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Isoperimetric upper bounds for the first eigenvalue

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Abstract. Let M be a closed hypersurface in a simply connected space form $\mathbb{M}(\kappa)$ where $\kappa = 0, 1$ or -1. In this paper, we give two isoperimetric upper bounds for the first eigenvalue of the Laplacian of M.

Keywords. Hypersurface; center of mass; simpy connected space form; Laplacian; eigenvalue.

1. Introduction

Let $(\mathbb{M}(\kappa), \mathrm{d}s^2)$ denote the simply connected space form of constant curvature κ, where $\kappa = 0, 1$ or -1 and dimension $n \geq 2$. Let M be a closed hypersurface of $\mathbb{M}(\kappa)$. Starting with the work of Bleecker and Weiner [4], there have been several works (see [6,7,9]) which give a sharp upper bound for the first eigenvalue $\lambda_1(M)$ of the Laplace operator of M. These upper bounds are all extrinsic, in the sense that they depend either on the length of the second fundamental form or the higher order mean curvatures of M.

In [10], for hypersurfaces in rank-1 symmetric spaces, an upper bound for the first eigenvalue of M was given in terms of the integral of the first eigenvalue of the geodesic spheres centred at the centre of gravity of M. For precious statements, see [10].

In this paper, we obtain isoperimetric upper bounds for the first eigenvalue $\lambda_1(M)$ of a hypersurface M in $\mathbb{M}(\kappa)$.

We refer to [2] and [8] for the basic Riemannian geometry used in this paper.

2. Statement of the results

To state our results, we need the notion of centre of mass and the result on the existence of centre of mass for measurable subsets of $\mathbb{M}(\kappa)$.

Let (M, g) be a complete Riemannian manifold. For a point $p \in M$, we denote by $c(p)$, the convexity radius of (M, g) at p. For a subset $A \subseteq B(q, c(q))$ for some $q \in M$, we let CA denote the convex hull of A. Let $\exp_q : T_q M \to M$ be the exponential map and $X = (x_1, x_2, \ldots, x_n)$ be the normal coordinates centered at q. We identify CA with $\exp_q^{-1}(CA)$ and we also denote $g_q(X, X)$ as $\| X \|_q^2$ for $X \in T_q M$.

Theorem 1. *Let A be a measurable subset of (M, g) contained in $B(q_0, c(q_0))$ for some point $q_0 \in M$. Let $G\colon [0, 2c(q_0)] \to \mathbb{R}$ be a continuous function such that G is positive on $(0, 2c(q_0))$. Then there exists a point $p \in CA$ such that*

$$\int_A G(\| X \|_p) X \mathrm{d}V = 0,$$

where $X = (x_1, x_2, \ldots, x_n)$ denotes the geodesic normal coordinate system centred at p.

For a proof, see [1,5–7] or [10].

DEFINITION 1

The point p in Theorem 1 is called a centre of mass of the subset A with respect to the mass distribution function or weight function G.

Now we state the main results.

Theorem 2. *Let M be a closed hypersurface in $\mathbb{M}(\kappa)$ contained in a ball of radius $\frac{i(\mathbb{M}(\kappa))}{4}$ bounding a convex domain Ω.*

(1) *If $\kappa = 0$ or 1, then*

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))},$$

where $R > 0$ is such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$.

(2) *If $\kappa = -1$, then*

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))} + \frac{1}{(n-1)\mathrm{Vol}(S(R))} \int_M \| \nabla^M \cosh r_p(q) \|^2,$$

where $R > 0$ is such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$ and $r_p(q)$ denote the distance from the centre of gravity p of M with respect to the mass distribution function $G(t) = \frac{\sinh t}{t}$.

Furthermore, equality holds in the two inequalities above iff M is a geodesic sphere of radius R.

If we do not assume that the hypersurface M bounds a convex domain Ω, then we have the following result for hypersurfaces M in $\mathbb{R}^n$ and $\mathbb{H}^n$.

Theorem 3. *Let M be a closed hypersurface in $\mathbb{M}(\kappa)$ and Ω be the bounded domain such that $M = \partial\Omega$.*

(1) *If $\kappa = 0$, then*

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))}\right)^2,$$

where $R > 0$ is such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$.

(2) *If* $\kappa = -1$, *then*

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))}\right)^2 + \frac{\mathrm{Vol}(M)}{(n-1)\mathrm{Vol}(S(R))^2} \int_M \| \nabla^M \cosh r \|^2,$$

where $R > 0$ *is such that* $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$ *and* $r_p(q)$ *denote the distance from the centre of gravity* p *of* M *with respect to the mass distribution function* $G(t) = \frac{\sinh t}{t}$.

Furthermore, equality holds in the two inequalities above iff M *is a geodesic sphere of radius* R.

Remark 1. By the isoperimetric inequality in $\mathbb{M}(\kappa)$, we know that $\mathrm{Vol}(M) \geq \mathrm{Vol}(S(R))$. Hence $\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))} \geq 1$. Thus the upper bound in Theorem 1 is better than the upper bound in Theorem 2. However this comes at the cost of our assumption that the domain Ω is convex.

3. Preliminaries

Given a point p in $\mathbb{M}(\kappa)$, let $S(r)$ denote the geodesic sphere of radius r with centre p. Let $\Delta_{S(r)}$ denote the Laplacian of $S(r)$ with induced metric and $\lambda_1(S(r))$ denote the first eigenvalue of $\Delta_{S(r)}$.

Let $i(\mathbb{M}(\kappa))$ denote the injectivity radius of $(\mathbb{M}(\kappa), \mathrm{d}s^2)$. Let M be a closed hypersurface in $\mathbb{M}(\kappa)$ contained in a ball of radius $\frac{i(\mathbb{M}(\kappa))}{2}$ and Ω be the bounded domain such that $\partial\Omega = M$. If p is a point in Ω, then for every point $q \in M$ there exists a unique geodesic segment γ such that $\gamma(0) = p$ and $\gamma(d(p,q)) = q$. We also write the geodesic γ as γ_u, if $\gamma'(0) = u$ where $u \in U_p\mathbb{M}(\kappa)$, the unit tangent sphere at the point p. Let us note that the geodesic segment γ_u joining the points p and q may meet the hypersurface M at points other than q.

If q is a point in M and γ_u is the geodesic segment joining p and q, we write $d(p,q)$ as $t_q(u)$.

For $r \geq 0$, we let

$$\sin_\kappa r = \begin{cases} \sin r, & \text{if } \kappa = 1 \\ r, & \text{if } \kappa = 0 \\ \sinh r, & \text{if } \kappa = -1 \end{cases} \quad \text{and} \quad \cos_\kappa r = \begin{cases} \cos r, & \text{if } \kappa = 1 \\ 1, & \text{if } \kappa = 0 \\ \cosh r, & \text{if } \kappa = -1. \end{cases}$$

DEFINITION 2

A subset S of a Riemannian manifold (M, g) is said to be star-shaped if there exists a point p in S such that for every point q in S there exists a unique geodesic segment γ_{pq} joining the points p and q such that the geodesic segment γ_{pq} is contained in S. In this case, we also say that S is star shaped about the point p.

Now we state a lemma which is crucial in our proof of the results.

Lemma 1. *Let* M *be a closed hypersurface in* $\mathbb{M}(\kappa)$ *contained in a ball of radius* $\frac{i(\mathbb{M}(\kappa))}{4}$ *and* Ω *be the bounded domain in* $\mathbb{M}(\kappa)$ *such that* $M = \partial\Omega$, *the boundary of* Ω. *Let* p *be a point in* Ω *such that* Ω *is star-shaped about the point* p. *Then*

$$\int_M \sin_\kappa^2 d(p,q) \geq \mathrm{Vol}(S(R)) \, \sin_\kappa^2 R,$$

where $R > 0$ *is such that* $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$. *Furthermore, equality holds iff* M *is a geodesic sphere of radius* R *centred at* p.

Proof. Let $p \in \Omega$ and $U_p\mathbb{M}(\kappa)$ denote the unit sphere in $T_p\mathbb{M}(\kappa)$, the tangent space at p. We let (t, u) denote the geodesic polar co-ordinate system centered at the point p.

Let us denote by $\mathrm{d}A(q)$, the volume density of M at the point $q \in M$ and by $\mathrm{d}S_r(q)$, the volume density of the geodesic sphere $S(p, d(p, q))$ at the point q. Let $\theta(q)$ be the angle between the unit outward normal ν to M and the radial vector field $\partial r(q)$. Then we know that $\mathrm{d}A(q) = \sec\theta(q)\,\mathrm{d}S_r(q)$. Let $\mathrm{d}u$ denote the spherical density of the unit sphere $U_p\mathbb{M}(\kappa)$. Then $\mathrm{d}S_r(q) = \sin_\kappa^{n-1} d(p, q)\mathrm{d}u$ and therefore we can write the volume density of M as $\mathrm{d}A(q) = \sec\theta(q)\, \sin_\kappa^{n-1} d(p, q)\mathrm{d}u$. For a point q in M, if $q = \exp_p(d(p, q)u)$ for $u \in U_p\mathbb{M}(\kappa)$, we write $d(p, q)$ as $t_q(u)$.

Therefore

$$\begin{aligned}
\int_M \sin_\kappa^2 d(p, q)\,\mathrm{d}A(q) &= \int_{U_p\mathbb{M}(\kappa)} \sin_\kappa^2 t_q(u) \sec\theta(q) \sin_\kappa^{n-1} t_q(u)\mathrm{d}u \\
&\geq \int_{U_p\mathbb{M}(\kappa)} \sin_\kappa^{n+1} t_q(u) \\
&= (n+1)\int_{U_p\mathbb{M}(\kappa)} \int_0^{t_q(u)} \sin_\kappa^n r \cos_\kappa r\, \mathrm{d}r\, \mathrm{d}u \\
&= (n+1)\int_\Omega \sin_\kappa r(q) \cos_\kappa r(q) \mathrm{d}V(q).
\end{aligned}$$

Let $R > 0$ be such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}\, B(p, R)$ and $f(r) = \sin_\kappa r \cos_\kappa r$ for $r \in \mathbb{R}$. Then $\mathrm{Vol}(\Omega \setminus (\Omega \cap B(p, R))) = \mathrm{Vol}(B(p, R) \setminus (\Omega \cap B(p, R)))$ and the function f is increasing in $[0, \frac{i(\mathbb{M}(\kappa))}{4}]$. We use these two facts in estimating $\int_\Omega f(r(q))\mathrm{d}V(q)$ below.

$$\begin{aligned}
\int_\Omega f(r(q))\mathrm{d}V(q) &= \int_{\Omega\cap B(p,R)} f(r(q))\mathrm{d}V(q) + \int_{\Omega\setminus\Omega\cap B(p,R)} f(r(q))\mathrm{d}V(q) \\
&= \int_{B(p,R)} f(r(q))\mathrm{d}V(q) - \int_{B(p,R)\setminus\Omega\cap B(p,R)} f(r(q))\mathrm{d}V(q) \\
&\quad + \int_{\Omega\setminus\Omega\cap B(p,R)} f(r(q))\,\mathrm{d}V(q) \\
&\geq \int_{B(p,R)} f(r(q))\,\mathrm{d}V(q) - \int_{B(p,R)\setminus\Omega\cap B(p,R)} f(r(q))\mathrm{d}V(q) \\
&\quad + \int_{\Omega\setminus\Omega\cap B(p,R)} f(R)\,\mathrm{d}V(q) \\
&= \int_{B(p,R)} f(r(q))\,\mathrm{d}V(q) \\
&\quad + \int_{B(p,R)\setminus\Omega\cap B(p,R)} (f(R) - f(r(q)))\mathrm{d}V(q) \\
&\geq \int_{B(p,R)} f(r(q))\,\mathrm{d}V(q)
\end{aligned}$$

$$= \int_{U_p\mathbb{M}(\kappa)} \int_0^R f(r) \sin_\kappa^{n-1} r \mathrm{d}r \, \mathrm{d}u$$

$$= \int_{U_p\mathbb{M}(\kappa)} \int_0^R \sin_\kappa^n r \cos_\kappa r \mathrm{d}r \, \mathrm{d}u$$

$$= \frac{1}{n+1} \int_{U_p\mathbb{M}(\kappa)} \sin_\kappa^{n+1} R \, \mathrm{d}u$$

$$= \mathrm{Vol}(S(p, R)) \frac{\sin_\kappa^2 R}{n+1}.$$

Therefore

$$\begin{aligned} \int_M \sin_\kappa^2 d(p,q) &\geq (n+1) \int_\Omega \sin_\kappa r(q) \cos_\kappa r(q) \mathrm{d}V(q) \\ &\geq \mathrm{Vol}(S(R)) \sin_\kappa^2 R. \end{aligned}$$

Further, equality holds in the above inequality iff $\sec\theta(q) = 1$ for all points $q \in M$ and $\mathrm{Vol}(B(p, R) \setminus (\Omega \cap B(p, R))) = 0$. But $\sec\theta(q) = 1$ iff $\theta(q) = 0$. This is true iff the outward unit normal $\nu(q) = \partial r(q)$. Therefore equality holds iff $\nu(q) = \partial r(q)$ for all points q in M. This proves that $\Omega = B(p, R)$ and M is a geodesic sphere of radius R. □

4. Proof of Theorem 2

We will now prove Theorem 2.

Proof. Let M be a closed hypersurface contained in a ball of radius $\frac{i(\mathbb{M}(\kappa))}{4}$. By the centre of mass Theorem 1, there exists a point p in Ω such that $\int_M \frac{\sin_\kappa r}{r} X = 0$ where $X = (x_1, x_2, \ldots, x_n)$ is the normal coordinate system at p and $\| X \| = r$.

Let $f_i = \frac{x_i}{r}$ and $\varphi_i = \sin_\kappa r \, f_i$ for $1 \leq i \leq n$. Then $\int_M \varphi_i = 0$ for $1 \leq i \leq n$ and we can use these functions as test functions in the Rayleigh quotient.

Therefore, for $1 \leq i \leq n$,

$$\lambda_1(M) \int_M \varphi_i^2 \leq \int_M \| \nabla^M \varphi_i \|^2,$$

where ∇^M denotes the gradient in M. Now $\sum_{i=1}^n \varphi_i^2 = \sin_\kappa^2 r$. Hence

$$\lambda_1(M) \int_M \sin_\kappa^2 r \leq \sum_{i=1}^n \int_M \| \nabla^M \varphi_i \|^2.$$

To prove the result, we need to estimate $\sum_{i=1}^n \int_M \| \nabla^M \varphi_i \|^2$.

This will be done in a series of steps:

- Let us write $\nabla^M \varphi_i = f_i \nabla^M \sin_\kappa r + \sin_\kappa r \nabla^M f_i$ for each i. Then

$$\begin{aligned} \| \nabla^M \varphi_i \|^2 &= f_i^2 \| \nabla^M \sin_\kappa r \|^2 + \sin_\kappa^2 r \| \nabla^M f_i \|^2 \\ &\quad + 2 f_i \sin_\kappa r \langle \nabla^M \sin_\kappa r, \nabla^M f_i \rangle \\ &= f_i^2 \| \nabla^M \sin_\kappa r \|^2 + \sin_\kappa^2 r \| \nabla^M f_i \|^2 \\ &\quad + \sin_\kappa r \langle \nabla^M \sin_\kappa r, \nabla^M f_i^2 \rangle. \end{aligned}$$

Since $\sum_{i=1}^n f_i^2 = 1$, we get $\nabla^M(\sum_i f_i^2) = 0$. Hence

$$\begin{aligned}\sum_{i=1}^n \| \nabla^M \varphi_i \|^2 &= \| \nabla^M \sin_\kappa r \|^2 + \sin_\kappa^2 r \sum_{i=1}^n \| \nabla^M f_i \|^2 \\ &\quad + \sin_\kappa r \left\langle \nabla^M \sin_\kappa r, \nabla^M \left(\sum_{i=1}^n f_i^2 \right) \right\rangle \\ &= \| \nabla^M \sin_\kappa r \|^2 + \sin_\kappa^2 r \sum_{i=1}^n \| \nabla^M f_i \|^2.\end{aligned}$$

- We will now compute $\| \nabla^M f_i \|^2$. Let us write $\nabla^M f_i = \nabla f_i - \langle \nabla f_i, \nu \rangle \nu$ where ν is the unit outward normal. Then $\| \nabla^M f_i \|^2 = \| \nabla f_i \|^2 - \langle \nabla f_i, \nu \rangle^2$.

  By Green's identity, $\| \nabla f_i \|^2 = f_i \Delta f_i - \Delta(f_i^2)$ for $1 \leq i \leq n$. Therefore, since $\sum_{i=1}^n f_i^2 = 1$,

$$\begin{aligned}\sum_{i=1}^n \| \nabla f_i \|^2 &= \sum_{i=1}^n f_i \Delta f_i - \Delta \left(\sum_{i=1}^n f_i^2 \right) \\ &= \sum_{i=1}^n f_i \Delta f_i.\end{aligned}$$

We decompose Δ as $\Delta = -\frac{\partial^2}{\partial r^2} - \mathrm{Tr}(A(r)) \frac{\partial}{\partial r} + \Delta_{S(r)}$, where Δ is the Laplacian of $(\mathbb{M}(\kappa), \mathrm{d}s^2)$ and $A(r)$ is the Weingarten map of the geodesic sphere $S(r)$ and $\frac{\partial}{\partial r}$ is the outward radial vector field.

Since $\frac{\partial f_i}{\partial r} = 0$, it follows that $\Delta f_i = \Delta_{S(r)} f_i$. It is well-known that $f_i = \frac{x_i}{r}$'s are eigenfunctions of $\Delta_{S(r)}$ with eigenvalue $\lambda_1(S(r)) = \frac{n-1}{\sin_\kappa^2 r}$ (see [3]). Therefore, for every $i \in \{1, 2, \ldots, n\}$,

$$f_i \Delta f_i = \frac{n-1}{\sin_\kappa^2 r} f_i^2$$

and

$$\begin{aligned}\sum_{i=1}^n f_i \Delta f_i &= \frac{n-1}{\sin_\kappa^2 r} \sum_{i=1}^n f_i^2 \\ &= \frac{n-1}{\sin_\kappa^2 r}.\end{aligned}$$

- We will now compute $\sum_{i=1}^n \left(\frac{\partial f_i}{\partial \nu} \right)^2$. First we write $\nu = \alpha \partial r + \beta v$, where $\| v \| = 1$ and $\langle v, \partial r \rangle = 0$. Then $\langle \nabla f_i, \nu \rangle = \beta \langle \nabla f_i, v \rangle$ for $1 \leq i \leq n$. We also note that $\beta^2 = \| \nabla^M r \|^2$. Let $v = d(\exp_{p_0}) \overline{v} \in T_q \mathbb{M}(\kappa)$ be orthogonal to ∇r. Then it follows from the standard Jacobi field estimate that $\| v \|^2 = \frac{\sin_\kappa^2 r}{r^2} \| \bar{v} \|^2$. Hence

$$\begin{aligned}\sum_{i=1}^n \langle \nabla f_i, \nu \rangle^2 &= \beta^2 \sum_{i=1}^n \langle \nabla f_i, v \rangle^2 \\ &= \| \nabla^M r \|^2 \frac{1}{r^2} \langle \nabla x_i, v \rangle^2\end{aligned}$$

$$= \| \nabla^M r \|^2 \frac{1}{r^2} \| \bar{v} \|^2$$
$$= \| \nabla^M r \|^2 \frac{1}{r^2} \frac{r^2}{\sin_\kappa^2 r}$$
$$= \| \nabla^M r \|^2 \frac{1}{\sin_\kappa^2 r}.$$

In the computations above, we have used the fact that $\langle \nabla x_i, v \rangle = v(x_i) = \bar{v}(x_i \circ \exp)$ is the i-th component of v with respect to the orthornormal basis defining the normal coordinates.

- We substitute these in $\sum_{i=1}^n \int_M \sin_\kappa^2 r \| \nabla^M f_i \|^2$ to get

$$\begin{aligned} \sum_{i=1}^n \int_M \sin_\kappa^2 r \| \nabla^M f_i \|^2 &= \sum_{i=1}^n \int_M \sin_\kappa^2 r f_i \Delta f_i - \int_M \sin_\kappa^2 r \left(\frac{\partial f_i}{\partial \nu} \right)^2 \\ &= \int_M \sin_\kappa^2 r \, \lambda_1(S(r)) \sum_{i=1}^n f_i^2 \\ &\quad - \int_M \sin_\kappa^2 r \frac{1}{\sin_\kappa^2 r} \| \nabla^M r \|^2 \\ &= \int_M \sin_\kappa^2 r \, \lambda_1(S(r)) - \int_M \| \nabla^M r \|^2 \\ &= (n-1)\mathrm{Vol}(M) - \int_M \| \nabla^M r \|^2. \end{aligned}$$

- Putting together all the computations above, we get

$$\begin{aligned} \sum_{i=1}^n \int_M \| \nabla^M \varphi_i \|^2 &= \int_M \| \nabla^M \sin_\kappa r \|^2 + \sum_{i=1}^n \int_M \sin_\kappa^2 r \| \nabla^M f_i \|^2 \\ &= \int_M \| \nabla^M \sin_\kappa r \|^2 + (n-1)\mathrm{Vol}(M) - \int_M \| \nabla^M r \|^2 \\ &= (n-1)\mathrm{Vol}(M) + \int_M ({\cos_\kappa}^2 r - 1) \| \nabla^M r \|^2 \\ &= (n-1)\mathrm{Vol}(M) - \kappa \int_M \sin_\kappa^2 r \| \nabla^M r \|^2. \end{aligned}$$

We have already shown in Lemma 1 that $\int_M \sin_\kappa^2 r \geq \sin_\kappa^2 R \,\mathrm{Vol}(S(R))$, where $R > 0$ is such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$. Therefore

$$\begin{aligned} \lambda_1(M)\,\mathrm{Vol}(S(R))\,\sin_\kappa^2 R &\leq \lambda_1(M) \int_M \sin_\kappa^2 d(p,q)\mathrm{d}q \\ &= \lambda_1(M) \sum_{i=1}^n \int_M \varphi_i^2 \\ &\leq \sum_{i=1}^n \int_M \| \nabla^M \varphi_i \|^2 \\ &\leq (n-1)\mathrm{Vol}(M) - \kappa \int_M \sin_\kappa^2 r \| \nabla^M r \|^2. \end{aligned}$$

If $\kappa = 0$ or 1, then $\kappa \sin_\kappa^2 r \geq 0$. Hence, in this case,

$$\begin{aligned}\lambda_1(M) &\leq \left(\frac{n-1}{\sin_\kappa^2 R}\right)\left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))}\right)\\ &= \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))}\right)\lambda_1(S(R)).\end{aligned}$$

If $\kappa = -1$, then $\kappa \sin_\kappa^2 r = -\sinh^2 r \leq 0$. Hence in this case,

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))} + \frac{1}{(n-1)\mathrm{Vol}(S(R))}\int_M \|\nabla^M \cosh r\|^2.$$

This completes the proof of the inequality.

In both the cases equality holds iff equality holds in Lemma 1 and φ_i's are eigenfunctions for $1 \leq i \leq n$. But equality holds in Lemma 1 iff M is a geodesic sphere of radius R. □

5. Proof of Theorem 3

Let M be a hypersurface in $\mathbb{M}(\kappa)$ and Ω be a bounded domain in $\mathbb{M}(\kappa)$ such that M is the boundary of Ω.

By center of mass Theorem 1 there exists a point p in the convex hull of M such that $\int_M \frac{\sin_\kappa r}{r} X = 0$, where $X = (x_1, x_2, \ldots, x_n)$ are the geodesic normal coordinates centred at the point p and $\|X\| = r$.

Then the computations done in the proof of Theorem 2 show that

$$\lambda_1(M)\int_M \sin_\kappa^2 r(q) \leq (n-1)\mathrm{Vol}(M) - \kappa\int_M \sin_\kappa^2 r\|\nabla^M r\|^2.$$

If $\kappa = 0$, then $\sin_\kappa r = r$ and if $k = -1$ then $\sin_\kappa r = \sinh r$.

First we complete the proof in the case when $\kappa = 0$. Let M be hypersurface in $\mathbb{R}^n$. We will now estimate $\int_M \sin_\kappa^2 r(q) = \int_M r^2(q)$ below. Using Cauch–Schwarz inequality and the divergence theorem we get

$$\begin{aligned}\int_M r^2(q) &\geq \frac{1}{\mathrm{Vol}(M)}\left(\int_M r(q)\right)^2\\ &\geq \frac{1}{\mathrm{Vol}(M)}\left(\frac{1}{2}\int_M \langle \nabla r^2, \nu\rangle\right)^2\\ &= \frac{1}{\mathrm{Vol}(M)}\left(\frac{1}{2}\int_\Omega \Delta r^2\right)^2\\ &= \frac{1}{\mathrm{Vol}(M)}\left(\int_\Omega n\right)^2\\ &= \frac{n^2}{\mathrm{Vol}(M)}\mathrm{Vol}(\Omega)^2.\end{aligned}$$

If $R > 0$ is such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$, then

$$\int_M r^2 \geq \frac{n^2}{\mathrm{Vol}(M)}\mathrm{Vol}(\Omega)^2$$

can be written as

$$\int_M r^2 \geq \frac{\mathrm{Vol}(S(R)^2}{\mathrm{Vol}(M)} R^2.$$

This proves that

$$\lambda_1(M) \leq \lambda_1(S(R)) \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))} \right)^2$$

and the equality holds iff M is a geodesic sphere of radius R.

We will now do the computations when M is a hypersurface in $\mathbb{H}^n$. In this case

$$\begin{aligned} \int_M \sinh^2 r(q) \;&\geq\; \frac{1}{\mathrm{Vol}(M)} \left(\int_M \sinh r(q) \right)^2 \\ &\geq\; \frac{1}{\mathrm{Vol}(M)} \left(\int_M \langle \nabla \cosh r, \nu \rangle \right)^2 \\ &=\; \frac{1}{\mathrm{Vol}(M)} \left(\int_\Omega \Delta \cosh r \right)^2 \\ &=\; \frac{1}{\mathrm{Vol}(M)} \left(\int_\Omega n \cosh r \right)^2 \\ &=\; \frac{n^2}{\mathrm{Vol}(M)} \left(\int_\Omega \cosh r \right)^2. \end{aligned}$$

Let $R > 0$ be such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B(R))$. Now using the fact that $\cosh r$ is an increasing function, we can prove as in Lemma 1 that

$$\int_\Omega \cosh r \geq \int_{B(R)} \cosh r = \frac{\sinh R}{n} \mathrm{Vol}(S(R)).$$

Therefore

$$\int_M \sinh^2 r(q) \geq \frac{\sinh^2 R}{\mathrm{Vol}(M)} \mathrm{Vol}(S(R))^2$$

and the equality holds iff M is a geodesic sphere of radius R. Substituting this in the inequality

$$\lambda_1(M) \int_M \sinh^2 r(q) \;\leq\; (n-1)\mathrm{Vol}(M) + \int_M \| \nabla^M \cosh r \|^2,$$

we get

$$\frac{\lambda_1(M)}{\lambda_1(S(R))} \;\leq\; \left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S(R))} \right)^2 + \frac{\mathrm{Vol}(M)}{(n-1)\mathrm{Vol}(S(R))^2} \int_M \| \nabla^M \cosh r \|^2.$$

Furthermore, equality holds in both the inequalities iff M is a geodesic sphere.

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