

Isometric coactions of compact quantum groups on compact quantum metric spaces

JOHAN QUAEGEBEUR and MARIE SABBE

Department of Mathematics, K.U. Leuven, Celestijnenlaan 200B, B–3001 Leuven, Belgium
E-mail: johan.quaeghebur@wis.kuleuven.be; marie.sabbe@wis.kuleuven.be

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Abstract. We propose a notion of isometric coaction of a compact quantum group on a compact quantum metric space in the framework of Rieffel, where the metric structure is given by a Lipnorm. Within this setting we study the problem of the existence of a quantum isometry group.

Keywords. Quantum metric space; isometric coaction; quantum isometry group.

1. Introduction

This paper pertains to the study of quantum symmetries of a (classical or quantum) space. This problem can be formulated and studied in various settings. In [16], Wang considers the case where the space is finite and carries very little extra structure (only a functional has to be preserved). The quantum symmetry groups he obtains, can thus be interpreted as quantum permutation groups.

The spaces we are interested in this paper are metric spaces, both classical and quantum. Symmetries of a (quantum) metric space should then preserve the extra metric structure of the space, i.e. they should be ‘isometric’ in a sense to be made precise. For finite metric spaces, the ‘quantum isometry group’ is therefore expected to be a quantum subgroup of the quantum permutation group of Wang. For finite classical metric spaces, this problem was studied by Banica [2]. He has given a definition for a quantum symmetry of a classical finite metric space. With this definition, he was able to construct a quantum isometry group as a subgroup of Wang’s quantum permutation group.

The framework of Banica in [2] is still half-classical since the metric spaces he considers, are classical. The concept of a metric space also has a quantum version. One approach of quantizing metric spaces is by spectral triples [7]. These triples are used in non-commutative geometry to quantize the differential structure of a manifold as well as the metrical structure of the space. The quantum isometries of spectral triplets have been studied successfully in a number of papers mainly by Goswami and his coworkers (e.g. [1,3–6,8,9]).

There exists another framework to describe quantum metric spaces. In [12], Rieffel introduces a framework in which he only quantizes the metrical information of the space, disregarding the differential structure. Rieffel considers quantum spaces that come only with a quantum metric (encoded by a so-called Lipnorm), and carry no further structure.

This is the framework we will be working in. In fact, here we want to address the problem suggested by Rieffel at the end of §6 of [13]: “It would be interesting to develop and study the notion of a ‘quantum isometry group’ for quantum metric spaces as quantum subgroups of the quantum permutation groups studied by Wang [16]”.

The quantum metric spaces considered by Rieffel, are compact. Classically, the isometry group of a compact space is a compact group. Hence, if we want to define a ‘quantum isometry group’ in Rieffel’s framework, we might expect it to be compact. Therefore, we gather some information on compact quantum groups and compact quantum metric spaces together with other preliminary notions in the second section.

Next, we introduce our notion of isometric coactions in the third section. We show in §4 that this notion extends the notion used in [2] to the setting where both the group acting, and the space acted upon are ‘quantum’.

In §5 we study the problem of the existence of a quantum isometry group for a given compact quantum metric space in the sense of Rieffel. For small concrete examples we can prove its existence. To show its existence for more general spaces (e.g. finite quantum metric spaces) we were required to strengthen the notion of ‘isometric coaction’ to ‘full-isometric coaction’. We prove that both notions coincide in some situations (e.g. the case where the acting group is classical, the case considered in [2] where the space acted upon is finite and classical, and some small full quantum cases). Actually, we do not have examples where both notions do not coincide. With this (possibly) stronger notion of ‘full-isometric coaction’ we can prove in Theorem 5.12 a quantum version of the classical result that any group acting on a metric space has a largest subgroup acting isometrically. Using that theorem, we get the existence of a quantum full-isometry group for finite quantum metric spaces preserving a given state on the space, as subgroups of the quantum permutation groups of Wang.

2. Preliminaries

First we need to recall some introductory definitions. We define compact quantum groups, compact quantum metric spaces and coactions.

2.1 Compact quantum groups

DEFINITION 2.1

A *compact quantum group* (CQG) is a pair (A, Δ) where

- (i) A is a unital C^* -algebra;
- (ii) The ‘comultiplication’ $\Delta : A \rightarrow A \otimes A$ is a $*$ -homomorphism such that
 - Δ is ‘coassociative’: $(\iota \otimes \Delta) \Delta = (\Delta \otimes \iota) \Delta$;
 - the sets $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$.

The notion of a compact quantum group generalizes the notion of a classical compact group. Indeed, any compact group G can be seen as a CQG: take $A = C(G)$ (the commutative C^* -algebra of continuous complex-valued functions on G) and define the comultiplication

$$\Delta : C(G) \rightarrow C(G) \otimes C(G) \cong C(G \times G) : f \mapsto \Delta f$$

by

$$(\Delta f)(s, t) = f(st) \text{ for } s, t \in G. \tag{1}$$

Conversely, if (A, Δ) is a CQG for which the C^* -algebra A happens to be commutative, then there is a compact group G such that A can be identified with $C(G)$ such that, under this identification, Δ is given by (1).

DEFINITION 2.2

Let (A, Δ) and $(\tilde{A}, \tilde{\Delta})$ be two compact quantum groups. We say that $\varphi : A \rightarrow \tilde{A}$ is a *morphism of CQGs* from (A, Δ) to $(\tilde{A}, \tilde{\Delta})$ if φ is a $*$ -homomorphism such that

$$(\varphi \otimes \varphi)\Delta = \tilde{\Delta}\varphi.$$

For more information on compact quantum groups, we refer to [11,18].

2.2 Compact quantum metric spaces

DEFINITION 2.3

A *compact quantum metric space* (CQMS) is a pair (B, L) where

- (i) B is a unital C^* -algebra;
- (ii) L is a *Lipnorm* on B , i.e. $L : B \rightarrow [0, +\infty]$ is a seminorm such that
 - $L(b) = L(b^*)$ for every $b \in B$;
 - $\forall b \in B : L(b) = 0 \Leftrightarrow b \in \mathbb{C}1$;
 - L is lower semicontinuous;
 - the ρ_L -topology coincides with the weak $*$ -topology on $\mathcal{S}(B)$, where ρ_L is the metric on the state space $\mathcal{S}(B)$ defined by

$$\rho_L(\mu, \nu) = \sup\{|\mu(b) - \nu(b)| \mid b \in B, L(b) \leq 1\}$$

for every $\mu, \nu \in \mathcal{S}(B)$.

The classical notion of a compact metric space fits into this framework. Indeed, let (X, d) be a compact metric space. Put $B = C(X)$ and consider the Lipschitz seminorm

$$L_d : C(X) \rightarrow [0, +\infty] : f \mapsto \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in X, x \neq y \right\}. \tag{2}$$

One can prove that this seminorm is a Lipnorm.

Another source of examples of compact quantum metric spaces is given by some spectral triples. A spectral triple $(\mathcal{B}, \mathcal{H}, D)$ is a $*$ -subalgebra \mathcal{B} of the bounded operators on a Hilbert space \mathcal{H} , together with a Dirac operator D on \mathcal{H} . This is a self-adjoint (possibly unbounded) operator on \mathcal{H} such that $[D, b]$ has a bounded extension for every $b \in \mathcal{B}$. In some cases, the normclosure of \mathcal{B} in $\mathcal{B}(\mathcal{H})$ will be a CQMS for the seminorm L determined by $L(b) = \|[D, b]\|$ for $b \in \mathcal{B}$.

For more information and examples of compact quantum metric spaces, we refer to the work of Rieffel [12,14]. Note that in [14] and later articles, the definition of a Lipnorm is without the requirement of lower semicontinuity. But Rieffel shows that, starting from any, possibly not lower semicontinuous Lipnorm L on B , one can always construct a

lower semicontinuous Lipnorm \tilde{L} such that L and \tilde{L} induce the same metric on the state space of B . Therefore, we prefer to have this requirement in the definition.

2.3 Coactions

DEFINITION 2.4

A *coaction* of a CQG (A, Δ) on a unital C^* -algebra B is a unital $*$ -homomorphism $\alpha : B \rightarrow B \otimes A$ such that

- (i) $(\iota \otimes \Delta) \alpha = (\alpha \otimes \iota) \alpha$,
- (ii) the set $\alpha(B)(1 \otimes A)$ is norm dense in $B \otimes A$.

We say that α is *faithful* if the set $\{(\psi \otimes \iota)\alpha(b) \mid b \in B, \psi \in B^*\}$ generates A as a C^* -algebra.

A classical action of a compact group G on a space X fits in this framework by taking $\alpha : C(X) \rightarrow C(X) \otimes C(G) \cong C(X \times G) : f \mapsto \alpha(f)$ determined by

$$\alpha(f)(x, s) = f(s \cdot x) \quad \text{for } x \in X, s \in G.$$

DEFINITION 2.5

If B is a C^* -algebra, (A, Δ) is a compact quantum group and $\alpha : B \rightarrow B \otimes A$ is a coaction, we call the triple (A, Δ, α) a *quantum transformation group* (QTG) of B . We say that the QTG (A, Δ, α) is *faithful* if the coaction α is faithful.

If ψ is a functional on B , we say that α *preserves* ψ if

$$(\psi \otimes \iota)\alpha(b) = \psi(b)1_A$$

for all $b \in B$.

If (A, Δ, α) and $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$ are two quantum transformation groups of B or (B, ψ) , we say that $\varphi : A \rightarrow \tilde{A}$ is a *morphism of QTGs* from (A, Δ, α) to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$ if φ is a morphism of compact quantum groups such that

$$(\iota \otimes \varphi)\alpha = \tilde{\alpha}.$$

3. Isometric coactions

Now the natural question arises: If in Definition 2.4, B happens to have the extra structure of a compact quantum metric space as in Definition 2.3, when would it make sense to call the coaction α isometric?

We introduce the following definition, which will turn out to generalize some existing notions. For later purposes, we formulate it not only for coactions but also for more general $*$ -homomorphisms.

DEFINITION 3.1

For a unital C^* -algebra C and a CQMS (B, L) , a $*$ -homomorphism $\alpha : B \rightarrow B \otimes C$ is called *isometric* if and only if

$$\forall \omega \in \mathcal{S}(C), \forall b \in B : L((\iota \otimes \omega)\alpha(b)) \leq L(b). \quad (3)$$

When writing this paper, we discovered that this notion already appeared in a recent paper of Li (Definition 8.8 of [10]). He calls a coaction invariant if the above inequality holds. However, he did not further explore this notion in the direction we are studying (the problem of the existence of a quantum isometry group).

The following lemma states that it is sufficient to check inequality (3) for a weakly $*$ -total set of states, i.e. a set of states whose convex hull is weakly $*$ -dense in the state space.

Lemma 3.2. *Suppose we have two unital C^* -algebras C and B , and a $*$ -homomorphism $\beta : B \rightarrow B \otimes C$. Assume that $S \subseteq \mathcal{S}(C)$ is a weakly $*$ -total set of states on C . Then, for any lower semicontinuous seminorm L on B , the following are equivalent:*

- (i) *for every $\omega \in S$ and every $b \in B$, we have $L((\iota \otimes \omega)\beta(b)) \leq L(b)$;*
- (ii) *for every $\omega \in \mathcal{S}(C)$ and every $b \in B$, we have $L((\iota \otimes \omega)\beta(b)) \leq L(b)$.*

Proof. Suppose (i) holds. Then one can immediately see that the inequality in (ii) holds if ω is taken from the convex hull of S . If ω is an arbitrary state on C , then ω is the weak $*$ -limit of a net of states ω_i in the convex hull of S . Using the lower semicontinuity of L , the inequality for ω follows from the inequality for ω_i . □

The following lemma, which can be proved by standard methods, describes a weakly $*$ -total set of states which will be useful later.

Lemma 3.3. *Let A be a unital C^* -algebra and Π be a set of $*$ -homomorphisms $\pi : A \rightarrow C_\pi$ where the C_π are unital C^* -algebras. We suppose Π separates the points of A , i.e. if $a \in A$ is non-zero, then there exists an element $\pi \in \Pi$ for which $\pi(a) \neq 0$. Then the convex hull of the set*

$$S = \{\omega \circ \pi \mid \pi \in \Pi, \omega \in \mathcal{S}(C_\pi)\}$$

is weakly $$ -dense in $\mathcal{S}(A)$.*

We study how Definition 3.1 specifies to coactions in half classical settings, i.e. cases where either the group or the space is classical. First we consider the case where A is commutative, and thus of the form $C(G)$ for a compact group G .

Theorem 3.4. *Let G be a compact group, (B, L) a CQMS and $\alpha : B \rightarrow B \otimes C(G)$ a coaction. For $s \in G$ we denote by ω_s the pure states that evaluates a function $f \in C(G)$ in s and by $\alpha_s : B \rightarrow B$ the automorphism given by $\alpha_s = (\iota \otimes \omega_s)\alpha$. Then the following are equivalent:*

- (i) *α is isometric (Definition 3.1);*
- (ii) *For every $s \in G$, we have $L \circ \alpha_s = L$.*

Proof. Suppose (i) holds. Fix elements $b \in B$ and $s \in G$. By (i) we know that $L(\alpha_s(b)) = L((\iota \otimes \omega_s)\alpha(b)) \leq L(b)$. On the other hand, we also have

$$L(b) = L(\alpha_e(b)) = L(\alpha_{s^{-1}}(\alpha_s(b))) \leq L(\alpha_s(b)).$$

Both inequalities together prove (ii). Conversely, notice that the states ω_s are exactly the pure states on $C(G)$. Since the convex hull of these states is weakly $*$ -dense in the state space of $C(G)$, (i) follows from (ii) using Lemma 3.2. □

More specifically, this theorem implies that in the double-classical case, where $A = C(G)$ for a compact group G and $B = C(X)$ for a metric space (X, d) , Definition 3.1 is equivalent to the usual notion of an isometric group action.

There is another half classical case of our setting, namely the case of a compact quantum group acting on a finite classical metric space. This is the situation that was studied by Banica in [2]. The setting is as follows:

Let $\alpha : C(X) \rightarrow C(X) \otimes A$ be a coaction of a CQG (A, Δ) on a finite metric space (X, d) with n points. Such a coaction is completely described by an $(n \times n)$ matrix $a = (a_{ij})_{i,j=1,\dots,n}$ over A . Indeed, consider the standard basis $\{\delta_1, \dots, \delta_n\}$ of $C(X)$, where δ_i is the function that is one in the i -th point of X , and zero elsewhere. The coaction α is determined by the elements $\alpha(\delta_j)$ for $j \in \{1, \dots, n\}$. As the element $\alpha(\delta_j)$ belongs to $C(X) \otimes A$ we can write

$$\alpha(\delta_j) = \sum_{i=1}^n \delta_i \otimes a_{ij}$$

for some elements a_{ij} in A . The properties of α being a coaction translate into properties of the elements a_{ij} (see [16]). The matrix $a = (a_{ij})$ has to be a so called *magic biunitary*, which means that its rows and columns are partitions of the unity of A with projections, or, explicitly, for all $i, j \in \{1, \dots, n\}$, we have

$$a_{ij}^* = a_{ij} = a_{ij}^2, \quad \sum_{k=1}^n a_{ik} = 1, \quad \sum_{k=1}^n a_{kj} = 1. \tag{4}$$

The comultiplication on the elements a_{ij} is given by

$$\Delta(a_{ij}) = \sum_{k=1}^n a_{ik} \otimes a_{kj}.$$

Note that (4) implies that

$$a_{ij}a_{ik} = 0 = a_{ji}a_{ki}$$

for all $i, j, k \in \{1, \dots, n\}$ with $j \neq k$. Moreover, if the coaction α is faithful, i.e. if A is generated by $\{a_{ij} \mid i, j = 1, \dots, n\}$, then (A, Δ) is of Kac type with bounded antipode S given by $S(a_{ij}) = a_{ji}$.

Now suppose d is a metric on $X = \{1, \dots, n\}$. Consider the $(n \times n)$ -matrix $(d(i, j))_{i,j=1,\dots,n}$ which we also denote by d . In [2], Banica calls the coaction α of A on X isometric if the matrices a and d commute:

$$ad = da. \tag{5}$$

Does Banica’s notion of isometric coactions coincide with the notion introduced in Definition 3.1 for the general ‘double-quantum’ setting? This question is answered in the affirmative in the following theorem, which we will prove in the next section.

Theorem 3.5. *Let (X, d) be a finite metric space with n points and (A, Δ) a CQG acting faithfully on X by the coaction $\alpha : C(X) \rightarrow C(X) \otimes A$.*

Take elements a_{ij} in A such that $\alpha(\delta_j) = \sum_{i=1}^n \delta_i \otimes a_{ij}$, where δ_j is 1 on the j -th point of X and 0 elsewhere. Write L_d for the Lipnorm defined by (2), a for the matrix $(a_{ij})_{i,j=1\dots n}$ and d for the matrix $(d(i, j))_{i,j=1\dots n}$. Then the following are equivalent:

- (a) α is isometric (Definition 3.1);
- (b) $ad = da$.

4. Proof of Theorem 3.5

4.1 Part 1: (a) implies (b)

Before we start the proof of the first implication of Theorem 3.5, we express in the following lemma in the commutation of a with d in a different, for our purposes more practical way.

Lemma 4.1. Using the notations of Theorem 3.5, the following are equivalent:

- (i) $ad = da$;
- (ii) $a_{ij}a_{kl} = 0$ for every $i, j, k, l \in X$ with $d(i, k) \neq d(j, l)$.

Proof. Suppose (i) holds. Take points i, j, k, l in X such that $d(i, k) \neq d(j, l)$. Because a and d commute, we have

$$\sum_{x=1}^n a_{ix}d(x, l) = \sum_{y=1}^n d(i, y)a_{yl}.$$

If we multiply this equality on the left by a_{ij} , we get

$$a_{ij}d(j, l) = \sum_{y=1}^n d(i, y)a_{ij}a_{yl}$$

because $a_{ix}a_{iy}$ is zero whenever $x \neq y$. Similarly, by multiplying this by a_{kl} on the right, we get

$$a_{ij}a_{kl}d(j, l) = d(i, k)a_{ij}a_{kl}.$$

Because $d(j, l) \neq d(i, k)$, this implies that $a_{ij}a_{kl} = 0$.

Conversely, suppose (ii) holds. Then, for every i, j in X , we have

$$\begin{aligned} (ad - da)_{ij} &= \sum_{x=1}^n a_{ix}d(x, j) - d(i, x)a_{xj} \\ &= \sum_{x,y=1}^n d(x, j)a_{ix}a_{yj} - d(i, y)a_{ix}a_{yj} = 0 \end{aligned}$$

which proves (i). □

In order to prove that (a) implies (b) in Theorem 3.5, we need another lemma. The formulation of that lemma needs some notations. Since we are working in a finite metric space, the set $D = \{d(i, j) \mid i, j \in X\}$ of all distances, is a finite set. So we can rearrange the numbers in D , writing $D = \{d_0, d_1, \dots, d_N\}$ where $d_k < d_l$ if $k < l$. For every distance $d_\gamma \in D$, and every point $i \in X$, we can look at the set of all points that are at a distance d_γ from i . We denote this set by $V_i^\gamma = \{j \in X \mid d(i, j) = d_\gamma\}$.

Lemma 4.2. We use the notations of Theorem 3.5 and assume (a). If a state ω on A is one on an element a_{ij} , then it is zero on every a_{kl} with $d(i, k) \neq d(j, l)$. In other words, for all $\gamma \in \{0, \dots, N\}$, all $i, j \in X$ and all $\omega \in \mathcal{S}(A)$ with $\omega(a_{ij}) = 1$, we have

$$\begin{aligned} \forall k \in V_i^\gamma, \forall l \notin V_j^\gamma : \omega(a_{kl}) &= 0, \\ \forall k \notin V_i^\gamma, \forall l \in V_j^\gamma : \omega(a_{kl}) &= 0. \end{aligned}$$

Proof. We prove this lemma by induction on γ .

Step 1. First of all, we want to check the lemma for $\gamma = 0$. We fix points $i, j \in X$ and a state ω on A such that $\omega(a_{ij}) = 1$. The value $\gamma = 0$ corresponds with the smallest distance in the metric space, which is zero.

First, we take $k \in V_i^0$. This means that $d(i, k) = d_0 = 0$, so $k = i$. We also take some $l \notin V_j^0$ (hence $l \neq j$) and we want to show that $\omega(a_{kl})$ is zero. But $\omega(a_{kl}) = \omega(a_{il}) \leq \sum_{x \neq j} \omega(a_{ix})$, because all a_{ix} are positive and ω is a positive map. Moreover, $\sum_{x \in X} a_{ix} = 1$ and since $\omega(1) = 1$, we have $\omega(a_{kl}) \leq 1 - \omega(a_{ij}) = 0$. Hence $\omega(a_{kl}) = 0$, which is the first item we had to prove.

Secondly, we take $k \notin V_i^0$, so $k \neq i$, and $l \in V_j^0$, so $l = j$. Then we have, by a similar argument as before, that $\omega(a_{kl}) = \omega(a_{kj}) \leq \sum_{x \neq i} \omega(a_{xj}) = 1 - \omega(a_{ij}) = 0$, so $\omega(a_{kl}) = 0$.

Step 2. We proceed by induction. We suppose that the lemma holds for every value in $\{0, \dots, \gamma\}$ for a certain γ . We want to prove that it also holds for $\gamma + 1$. Again, we fix points i and j in X , and a state ω on A such that $\omega(a_{ij}) = 1$. We take $k \in V_i^{\gamma+1}$, and $l \notin V_j^{\gamma+1}$.

We are going to use the inequality

$$L_d((t \otimes \omega)\alpha(f)) \leq L_d(f) \tag{6}$$

for the chosen state ω and for $f = D_j$, where $D_j : X \rightarrow \mathbb{C} : x \mapsto d(x, j)$. Using the triangle inequality, it is easy to check that $L_d(D_j) = 1$. So inequality (6) implies that

$$\frac{|(t \otimes \omega)\alpha(D_j)(k) - (t \otimes \omega)\alpha(D_j)(i)|}{d(k, i)} \leq 1.$$

Notice that $D_j = \sum_{x \in X} d(j, x)\delta_x$, so, using the formula from Theorem 3.5 for the $\alpha(\delta_x)$, we get

$$\left| \sum_{x \in X} d(j, x)\omega(a_{kx}) - \sum_{x \in X} d(j, x)\omega(a_{ix}) \right| \leq d(k, i) = d_{\gamma+1}.$$

In the first step, we have proven that $\omega(a_{ix}) = 0$ whenever $x \neq j$. This gives

$$\left| \sum_{x \in X} d(j, x)\omega(a_{kx}) \right| \leq d_{\gamma+1}. \tag{7}$$

Because of the induction hypothesis, we have that $\omega(a_{kx}) = 0$ if $d(j, x) < d_{\gamma+1}$. Indeed, in that case, we can find a value $c \in \{0, \dots, \gamma\}$ such that $d(j, x) = d_c$ and hence $x \in V_j^c$.

On the other hand, since $k \in V_i^{\gamma+1}$, we know that $k \notin V_i^c$. As $c \leq \gamma$, it follows from the induction hypothesis that $\omega(a_{kx}) = 0$. This means that the left-hand side of inequality (7) is a convex combination of those distances $d(j, x)$ that are at least $d_{\gamma+1}$, although the convex combination itself is smaller than $d_{\gamma+1}$. It is clear that it is only possible if the coefficients of the distances $d(j, x) > d_{\gamma+1}$ are zero. So, $\omega(a_{kx}) = 0$ unless $d(j, x) = d_{\gamma+1}$. In particular, since $l \notin V_j^{\gamma+1}$, this implies that $\omega(a_{kl}) = 0$.

We still have to prove the second case, for $k \notin V_i^{\gamma+1}$ and $l \in V_j^{\gamma+1}$. In this case, we define a new state

$$\omega' : A \rightarrow \mathbb{C} : a \mapsto \omega(S(a)),$$

where S is the antipode of (A, Δ) . Since $\omega'(a_{ji}) = \omega(S(a_{ji})) = \omega(a_{ij}) = 1$, we can use the first case of this lemma on the distance $\gamma + 1$, the points j and i in X and the state ω' . Since $l \in V_j^{\gamma+1}$ and $k \notin V_i^{\gamma+1}$, the first case states that $0 = \omega'(a_{lk}) = \omega(a_{kl})$ and this concludes the proof. \square

Now we are ready to prove the first implication of Theorem 3.5. Assume that (a) holds. Choose any $i, j, k, l \in X$ with $d(i, k) \neq d(j, l)$. By Lemma 4.1, it suffices to show that $a_{ij}a_{kl} = 0$. Suppose that the product $a_{ij}a_{kl}$ is nonzero. Then also $a_{ij}a_{kl}a_{ij} = (a_{ij}a_{kl})(a_{ij}a_{kl})^*$ is nonzero. Hence we can find a state ω on A such that $\omega(a_{ij}a_{kl}a_{ij})$ is nonzero. But then also $\omega(a_{ij})$ is nonzero, because of the Cauchy–Schwarz inequality. Using this state ω , we define a new state

$$\omega_{ij} : A \rightarrow \mathbb{C} : x \mapsto \frac{\omega(a_{ij}xa_{ij})}{\omega(a_{ij})}.$$

One can easily check that this is indeed a state and that $\omega_{ij}(a_{ij}) = 1$. So we can use the first case of Lemma 4.2 on the number γ for which $d(i, k) = d_\gamma$, on the points i and j , and on the state ω_{ij} , and conclude that $\omega_{ij}(a_{kl}) = 0$. But this clearly contradicts the fact that $\omega_{ij}(a_{kl}) = \frac{\omega(a_{ij}a_{kl}a_{ij})}{\omega(a_{ij})}$ is nonzero because of the choice of ω . Therefore $a_{ij}a_{kl} = 0$, and this concludes the proof of this first implication.

4.2 Part 2: (b) implies (a)

Since this part of the theorem will not use anything specific about the CQG-structure we have on A (except for the fact that the matrix a is a magic biunitary), we can reformulate this part in a more general setting.

Theorem 4.3. *Let (X, d) be a finite metric space with n points, A a unital C^* -algebra and $\alpha : C(X) \rightarrow C(X) \otimes A$ a $*$ -homomorphism.*

Take elements a_{ij} in A such that $\alpha(\delta_j) = \sum_{i=1}^n \delta_i \otimes a_{ij}$. We use the same notations as in Theorem 3.5 for δ_i , L_d , a and d . Suppose a is a magic biunitary matrix. Then, if $ad = da$, it follows that α is isometric.

First note that it is sufficient to prove the following lemma:

Lemma 4.4. Using the notations and assumptions of Theorem 4.3, we fix elements $x, y \in X$ and a state ω on A . Then there exist positive numbers λ_{ij} ($i, j \in X$) such that for all $i, j \in X$, we have

$$\begin{cases} \sum_{j \in X} \lambda_{ij} = \omega(a_{xi}) \\ \sum_{i \in X} \lambda_{ij} = \omega(a_{yj}) \\ \lambda_{ij} = 0 \text{ if } d(i, j) \neq d(x, y). \end{cases}$$

Suppose the lemma holds, then we can prove Theorem 4.3. Indeed, we fix a state $\omega \in \mathcal{S}(A)$ and a function $f \in C(X)$. Now we can rewrite

$$(\iota \otimes \omega)\alpha(f) = \sum_{i \in X} f(i)(\iota \otimes \omega)\alpha(\delta_i) = \sum_{i, j \in X} f(i)\omega(a_{ji})\delta_j.$$

Choose elements $x, y \in X$ with $x \neq y$. Using Lemma 4.4, we can write

$$\begin{aligned} \frac{|(\iota \otimes \omega)\alpha(f)(x) - (\iota \otimes \omega)\alpha(f)(y)|}{d(x, y)} &= \frac{|\sum_{i \in X} f(i)(\omega(a_{xi}) - \omega(a_{yi}))|}{d(x, y)} \\ &= \frac{|\sum_{i \in X} f(i) \left(\sum_{j \in X} \lambda_{ij} - \sum_{j \in X} \lambda_{ji} \right)|}{d(x, y)} = \frac{|\sum_{i, j \in X} \lambda_{ij}(f(i) - f(j))|}{d(x, y)} \\ &= \left| \sum_{i, j \in X} \lambda_{ij} \frac{f(i) - f(j)}{d(i, j)} \right| \leq \sum_{i, j \in X} \lambda_{ij} L_d(f) = L_d(f). \end{aligned}$$

Because this holds for all elements x and y in X with $x \neq y$, we find that $L_d((\iota \otimes \omega)\alpha(f)) \leq L_d(f)$ for every $f \in C(X)$, which would conclude the proof of Theorem 4.3.

Of course, we still have to prove Lemma 4.4. This lemma is combinatorial, and its proof will use a famous theorem in graph theory. To formulate this theorem, we need to introduce some notations.

The graph theory we need concerns flow networks. A flow network is a set V of vertices, and a set $E \subseteq V \times V$ of directed edges. Every edge (u, v) in E has a positive capacity $c(u, v)$. We also fix two vertices s and t , called the source and the sink.

An s - t -flow in such a network is a mapping from the set of edges to the positive numbers, which maps every edge $(v, w) \in E$ on $f(v, w)$, the ‘flow from v to w ’. The conditions are that the flow $f(v, w)$ through an edge is always smaller than the capacity $c(v, w)$ of the edge. Secondly, the flow can not stay inside a vertex (except for the source and the sink). So the flow entering a vertex must equal the flow leaving the vertex, or $\sum_{u \in V} f(u, v) = \sum_{w \in V} f(v, w)$ for every vertex $v \in V \setminus \{s, t\}$. In this mathematical notation we suppose that the capacity $c(v, w)$ (and hence also the flow $f(v, w)$) is zero when there is no edge (v, w) in E .

The value of an s - t -flow is then the sum of all the flow leaving the source $\sum_{v \in V} f(s, v)$. This equals the flow entering the sink.

An s - t -cut is a partition of the vertices. We cut the graph into two pieces, partitioning the set of vertices in two sets, S and T in such a way that S contains the source s and T contains the sink t . The capacity of such a cut is then the sum of the capacities of all edges crossing the cut from S to T . Mathematically, this is $\sum_{v \in S, w \in T} c(v, w)$.

Now we can formulate the famous max-flow min-cut theorem briefly.

Theorem 4.5 (Max-flow min-cut theorem). *The maximal value of an s - t -flow is equal to the minimal capacity of an s - t -cut.*

This theorem will be used to prove the following combinatorial lemma.

Lemma 4.5. Suppose we have positive numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ such that $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$. Next, suppose that for every $i \in \{1, \dots, n\}$, there exists a set $V_i \subseteq \{1, \dots, n\}$ such that for every subset Z of $\{1, \dots, n\}$,

$$\sum_{i \in Z} \alpha_i \leq \sum_{j \in \bigcup_{i \in Z} V_i} \beta_j. \tag{8}$$

Then, for $i, j \in \{1, \dots, n\}$, we can take positive numbers λ_{ij} such that for all $i, j \in \{1, \dots, n\}$,

$$\begin{cases} \sum_{j=1}^n \lambda_{ij} = \alpha_i, \\ \sum_{i=1}^n \lambda_{ij} = \beta_j, \\ \lambda_{ij} = 0 \text{ if } j \notin V_i. \end{cases}$$

Proof. We construct a flow network. We have n vertices on the left-hand side, which we label l_1, l_2, \dots, l_n , and n vertices on the right-hand side, labeled r_1, r_2, \dots, r_n . In short, we denote L for $\{l_1, \dots, l_n\}$ and R for $\{r_1, \dots, r_n\}$. For every $i \in \{1, \dots, n\}$, we have an edge from the source s to vertex l_i with capacity α_i , and an edge from vertex r_i to the sink t with capacity β_i . In the center, we have an edge from vertex l_i to every vertex r_j with $j \in V_i$. We give those edges capacity 1. We denote the set of vertices by V and the set of edges by E . From now on, we will see V_i as the subset of all vertices r_j with $j \in V_i$.

We want to use the max-flow min-cut theorem to find a maximal flow in this flow network. For this end, we investigate the minimal capacity of an s - t -cut. An example of an s - t -cut with capacity one is $S = \{s\}, T = V \setminus \{s\}$. We claim that this capacity is minimal. So we want to prove that every other s - t -cut has a capacity of at least one. Let us consider all possible cases.

- First we suppose S contains none of the vertices on the left hand side: $S \cap L = \emptyset$. Then the capacity of the cut is

$$\sum_{\substack{v \in S \\ w \in T}} c(v, w) \geq \sum_{w \in L} c(s, w) = \sum_{i=1}^n c(s, l_i) = \sum_{i=1}^n \alpha_i = 1.$$

- If one of the central edges is crossing the cut from S to T , then the capacity of the cut is bigger than one. This is, if $S \cap L \neq \emptyset$ and $T \cap \bigcup_{i \in S} V_i \neq \emptyset$, then we have a vertex $r_j \in T$ in one of the V_i , with $l_i \in S$. Then l_i belongs to S and r_j to T , so the capacity of the cut is bigger than $c(l_i, r_j)$, which is 1 because $r_j \in V_i$.
- The last case is the case where $S \cap L \neq \emptyset$ but $T \cap \bigcup_{i \in S} V_i = \emptyset$. This means that $V_i \subseteq S$ when $l_i \in S$. If we now use the given inequality (8) on the set $S \cap L$, we get

$$\sum_{l_i \in S} \alpha_i \leq \sum_{r_j \in \bigcup_{l_i \in S} V_i} \beta_j.$$

For the total capacity of the cut, this gives

$$\begin{aligned} \sum_{\substack{v \in S \\ w \in T}} c(v, w) &\geq \sum_{w \in (T \cap L)} c(s, w) + \sum_{v \in (S \cap R)} c(v, t) \\ &\geq \sum_{l_i \in T} \alpha_i + \sum_{r_j \in \bigcup_{l_i \in S} V_i} \beta_j \geq \sum_{l_i \in T} \alpha_i + \sum_{l_i \in S} \alpha_i = \sum_{i=1}^n \alpha_i = 1. \end{aligned}$$

We have proven that the minimal capacity of an s - t -cut in the flow network we consider is one. Because of the max-flow min-cut theorem, this implies that we can have a maximal flow f of value one through this network. If we denote the flow in the edge (l_i, r_j) by λ_{ij} , we have found the positive numbers we were looking for:

- It is obvious that all $\lambda_{ij} = f(l_i, r_j)$ are positive.
- For the first condition, we have to calculate $\sum_{j=1}^n \lambda_{ij} = \sum_{j=1}^n f(l_i, r_j)$. This is the total amount of flow leaving the vertex l_i , so this has to equal the amount of flow entering l_i . The only edge through which the flow can enter l_i , is the edge (s, l_i) , which has capacity α_i . In order to have a total flow of value one, the flow in every edge (s, l_k) must reach its capacity α_k (since $\sum_{k=1}^n \alpha_k = 1$). Hence, in particular, the flow through the edge (s, l_i) equals its capacity α_i . So we have proven that $\sum_{j=1}^n \lambda_{ij} = f(s, l_i) = \alpha_i$.
- For the second condition, we calculate $\sum_{i=1}^n \lambda_{ij} = \sum_{i=1}^n f(l_i, r_j)$. Similarly we see that this is the total amount of flow entering the vertex r_j . This is equal to the amount of flow leaving r_j , which is the flow through the edge (r_j, t) . With a similar reasoning as for the first condition, we see that the flow $f(r_j, t)$ equals the capacity β_j of the edge.
- If $j \notin V_i$, there is no edge from l_i to r_j . Clearly, there cannot be any flow going from l_i to r_j , so $\lambda_{ij} = f(l_i, r_j)$ is zero. \square

We want to use this purely combinatorial lemma to prove Lemma 4.4.

Proof of Lemma 4.4. Fix elements x and y in X and a state ω on A . It is obvious that we want to use Lemma 4.6 for the numbers $\alpha_i = \omega(a_{xi})$ and $\beta_i = \omega(a_{yi})$. To make the third condition of Lemma 4.6 match with the third condition of Lemma 4.4, we take V_i to be the set $\{j \in X \mid d(i, j) = d(x, y)\}$.

In order to use Lemma 4.6, we need to check the inequality

$$\sum_{i \in Z} \omega(a_{xi}) \leq \sum_{\substack{j \in X: \\ \exists i \in Z: d(i, j) = d(x, y)}} \omega(a_{yj})$$

for every subset $Z \subseteq X$. But this follows from the fact that a and d commute. Indeed, by using the commutation relation in the way we obtained in Lemma 4.1, we get for $Z \subseteq X$,

$$\sum_{i \in Z} a_{xi} = \sum_{i \in Z} a_{xi} \sum_{j \in X} a_{yj} = \sum_{i \in Z} a_{xi} \sum_{\substack{j \in X: \\ \exists i \in Z: d(i, j) = d(x, y)}} a_{yj}$$

because $a_{xi} a_{yj} = 0$ whenever $d(i, j) \neq d(x, y)$.

Since both factors in the previous product are projections, we have

$$\sum_{i \in Z} a_{xi} \leq \sum_{\substack{j \in X: \\ \exists i \in Z: d(i, j) = d(x, y)}} a_{yj}.$$

Since ω is a positive map, we immediately get the desired inequality. Hence we can apply Lemma 4.6 to obtain the desired positive numbers λ_{ij} . \square

5. Existence of isometry groups?

5.1 Statement of the problem

In the previous sections, we introduced a notion of isometries in the general quantum context. We proved that this notion generalizes the existing notions of isometries in the half-classical cases. It would be nice if this notion would allow to prove the *quantum isometry group* of a given compact quantum metric space (B, L) . This quantum isometry group would have to be a universal object in the category of all faithful quantum transformation groups of B that preserve the metric structure given by L . We already know in what way the structure should be preserved.

DEFINITION 5.1

Let (B, L) be a compact quantum metric space. We call a quantum transformation group (A, Δ, α) of B *isometric* if α is an isometric coaction.

For a CQMS (B, L) , we will be considering the category $\mathcal{C}(B, L)$ of all faithful and isometric quantum transformation groups (A, Δ, α) of (B, L) . The morphisms are morphisms of QTGs, as defined in Definition 2.5.

It is also possible that B has some additional structure that has to be preserved, on top of the Lipnorm. For example, we can consider a functional ψ on B that should be preserved by the coaction. For this purpose, we introduce a second category. For a CQMS (B, L) and a functional ψ on B , we consider the category $\mathcal{C}(B, L, \psi)$ of all faithful and isometric quantum transformation groups of (B, L) , preserving the functional ψ as defined in Definition 2.5. The morphisms are still the morphisms of QTGs.

DEFINITION 5.2

The *quantum isometry group of a CQMS (B, L)* is the universal object in the category $\mathcal{C}(B, L)$, if such an object exists. Hence it is an object (A, Δ, α) in $\mathcal{C}(B, L)$ such that, for every object $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$ in $\mathcal{C}(B, L)$, there is a unique morphism of quantum transformation groups from (A, Δ, α) to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$.

If ψ is a functional on B , we define the *quantum isometry group of (B, L, ψ)* to be the universal object in $\mathcal{C}(B, L, \psi)$, if such an object exists.

It is not immediately clear under what conditions such a quantum isometry group exists. In what follows, we want to find a quantum isometry group for a certain class of quantum metric spaces.

We use the classical case as a source of inspiration.

Classical fact 5.3. The isometry group of a metric space is the largest, isometrically acting subgroup of the permutation group.

We want to use this idea in the quantum setting too and therefore, we need spaces that have a quantum permutation group. Hence we will restrict ourselves to finite compact quantum metric spaces. For those spaces, Wang has defined the quantum permutation group preserving a functional [16,17].

In order to get to the actual quantum isometry groups, we first want to generalize the following result to the quantum setting. Classically, when a group G acts on a metric space (X, d) , there exists a largest subgroup of G that acts isometrically on X , namely the subset H consisting of those elements g of G for which the action $X \rightarrow X : x \mapsto gx$ is isometric. To formulate this result in the quantum setting, we recall the definition of a quantum subgroup.

DEFINITION 5.4

Let (A, Δ) be a compact quantum group and $I \subseteq A$ a subset of A . We say that I is a *Woronowicz Hopf C^* -ideal* if I is a two-sided closed $*$ -ideal in A such that

$$(p_I \otimes p_I)\Delta(I) = \{0\},$$

where p_I is the canonical projection $A \rightarrow A/I : a \mapsto a + I$.

When I is a Woronowicz Hopf C^* -ideal, A/I becomes a compact quantum group for the comultiplication defined by

$$\Delta_I : A/I \rightarrow A/I \otimes A/I : p_I(a) \mapsto (p_I \otimes p_I)\Delta(a). \quad (9)$$

The CQG $(A/I, \Delta_I)$ can be considered a *compact quantum subgroup* of (A, Δ) .

With the notion of quantum subgroups, we can try to generalize the aforementioned classical result. We fix a coaction α of a CQG (A, Δ) on a CQMS (B, L) . We want to prove the existence of a smallest Woronowicz Hopf C^* -ideal of A , such that the induced coaction

$$\alpha_I : B \rightarrow B \otimes A/I : b \mapsto (\iota \otimes p_I)\alpha(b) \quad (10)$$

is isometric, where p_I is the canonical projection of A onto A/I .

Notation 5.5. Let (B, L) be a CQMS. We write Π for the class of all morphisms $\pi : (A, \Delta, \alpha) \rightarrow (C_\pi, \Delta_\pi, \alpha_\pi)$ of QTGs of B such that $\alpha_\pi = (\iota \otimes \pi)\alpha$ is an isometric coaction.

Using this notation, one can describe the natural candidate for I as the intersection of all kernels of morphisms in Π . Although this natural ideal I , turns out to be a Woronowicz Hopf C^* -ideal in several (small) examples, we have not been able to show this fact in the general setting. However, the following result shows that if this I is a Woronowicz Hopf C^* -ideal, then it is indeed the ideal we were looking for.

Theorem 5.6. *Let (B, L) be a CQMS and (A, Δ, α) be a QTG of B . Define an ideal I by*

$$I = \bigcap_{\pi \in \Pi} \ker(\pi).$$

If I is a Woronowicz Hopf C^ -ideal, then $(A/I, \Delta_I, \alpha_I)$, defined by (9) and (10), is an isometric QTG of (B, L) . By construction I is the smallest Woronowicz Hopf C^* -ideal for which this statement holds.*

Proof. We already know that $(A/I, \Delta_I, \alpha_I)$ is a QTG. We need to check that α_I is isometric. For a morphism $\pi : A \rightarrow C_\pi$ in Π , we know that $\pi(I) = \{0\}$. Hence the map

$$\bar{\pi} : A/I \rightarrow C_\pi : (a + I) \mapsto \pi(a)$$

is well-defined. Remark that $\bar{\pi} \circ p_I = \pi$, where p_I is the projection of A onto A/I . It is clear that these maps $\bar{\pi}$ (with $\pi \in \Pi$) separate the points of A/I . Hence, by Lemma 3.3, we know that the convex hull of the set

$$S = \{\omega \circ \bar{\pi} \mid \pi \in \Pi, \omega \in \mathcal{S}(C_\pi)\}$$

is weakly $*$ -dense in $\mathcal{S}(A/I)$.

Choose an element $b \in B$, a morphism $\pi \in \Pi$ and a state ω on C_π . By Lemma 3.2, it is sufficient to prove that $L((\iota \otimes \omega \circ \bar{\pi})\alpha_I(b)) \leq L(b)$. But we have

$$L((\iota \otimes \omega \circ \bar{\pi})\alpha_I(b)) = L((\iota \otimes \omega \circ \pi)\alpha(b)) = L((\iota \otimes \omega)\alpha_\pi(b)) \leq L(b),$$

where the last inequality holds since $\pi \in \Pi$. □

Before we get back to the general setting, we give an example of a ‘small’ compact quantum metric space, where we can directly compute the quantum isometry group, using the previous theorem.

5.2 Example

We consider the quantum space $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. We will write the elements of $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ as triples. We use the notation e_{ij} ($i, j = 1, 2$) for the matrix units in $M_2(\mathbb{C})$.

One can check that the quantum permutation group of Wang [16], preserving the trace, reduces to the C^* -algebra A generated by four generators x, y, z, p with relations

$$\begin{cases} x^2 = -yz \\ 2xx^* + yy^* + zz^* = 1 \\ p^* = p = p^2 \end{cases} \quad (11)$$

and such that the $*$ -algebra generated by x, y, z is commutative.

The comultiplication is defined by

$$\begin{aligned} \Delta(x) &= (zz^* - yy^*) \otimes x + x \otimes z + x^* \otimes y, \\ \Delta(y) &= (yx^* - xz^*) \otimes 2x + y \otimes z + z^* \otimes y, \\ \Delta(z) &= (xy^* - zx^*) \otimes 2x + z \otimes z + y^* \otimes y, \\ \Delta(p) &= p \otimes p + (1 - p) \otimes (1 - p). \end{aligned} \quad (12)$$

The coaction of A on $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ is the unital $*$ -homomorphism defined by

$$\begin{aligned} \alpha(e_{12}, 0, 0) &= (e_{11} - e_{22}, 0, 0) \otimes x + (e_{12}, 0, 0) \otimes z + (e_{21}, 0, 0) \otimes y, \\ \alpha(0, 1, 0) &= (0, 1, 0) \otimes p + (0, 0, 1) \otimes (1 - p). \end{aligned} \quad (13)$$

There are several Lipnorms that make this space into a CQMS. The Lipnorm L we will consider here will be the following:

$$L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e, f\right) = |a - d| + |b| + |c| + |a - e| + |a - f| \quad (14)$$

for all $a, b, c, d, e, f \in \mathbb{C}$.

In search of the ideal I , we investigate when a coaction of the form (13) is isometric.

Theorem 5.7. *Let (C, Δ_C) be a CQG with a coaction α_C on $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$, preserving the trace. Hence α_C is of the form (13) for some elements x, y, z, p in C , with relations (11) and such that the $*$ -algebra D generated by x, y, z is commutative. Then α_C is isometric if and only if $x = y = 0$.*

Proof. By direct computation one can check that every coaction of the form (13) with $x = y = 0$, is isometric.

Conversely, suppose that α_C is isometric. To prove that $x = 0$, we use the isometric inequality (3) on the element $(e_{12}, 0, 0)$. Let ω be a state on C . By the definition of α_C , we get

$$L\left(\begin{pmatrix} \omega(x) & \omega(z) \\ \omega(y) & -\omega(x) \end{pmatrix}, 0, 0\right) = L((\iota \otimes \omega)\alpha_C(e_{12}, 0, 0)) \leq L(e_{12}, 0, 0).$$

Using the definition of L , we see that this is equivalent to

$$2|\omega(x)| + |\omega(y)| + |\omega(z)| \leq 1. \quad (15)$$

On the other hand, we have some relations on x, y, z . First, suppose that the restriction of ω to D is a $*$ -homomorphism. Then the relations (11) give us

$$\begin{cases} \omega(x)^2 = -\omega(y)\omega(z) \\ 2|\omega(x)|^2 + |\omega(y)|^2 + |\omega(z)|^2 = 1. \end{cases}$$

Take the absolute value of the first relation and substitute this in the second relation to get

$$(|\omega(y)| + |\omega(z)|)^2 = 1.$$

But then the inequality (15) can only hold if $\omega(x) = 0$. So, we have proven that any state that restricts to a $*$ -homomorphism on C , should be zero on x .

Now consider an arbitrary state ω on C . If we restrict ω to D , we get a state φ on D . We can write φ as a weak $*$ -limit of convex combinations of pure states φ_i on D . But since D is commutative, these pure states will be $*$ -homomorphisms. Let ω_i be any extension of φ_i to a state on C . By construction, every ω_i is a state on C that restricts to a $*$ -homomorphism on D . Hence, every ω_i (and every φ_i) is zero on x . But this implies that $\omega(x) = \varphi(x) = 0$.

Similarly as before, in order to show that $y = 0$, it is sufficient to prove that $\omega(y) = 0$ for all states ω on C that restrict to a $*$ -homomorphism on D . Fix such a state ω . This time, we use the isometric inequality (3) for the element $(e_{22}, 0, 0)$. The formula for α_C (with $x = 0$) gives us

$$L\left(\begin{pmatrix} \omega(y^*y) & 0 \\ 0 & \omega(z^*z) \end{pmatrix}, 0, 0\right) = L((\iota \otimes \omega)\alpha_C(e_{22}, 0, 0)) \leq L(e_{22}, 0, 0).$$

Using the definition of L , we get

$$|\omega(y^*y) - \omega(z^*z)| + 2|\omega(y^*y)| \leq 1.$$

Suppose $\omega(y)$ is non-zero. Since $x = 0$ and since we assumed ω restricts to a $*$ -homomorphism on D , the first relation of (11) would imply that $\omega(z)$ is zero. But then, by the second relation of (11), we get that $|\omega(y)| = 1$ which would contradict the above inequality. We conclude that $x = y = 0$. \square

We have proven that a morphism $\pi : (A, \Delta, \alpha) \rightarrow (C_\pi, \Delta_\pi, \alpha_\pi)$ of QTGs of (B, L) belongs to Π (Notation 5.5) if and only if $\pi(x) = \pi(y) = 0$. So, the ideal I in Theorem 5.6 is the ideal generated by x and y . Now I is a Woronowicz Hopf C^* -ideal. Indeed, by (12), we can immediately see that $(p_I \otimes p_I)\Delta(x) = (p_I \otimes p_I)\Delta(y) = 0$.

Hence A/I is the C^* -algebra generated by a unitary \bar{z} and a projection \bar{p} . This is a CQG for the comultiplication given by

$$\begin{aligned} \Delta_I(\bar{z}) &= \bar{z} \otimes \bar{z}, \\ \Delta_I(\bar{p}) &= \bar{p} \otimes \bar{p} + (1 - \bar{p}) \otimes (1 - \bar{p}). \end{aligned} \tag{16}$$

Notice that A/I is isomorphic to $C(\mathbb{T}) * C(\mathbb{Z}_2)$. The coaction α_I of A/I on $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ is given by

$$\begin{aligned} \alpha_I(e_{12}, 0, 0) &= (e_{12}, 0, 0) \otimes \bar{z}, \\ \alpha_I(0, 1, 0) &= (0, 1, 0) \otimes \bar{p} + (0, 0, 1) \otimes (1 - \bar{p}). \end{aligned} \tag{17}$$

By Theorem 5.6 (or Theorem 5.7) this coaction is isometric. And because of the construction, the QTG $(A/I, \Delta_I, \alpha_I)$ is the desired universal object in the category of all QTGs acting isometrically on $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$ and preserving the trace.

5.3 Full-isometric coactions

In the above ‘small’ example, we have described a way to obtain quantum isometry groups using Theorem 5.6. However, as we mentioned earlier, we have not been able to show that the ideal I is a Woronowicz Hopf C^* -ideal in the general setting. To solve this problem, we introduce a new, possibly stronger notion of isometric coactions, that we will call *full-isometric*. This new notion will be more technical, but it will give an easy solution to the previous problem. Also, we will see that in the described half-classical cases (as well as in some easy examples), these full-isometric coactions will turn out to be the same as the isometric coactions we introduced before. Actually, we do not have an example where the notion of a full-isometry is indeed stronger than the described notion of an isometry.

So, the goal is to develop a notion of full-isometric coactions that makes sure that every CQG, acting on a CQMS, has a largest subgroup that acts full-isometrically. That is, the intersection I of the kernels of all morphisms of CQGs that have a full-isometric induced coaction, should be a Woronowicz Hopf C^* -ideal. To formulate the correct notion, we introduce the following notation.

DEFINITION 5.8

Let B be a C^* -algebra and (C_i, Δ_i, β_i) be QTGs of B for $i = 1, \dots, n$. Then we denote $\beta_1 * \beta_2$ for the $*$ -homomorphism $(\beta_1 \otimes \iota)\beta_2 : B \rightarrow B \otimes C_1 \otimes C_2$. We call it the *composition* of β_1 and β_2 . Inductively we can define $\beta := \beta_1 * \dots * \beta_n : B \rightarrow B \otimes C_1 \otimes \dots \otimes C_n$.

DEFINITION 5.9 (Full-isometric coaction)

A coaction α of a CQG (A, Δ) on a CQMS (B, L) is called *full-isometric* if and only if for all $n \in \mathbb{N}$ and all faithful isometric QTGs (C_i, Δ_i, β_i) ($i = 1, \dots, n$) such that $\beta := \beta_1 * \dots * \beta_n$ is isometric, we have that $\alpha * \beta$ is isometric, i.e.

$$\forall \omega \in \mathcal{S}(A \otimes C_1 \otimes \dots \otimes C_n), \forall b \in B : L((\iota \otimes \omega)(\alpha \otimes \iota)\beta(b)) \leq L(b).$$

Notice that the term isometric was defined in Definition 3.1 not only for coactions but the more general setting of *-homomorphisms; so we can use it for the mappings of the form $\beta_1 * \dots * \beta_n$ with β_i coactions.

As the terminology suggests, being full-isometric implies being isometric. We show that the full-isometric and the isometric coactions are the same in both half-classical cases.

Theorem 5.10. *Let G be a compact group, (B, L) a CQMS and $\alpha : B \rightarrow B \otimes C(G)$ a coaction. Then α is isometric if and only if α is full-isometric.*

Proof. Suppose α is an isometric coaction. Choose faithful isometric QTGs (C_i, Δ_i, β_i) for $i = 1, \dots, n$ such that $\beta = \beta_1 * \dots * \beta_n$ is isometric. We will write C for $C_1 \otimes \dots \otimes C_n$. To prove that $\alpha * \beta$ is isometric, we fix an element $b \in B$ and a pure state ω on $C(G) \otimes C$. Since $C(G)$ is commutative, we can use Theorem 4.14 of [15] to write $\omega = \omega_s \otimes \varphi$ for pure states ω_s on $C(G)$ and φ on C . But then we have

$$\begin{aligned} L((\iota \otimes \omega)(\alpha * \beta)(b)) &= L((\iota \otimes \omega_s)\alpha((\iota \otimes \varphi)\beta(b))) \\ &\leq L((\iota \otimes \varphi)\beta(b)) \leq L(b) \end{aligned}$$

since both α and β are isometric. Lemma 3.2 shows that this inequality holds for all states on $C(G) \otimes C$ since it holds for all pure states. \square

Also in the context of Banica, described in and before Theorem 3.5, full-isometric turns out to be equivalent with isometric.

Theorem 5.11. *Let (X, d) be a finite metric space with n points and (A, Δ) a CQG acting faithfully on X by a coaction $\alpha : C(X) \rightarrow C(X) \otimes A$. Consider the Lipnorm L_d on $C(X)$, induced by the distance function d on X . Then α is isometric if and only if α is full-isometric.*

Proof. To prove that an isometric coaction α is also full-isometric, we first choose faithful isometric QTGs (C_i, Δ_i, β_i) for $i = 1, \dots, m$ such that $\beta = \beta_1 * \dots * \beta_m$ is isometric. We want to show that $\alpha * \beta$ is isometric. We write Y for $A \otimes C_1 \otimes \dots \otimes C_m$, and we use y_{ij} for the elements in Y such that, for every $j \in \{1, \dots, n\}$ we have

$$(\alpha * \beta)(\delta_j) = \sum_{i=1}^n \delta_i \otimes y_{ij}.$$

Then, by Theorem 4.3, it is sufficient to check that the matrix $y = (y_{ij})_{ij}$ is a magic biunitary that commutes with the distance-matrix d .

We need to compute the elements y_{ij} . For every $k \in \{1, \dots, m\}$ we can take elements $c_{ij}^{(k)}$ in C_k such that $\beta_k(\delta_j) = \sum_{i=1}^n \delta_i \otimes c_{ij}^{(k)}$ for every $j \in \{1, \dots, n\}$. We can also take

elements a_{ij} in A such that for every $j \in \{1, \dots, n\}$ we have $\alpha(\delta_j) = \sum_{i=1}^n \delta_i \otimes a_{ij}$. Then one can see that for every $i, j \in \{1, \dots, n\}$, we have

$$y_{ij} = \sum_{k_1, \dots, k_m=1}^n a_{ik_1} \otimes c_{k_1 k_2}^{(1)} \otimes c_{k_2 k_3}^{(2)} \otimes \dots \otimes c_{k_{m-1} k_m}^{(m-1)} \otimes c_{k_m j}^{(m)}. \tag{18}$$

Now we check whether the elements y_{ij} satisfy all the conditions needed to apply Theorem 4.3. Firstly, since α and all the β_k are coactions, we know that the matrix $a = (a_{ij})_{ij}$ and all matrices $c^{(k)} = (c_{ij}^{(k)})_{ij}$ are magic biunitary matrices. From this, one can easily see that y also will be a magic biunitary.

Secondly, since α and all the β_k are supposed to be faithful, isometric coactions, we know by Theorem 3.5 that a and all $c^{(k)}$ commute with d . Then, using (18), it is straightforward to verify that d commutes with y as well. \square

5.4 Largest subgroup acting full-isometrically

Now we are ready to generalize the classical fact 5.3 to the quantum setting for the notion of full-isometric coactions.

Theorem 5.12. *Let (A, Δ) be a CQG with bounded counit, and (B, L) a CQMS. If α is a faithful coaction of (A, Δ) on (B, L) , then there exists a proper Woronowicz Hopf C^* -ideal I in A such that*

$$\alpha_I : B \rightarrow B \otimes A/I : b \mapsto (\iota \otimes p_I)\alpha(b)$$

is a faithful and full-isometric coaction, where p_I is the canonical projection of A onto A/I . The ideal I can be chosen to be the smallest one possible, in the sense that for any other Woronowicz Hopf C^ -ideal J for which $\alpha_J = (\iota \otimes p_J)\alpha$ is a faithful and full-isometric coaction, we have that $I \subseteq J$.*

To be able to construct the desired (and natural) Woronowicz Hopf C^* -ideal, we introduce some notations.

DEFINITION 5.13

Let B be a unital C^* -algebra and take QTGs (A, Δ, α) and $(C_i, \Delta_i, \alpha_i)$ for $i = 1, \dots, n$. We say that $\pi : A \rightarrow C_1 \otimes \dots \otimes C_n$ is a *multi-morphism* of QTGs if π is of the form $\pi_1 * \dots * \pi_n$ for some morphisms of QTGs $\pi_i : A \rightarrow C_i$ ($i = 1, \dots, n$). We used the notation $\pi_1 * \pi_2$ to denote the $*$ -homomorphism $(\pi_1 \otimes \pi_2)\Delta : A \rightarrow C_1 \otimes C_2$, and inductively one can define $\pi_1 * \dots * \pi_n$.

Notation 5.14. Let (B, L) be a CQMS and (A, Δ, α) a QTG of B . Let $\pi_i : (A, \Delta, \alpha) \rightarrow (C_{\pi_i}, \Delta_{\pi_i}, \alpha_{\pi_i})$ be morphisms of QTGs for $i = 1, \dots, n$. We write Π_f for the class of all multi-morphisms of QTGs $\pi = \pi_1 * \dots * \pi_n$ such that every coaction α_{π_i} is faithful and full-isometric ($i = 1, \dots, n$).

Proof of Theorem 5.12. We claim that the desired Woronowicz Hopf C^* -ideal will be

$$I = \bigcap_{\pi \in \Pi_f} \ker(\pi). \tag{19}$$

First, we show that the set I is indeed a Woronowicz Hopf C^* -ideal, i.e. $(p_I \otimes p_I)\Delta(I) = \{0\}$. To do so, it is sufficient to prove that $(\pi_1 \otimes \pi_2)\Delta(I) = \{0\}$ for every $\pi_1, \pi_2 \in \Pi_f$. So we choose $\pi_1 = \pi_1^{(1)} * \cdots * \pi_n^{(1)} : A \rightarrow C_1^{(1)} \otimes \cdots \otimes C_n^{(1)}$ and $\pi_2 = \pi_1^{(2)} * \cdots * \pi_m^{(2)} : A \rightarrow C_1^{(2)} \otimes \cdots \otimes C_m^{(2)}$ in Π_f . This means that all $(C_i^{(j)}, \Delta_i^{(j)}, \alpha_i^{(j)})$ are faithful QTGs of B such that the coactions $\alpha_i^{(j)}$ are full-isometric and $\pi_i^{(j)}$ are morphisms of QTGs. But then obviously

$$\pi_1 * \pi_2 = \pi_1^{(1)} * \cdots * \pi_n^{(1)} * \pi_1^{(2)} * \cdots * \pi_m^{(2)}$$

is still a multi-morphism of QTGs, and since all $\alpha_i^{(j)}$ are faithful and full-isometric, we know that $\pi_1 * \pi_2$ belongs to Π_f . Hence $(\pi_1 \otimes \pi_2)\Delta(I) = (\pi_1 * \pi_2)(I) = \{0\}$, so I is a Woronowicz Hopf C^* -ideal. Notice that I is proper since the counit ϵ belongs to Π_f .

Now it makes sense to consider the CQG A/I with comultiplication

$$\Delta_I : A/I \rightarrow A/I \otimes A/I : a + I \mapsto (p_I \otimes p_I)\Delta(a)$$

and the coaction α_I of $(A/I, \Delta_I)$ on B , as defined in the theorem. Since α is faithful, it is clear that also the induced coaction α_I is faithful.

To prove that α_I is full-isometric, we first choose faithful, isometric QTGs (C_i, Δ_i, β_i) for $i = 1, \dots, n$ such that $\beta = \beta_1 * \cdots * \beta_n$ is isometric. Denote $C_1 \otimes \cdots \otimes C_n$ by C . We want to prove that $\alpha_I * \beta : B \rightarrow B \otimes A/I \otimes C$ is isometric, using Lemma 3.2.

We choose a multi-morphism $\pi \in \Pi_f$. Then we can write $\pi = \pi_1 * \cdots * \pi_m$ for some morphisms $\pi_i : (A, \Delta, \alpha) \rightarrow (C_{\pi_i}, \Delta_{\pi_i}, \alpha_{\pi_i})$ of QTGs. We use C_π to denote $C_{\pi_1} \otimes \cdots \otimes C_{\pi_m}$. We introduce some more notations: for $i \in \{1, \dots, m\}$ we write $\bar{\pi}_i$ for the unique $*$ -homomorphism from A/I to C_{π_i} such that $\bar{\pi}_i \circ p_I = \pi_i$. We remark that this is well-defined since $\pi_i(I) = \{0\}$. We denote $\bar{\pi}_1 * \cdots * \bar{\pi}_m$ by $\bar{\pi}$, where the $*$ -product is defined using the comultiplication Δ_I . We will also write $\alpha^{(j)}$ for $\alpha * \cdots * \alpha$, the $*$ -product of j factors α (hence $\alpha^{(j)}$ is a mapping from B to $B \otimes A^{\otimes j}$). The symbol $\Delta^{(j)}$ denotes $(\Delta \otimes \iota \otimes \cdots \otimes \iota) \dots (\Delta \otimes \iota)\Delta$, where we have j factors in the composition (hence $\Delta^{(j)}$ is a mapping from A to $A^{\otimes(j+1)}$).

Notice that for any state ω on $C_\pi \otimes C$, the mapping $\omega(\bar{\pi} \otimes \iota)$ is a state on $A/I \otimes C$. Moreover, for any $b \in B$, we have

$$\begin{aligned} & L((\iota \otimes \omega(\bar{\pi} \otimes \iota))(\alpha_I * \beta)(b)) \\ &= L((\iota \otimes \omega)(\iota \otimes \bar{\pi} \otimes \iota)(\iota \otimes p_I \otimes \iota)(\alpha \otimes \iota)\beta(b)) \\ &= L((\iota \otimes \omega)(\iota \otimes \pi \otimes \iota)(\alpha \otimes \iota)\beta(b)) \\ &= L((\iota \otimes \omega)(\iota \otimes \pi_1 \otimes \cdots \otimes \pi_m \otimes \iota)(\iota \otimes \Delta^{(m-1)} \otimes \iota)(\alpha \otimes \iota)\beta(b)) \\ &= L((\iota \otimes \omega)(\iota \otimes \pi_1 \otimes \cdots \otimes \pi_m \otimes \iota)(\alpha^{(m)} \otimes \iota)\beta(b)) \\ &= L((\iota \otimes \omega)(\alpha_{\pi_1} * \cdots * \alpha_{\pi_m} \otimes \iota)\beta(b)) \\ &= L((\iota \otimes \omega)(\alpha_{\pi_1} * \cdots * \alpha_{\pi_m} * \beta)(b)). \end{aligned}$$

Now, since π belongs to Π_f , we know that every coaction α_{π_i} is faithful and full-isometric. In particular, α_{π_m} is full-isometric, so, because of the choice of β , we now know that $\alpha_{\pi_m} * \beta$ is isometric. Since also α_{π_m} and all β_i are faithful and isometric, and $\alpha_{\pi_{m-1}}$ is full-isometric, this implies that $\alpha_{\pi_{m-1}} * \alpha_{\pi_m} * \beta$ is isometric. We can continue this argument to conclude that $\alpha_{\pi_1} * \cdots * \alpha_{\pi_m} * \beta$ is isometric. Together with the above calculation, it follows that $L((\iota \otimes \omega(\bar{\pi} \otimes \iota))(\alpha_I * \beta)(b)) \leq L(b)$ for any element $b \in B$, any $\pi : A \rightarrow C_\pi$ in Π_f and any state ω on $C_\pi \otimes C$.

By Lemma 3.2, in order to conclude that $\alpha_I * \beta$ is isometric, it suffices to show that the convex hull of the set

$$S = \{\omega(\bar{\pi} \otimes \iota) \mid \pi \in \Pi_f : A \rightarrow C_\pi, \omega \in \mathcal{S}(C_\pi \otimes C)\}$$

is weakly $*$ -dense in the set of states on $A/I \otimes C$. This will follow by applying Lemma 3.3 to the C^* -algebra $A/I \otimes C$ and the set Π of $*$ -homomorphisms $\bar{\pi} \otimes \iota : A/I \otimes C \rightarrow C_\pi \otimes C$ with $\pi \in \Pi_f$, if we show that this set Π separates the points of $A/I \otimes C$. So, choose an element x in $A/I \otimes C$ and suppose x is nonzero. Then, since the product states separate the points of the minimal tensor product, there exists a state ψ on C such that $(\iota \otimes \psi)(x)$ is non-zero. We can take an element y in A such that $p_I(y) = (\iota \otimes \psi)(x)$. Since this element is non-zero in A/I , we know that $y \notin I$. Hence, there exists a morphism π in Π_f such that $\pi(y)$ is non-zero. But then $\bar{\pi}((\iota \otimes \psi)(x)) = \bar{\pi}(p_I(y)) = \pi(y)$ is non-zero in C_π . So we may conclude that $(\bar{\pi} \otimes \iota)(x)$ is non-zero, which proves that Π separates the points of $A/I \otimes C$. This completes the proof of the fact that α_I is full-isometric.

For the final part of the theorem, let J be any Woronowicz Hopf C^* -ideal in A , such that the coaction

$$\alpha_J : B \rightarrow B \otimes A/J : b \mapsto (\iota \otimes p_J)\alpha(b)$$

is faithful and full-isometric. We denote the canonical projection of A onto A/J by p_J .

We want to prove that $I \subseteq J$. Since J is a Woronowicz Hopf C^* -ideal, we know that A/J is a compact quantum group with comultiplication

$$\Delta_J : A/J \rightarrow A/J \otimes A/J : p_J(a) \mapsto (p_J \otimes p_J)\Delta(a).$$

Hence, by definition, the mapping p_J is a morphism of QTGs from (A, Δ, α) to $(A/J, \Delta_J, \alpha_J)$. Moreover, since α_J is a faithful and full-isometric coaction, we know that p_J belongs to Π_f . But then, by construction, I is a subset of the kernel of p_J , which is exactly J . □

5.5 Existence of a full-isometry group for a finite CQMS

Now we can put everything together to prove the existence of a full-isometry group in cases where there is a quantum permutation group. For example, for finite compact quantum metric spaces, we can define a quantum isometry group fixing the trace (or another functional), since Wang defined the quantum permutation group [16,17].

Theorem 5.15. *Let (B, L) be a finite compact quantum metric space and let ψ be a functional on B . Then there exists a quantum isometry group of (B, L, ψ) .*

Proof. Denote by (A, Δ, α) the quantum permutation group of (B, ψ) , according to Wang's definition [16]. This means that (A, Δ, α) is a universal object in the category $\mathcal{C}(B, \psi)$ of all faithful transformation groups of the pair (B, ψ) .

Let I be the Woronowicz Hopf C^* -ideal as in (the proof of) Theorem 5.12, given by (19). Then we claim that $(A/I, \Delta_I, \alpha_I)$ is the desired quantum isometry group of (B, L, ψ) . As before, we used Δ_I to denote the induced comultiplication on A/I , and α_I to denote the induced coaction $(\iota \otimes p_I)\alpha$.

From Theorem 5.12 we already know that $(A/I, \Delta_I, \alpha_I)$ is a faithful and full-isometric quantum transformation group. To prove the universality, let $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$ be a second faithful and full-isometric quantum transformation group of (B, ψ) . We want to show that

there exists a unique morphism of quantum transformation groups from $(A/I, \Delta_I, \alpha_I)$ to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$.

Because of the universality of the object (A, Δ, α) in the category $\mathcal{C}(B, \psi)$, we know that there is a unique morphism of QTGs $\theta : A \rightarrow \tilde{A}$ from (A, Δ, α) to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$.

Since θ is a morphism of QTGs, we know that $\alpha_\theta = (\iota \otimes \theta)\alpha$ equals $\tilde{\alpha}$, hence α_θ is full-isometric. But then, by definition, θ belongs to Π_f . It is now clear that $\theta(I) = \{0\}$, by the construction of I .

We have verified that the mapping

$$\bar{\theta} : A/I \rightarrow \tilde{A} : (a + I) \mapsto \theta(a) \quad (20)$$

is well defined. Remark that $\bar{\theta}p_I = \theta$. It is easy to see that $\bar{\theta}$ is the desired morphism of quantum transformation groups from $(A/I, \Delta_I, \alpha_I)$ to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$.

To conclude the proof, we have to check the uniqueness of $\bar{\theta}$. But if we take any morphism of quantum transformation groups σ from $(A/I, \Delta_I, \alpha_I)$ to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$, then σp_I would be a morphism of quantum transformation groups from (A, Δ, α) to $(\tilde{A}, \tilde{\Delta}, \tilde{\alpha})$. Because of the uniqueness of θ , this implies that $\sigma p_I = \theta$. But then of course $\sigma = \bar{\theta}$, which proves the uniqueness of $\bar{\theta}$.

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