenvalues in Riemannian geometry (New York: Academic Press) (1984)

[6] Chern S S, On the curvatures of a piece of hypersurface in Euclidean space, *Abh. Math. Sem. Univ. Hamburg* **29** (1965) 77–91

[7] Flanders H, Remark on mean curvature, *J. London Math. Soc.* **41** (1966) 364–366

[8] Heinz E, Über flächen mit eineindeutigen projektion auf eine ebene, deren Krümmungen durch Ungleichungen Eingeschränkt Sind, *Math. Ann.* **129** (1955) 451–454

[9] Kobayashi S and Nomizu K, Foundations of differential geometry, Vol. I (New York: Wiley (Interscience)) (1963)

[10] Lang S, Real and functional analysis (New York: Springer-Verlag) (1993)

[11] Martin D, Manifold theory: an introduction for mathematical physicists (New York: Ellis Horwood Ltd) (1991)

[12] O'Neill B, Semi-Riemannian geometry with applications to relativity (London: Academic Press) (1983)

[13] Romero A and Rubio R M, On the mean curvature of space-like surfaces in certain three-dimensional Robertson–Walker spacetimes and Calabi–Bernstein's type problems, *Ann. Glob. Anal. Geom.* **37** (2010) 21–31

[14] Salavessa I M C, Graphs with parallel mean curvature, *Proc. Amer. Math. Soc.* **107** (1989) 449–458

[15] Salavessa I M C, Space-like graphs with parallel mean curvature, *Bull. Bell. Math. Soc.* **15** (2008) 65–76

[16] Zhang Z, A remark on the mean curvature of a graph-like hypersurface in hyperbolic space, *J. Math. Anal. Appl.* **305** (2005) 491–501