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Quotient semigroups and extension semigroups

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Abstract. We discuss properties of quotient semigroup of abelian semigroup from the viewpoint of C^*-algebra and apply them to a survey of extension semigroups. Certain interrelations among some equivalence relations of extensions are also considered.

Keywords. Quotient semigroup; extension semigroup; congruence relation.

1. Introduction

Abelian groups and semigroups play an important role in the classification of C^*-algebras and their extensions. Especially, equipped with addition, the sets of equivalence classes of extensions become quotient semigroups. Therefore, the theory of quotient semigroups is essential for studying C^*-algebras and their extensions, though it is not a new object from the viewpoint of pure algebra. But the existing theory of semigroups in algebra is not suitable to being applied to C^*-algebra because of lack of details. In order to understand and apply C^*-algebra extension theory and KK-theory, it is crucial to study the theory of quotient semigroups from the viewpoint of C^*-algebra.

Classifications of C^*-algebras and their extensions are two main aspects in the field of operator algebra since 1970's. One can see [1] for details of their developments (see also [2,5–8,12–15]). In the recent past, interest in extension algebras has resulted in classification of such algebras [10,11,17–19]. Unlike classification of extensions, Ext-groups in KK-theory provide little information for classification of extension algebras, and so the isomorphism equivalence of extensions has not been studied.

Besides extension semigroups, the semigroups of Murray–von Neumann equivalence classes of projections have already been applied to classifying C^*-algebras. Therefore, we need to study systematically the questions of semigroups relating to C^*-algebras and their extensions.

In §2, we discuss some fundamental questions of quotient semigroups relating to classifying C^*-algebras. In §3, we give a survey of extension semigroups. Most of them are not new. We discuss certain interrelations among some equivalence relations of C^*-algebra extensions.

2. Quotient semigroups

In this section, we discuss several fundamental questions on semigroups. These questions are important in studying extension semigroups of C^*-algebras, but they are not considered systematically in existing literature. We also give some examples to make these questions clear.

Let us firstly recall some definitions on semigroups. Suppose that S is a semigroup. Then an equivalence relation $\sim$ on S is said to be congruent if $a \sim b$ and $c \sim d$ implies $ac \sim bd$ for any elements a, b, c and d in S. Let $S/\sim$ be the quotient set and $\pi : S \to S/\sim$ the quotient mapping which maps a to $\tilde{a}$, where $\tilde{a}$ is the equivalence class of S containing a.

The following results are easy to prove (see also [4]).

PROPOSITION 2.1

Let S be a semigroup and $\sim$ an equivalence relation on S. Then the following statements are equivalent:

(1) *The relation $\sim$ is a congruence relation*;
(2) *If $a \sim b$, then we have $ac \sim bc$ and $ca \sim cb$ for each element $c \in S$*;
(3) *The set $S/\sim$ becomes a semigroup, which is called quotient semigroup, equipped with the multiplication defined by $\tilde{a} \cdot \tilde{b} = \widetilde{ab}$.*

PROPOSITION 2.2

If the quotient semigroup $S/\sim$ admits a unit element E, then $\ker \pi = E$ and $\ker \pi$ is a sub-semigroup, where $\pi : S \to S/\sim$ is the quotient homomorphism.

In the rest of this section, we only discuss the properties of quotient semigroup of abelian semigroup. The semigroups we considered in the following are always abelian if they are not specialized.

Similar to the construction of quotient group and quotient linear space, quotient semigroup may be induced by the equivalence relation which is induced by sub-semigroup.

Let $(S, +)$ be an abelian semigroup and S_1 a sub-semigroup of S. Define a relation $\overset{S_1}{\sim}$ on S by $a \overset{S_1}{\sim} b$ if and only if there exist elements $x_1, x_2 \in S_1$ such that $a + x_1 = b + x_2$.

Remarks.

(1) The relation $\overset{S_1}{\sim}$ is a congruence relation on S. Let $S/S_1 = S/\overset{S_1}{\sim}$. Then we call it the quotient semigroup induced by S_1.
(2) It should be pointed out, however, that $a \overset{S_1}{\sim} b$ is not equivalent to that and there exists an element $x \in S_1$ such that $a + x = b$ or $b + x = a$. See Example 2.

There are some essential questions we have to consider. Suppose that $\sim$ is a congruence relation on S.

Question 1. Does $S/\sim$ admit a unit element? If $S/\sim$ has a unit element E, is E a sub-semigroup? What is the relation between E and $\ker \pi$? What about S/S_1?

Question 2. When is the congruence relation $\sim$ induced by a sub-semigroup S_1, that is, when do we have $S/\!\sim\, = S/S_1$?

Question 3. If $S/\!\sim$ is a group, does $S/\ker\pi = S/\!\sim$ hold? Does the element in $S/\!\sim$ have the form $a\ker\pi$?

Question 4. In S/S_1, when does $\ker\pi = S_1$ hold?

Conclusion 1.

(1) In general, $S/\!\sim$ does not admit a unit element (see Example 1). $S/\!\sim$ admits a unit element if and only if there exists an equivalence class E such that E is a sub-semigroup and $a + e \sim a$ for any $a \in S$ and $e \in E$.
(2) If E is the unit element of $S/\!\sim$, then E is a sub-semigroup and $E = \ker\pi$.
(3) The quotient semigroup S/S_1 induced by a sub-semigroup S_1 admits a unit element E, that is, $E = \ker\pi$.

One can see that $S_1 \subset E$, but the converse does not hold in general (see Example 2).

Example 1. Let $S = \mathbb{N} \oplus \mathbb{N}$ and define $(a, b) \sim (c, d) \Leftrightarrow b = d$. Then $\sim$ is a congruence relation and $S/\!\sim\, = \{\mathbb{N} \oplus 1, \mathbb{N} \oplus 2, \ldots\}$. Hence $S/\!\sim$ does not admit a unit element. This congruence relation can not be induced by any sub-semigroup of S.

Example 2. Let $S = \mathbb{N}$ and $S_1 = \{6, 8, 10, \ldots\}$. Since $1 + (2n+8) = (2n-1) + 10$, $2n + 8 \in S_1$ and $10 \in S_1$, $\tilde{1} = \widetilde{2n-1}$ ($n \in \mathbb{N}$). We also have $\tilde{2} = \widetilde{2n}$ ($n \in \mathbb{N}$). Therefore $S/S_1 = \{\tilde{1}, \tilde{2}\}$.

(1) It is easy to see that $\ker\pi = \{2, 4, 6, \ldots\}$, so $\ker\pi \neq S_1$.
(2) We have $2 \sim 4$. However, there is not an element $x \in S_1$ such that $2 + x = 4$ or $4 + x = 2$.
(3) One can also see that $2 \sim 6$ and $6 \in S_1$, but $2 \notin S_1$.

PROPOSITION 2.3

If there is a sub-semigroup S_1 such that $S/\!\sim\, = S/S_1$, then $S/S_1 = S/\ker\pi$.

Proof. It is equivalent to show that $a \overset{S_1}{\sim} b$ if and only if $a \overset{\ker\pi}{\sim} b$. If $a \overset{S_1}{\sim} b$, then there exist elements $x_1, x_2 \in S_1$ such that $a + x_1 = b + x_2$, and hence $a \overset{\ker\pi}{\sim} b$ (since $S_1 \subset \ker\pi$).

Conversely, if $a \overset{\ker\pi}{\sim} b$, then there exist elements $x_1', x_2' \in \ker\pi$ such that $a + x_1' = b + x_2'$, and hence $\pi(a) = \pi(b)$. It follows that $a \overset{S_1}{\sim} b$. $\square$

Conclusion 2. If $\sim$ is induced by a sub-semigroup S_1, i.e. $\sim\, \Leftrightarrow \overset{S_1}{\sim}$, then $S_1 \subset \ker\pi$ and $S/\!\sim\, = S/S_1 = S/\ker\pi$ by Proposition 2.3. Hence $\ker\pi$ is the maximal sub-semigroup with $S/\!\sim\, = S/S_1$.

From the above discussion, if $S/\!\sim$ admits a unit element, the unit element must be $\ker\pi$, and $a \overset{\ker\pi}{\sim} b$ implies $a \sim b$. The remaining question is that when are $\overset{\ker\pi}{\sim}$ and $\sim$ equivalent.

Conclusion 3.

(1) If $S/\sim$ is a group, then $a \overset{\ker \pi}{\sim} b \Leftrightarrow a \sim b$, so we have $S/\sim = S/\ker \pi$.
(2) It is not necessary that each element of $S/\sim$ has the form $a \ker \pi$. See Example 3 below.

Proof. To prove Conclusion 3(1), it is easy to see that $\pi(a) = \pi(b)$ from $a \sim b$. Suppose $\pi(c)$ is the inverse of $\pi(a)$ in $S/\sim$. Then $\pi(a) + \pi(c) = \pi(b) + \pi(c)$ is the unit element of $S/\sim$. Hence $a + c,\ b + c \in \ker \pi$. Since $a + (b + c) = b + (a + c)$, $a \overset{\ker \pi}{\sim} b$. Thus $S/\sim = S/\ker \pi$ if $S/\sim$ is a group. □

Example 3. Let $S = \mathbb{N} \oplus \mathbb{Z}$ and $\sim$ the congruence relation induced by the sub-semigroup $\mathbb{N} \oplus 0$. Then $S/\sim$ has the unit E and $E = \ker \pi = \mathbb{N} \oplus 0$. Hence $S/\sim$ is a group. Consider the equivalence class $A = \{n \oplus 5 : n \in \mathbb{N}\}$ of $a = (3, 5) \in A$. Then $aE = \{(3 + n, 5) : n \in \mathbb{N}\}$, but $aE \neq A$.

Conclusion 4. In order to find the condition for $\ker \pi = S_1$, we give the following definition. Let S be an abelian semigroup and S_1, S_2 be sub-semigroups of S. Put $S_2 - S_1 = \{a \in S : \exists x_i \in S_i,\ s.t.\ a + x_1 = x_2\}$. Then $S_2 - S_1$ is a sub-semigroup of S.

We have the following results for Question 4.

Theorem 2.4. *Let $\pi : S \to S/S_1$ be the quotient homomorphism. Then* $\ker \pi = S_1 - S_1$ *and hence* $\ker \pi = S_1$ *if and only if* $S_1 = S_1 - S_1$.

Proof. By the definition of $\pi : S \to S/S_1$, if $a \in \ker \pi$, then $\pi(a) = \tilde{0}$. This is equivalent to $a \sim x \in S_1$, i.e. there are elements $y_1, y_2 \in S_1$ such that $a + y_1 = x + y_2$. This occurs if and only if there exists an element $y \in S_1$ such that $a + y \in S_1$, so there exist elements $z_1, z_2 \in S_1$ such that $a + z_1 = z_2$. Hence $a \in \ker \pi$ if and only if $a \in S_1 - S_1$. Then we have $\ker \pi = S_1 - S_1$. Therefore $\ker \pi = S_1$ if and only if $S_1 = S_1 - S_1$. □

Remark. Let S_1 be a semigroup. Then $S_1 \subset S_1 - S_1$, but the converse does not hold in general.

Theorem 2.5. *Let S be an abelian semigroup with cancelation property. Then $S_1 - S_1 = G(S_1) \cap S$, where $G(S_1)$ is the Grothendieck group of S_1.*

Proof. Let $\gamma : S \to G(S)$ be the Grothendieck mapping of S. Then

$$G(S) = \{\gamma(x) - \gamma(y) :\ x,\ y \in S\}.$$

Since S has cancelation property, γ is injective. If we identify x with $\gamma(x)$, then S is a sub-semigroup of $G(S)$ and $G(S) = \{x - y :\ x,\ y \in S\}$. For any $a \in S_1 - S_1$ there exist elements $x_1, x_2 \in S_1$ such that $a + x_1 = x_2$, so $a = x_2 - x_1 \in G(S_1)$. Thus $S_1 - S_1 \subset S \cap G(S_1)$.

Conversely, for any $b \in S \cap G(S_1)$ there exist elements $x_1,\ x_2 \in S$ such that $b = x_2 - x_1$, then $b + x_1 = x_2$. Hence $b \in S_1 - S_1$. Therefore $S_1 - S_1 = G(S_1) \cap S$. □

COROLLARY 2.6

Let S be an abelian semigroup with cancelation property. If $G(S_1) \cap S \subset S_1$, then $\ker \pi = S_1$.

3. Extension semigroups

Let A and B be C^*-algebra, and let $\tau : A \to \mathcal{Q}$ be an extension of A by B, where $\mathcal{Q} = M(B)/B$ is the corona algebra of B. If A is unital, τ is said to be an unital extension if τ is unital.

Let $\tau_1, \tau_2 : A \to \mathcal{Q}$ be two extensions of A by B.

DEFINITION I (Weak equivalence) [2]

Two extensions τ_1 and τ_2 are called weakly equivalent, denoted by $\tau_1 \overset{I}{\sim} \tau_2$, if there is a partial isometry $v \in \mathcal{Q}$ such that v^*v is the unit of $\tau_1(A)$ and $\tau_2 = Adv \circ \tau_1$. The set of all weak equivalence classes of extensions of A by B is denoted by $\mathrm{Ext}_I(A, B)$.

DEFINITION II (Weak equivalence) [3]

Two extensions τ_1 and τ_2 are called weakly (unitarily) equivalent, denoted by $\tau_1 \overset{w}{\sim} \tau_2$, if there is a unitary $u \in \mathcal{Q}$ such that $\tau_2 = Adu \circ \tau_1$. The set of all weak unitary equivalence classes of extensions of A by B is denoted by $\mathrm{Ext}_w(A, B)$.

DEFINITION III (Strong equivalence) [3]

Two extensions τ_1 and τ_2 are called strongly (unitarily) equivalent, denoted by $\tau_1 \overset{s}{\sim} \tau_2$, if there is a unitary $u \in M(B)$ such that $\tau_2 = Ad\pi(u) \circ \tau_1$. The set of all strong equivalence classes of extensions of A by B is denoted by $\mathrm{Ext}_s(A, B)$.

DEFINITION IV [14]

Let A be a unital C^*-algebra. Then τ_1 and τ_2 are called equivalent if there is a partial isometry $v \in \mathcal{Q}$ such that $v^*v = \tau_1(1_A)$, $vv^* = \tau_2(1_A)$ and $\tau_2 = Adv \circ \tau_1$. Moreover, v is a unitary if both τ_1 and τ_2 are unital.

Remark. In Definition I, it follows that vv^* is the unit of $\tau_2(A)$. Then τ_1 is equivalent to τ_2 if and only if τ_2 is equivalent to τ_1. However, it does not follow that v^*v is the unit of $\tau_1(B)$ from the fact that $\tau_2 = Adv \circ \tau_1$ and vv^* is the unit of $\tau_2(A)$.

These four equivalence relations appeared in some occasions, and the existing references have not discussed their interrelation in detail. We now summarize their relationships in the case of $B = \mathcal{K}$ by using the results in [2] and [3].

1. *Relationships between weak equivalences.*

(1) They are the same if A is nonunital.

(2) If A is unital, there are three cases:

(i) If both τ_1 and τ_2 are nonunital, then $\tau_1 \overset{I}{\sim} \tau_2 \Leftrightarrow \tau_1 \overset{w}{\sim} \tau_2$.

(ii) If both τ_1 and τ_2 are unital, we also have $\tau_1 \overset{I}{\sim} \tau_2 \Leftrightarrow \tau_1 \overset{w}{\sim} \tau_2$.

(iii) If either τ_1 or τ_2 is unital, but another is nonunital, then $\tau_1 \overset{I}{\sim} \tau_2 \not\Leftrightarrow \tau_1 \overset{w}{\sim} \tau_2$.

2. *Relationships between weak equivalences and strong equivalence.*

(1) If A is nonunital, then Definition II $\Leftrightarrow$ Definition III, and hence

$$\text{Definition I} \Leftrightarrow \text{Definition II} \Leftrightarrow \text{Definition III}.$$

(2) If A is unital, there are two cases:

(i) If both τ_1 and τ_2 are unital, then $\tau_1 \overset{s}{\sim} \tau_2 \not\Leftrightarrow \tau_1 \overset{w}{\sim} \tau_2$.

(ii) If both τ_1 and τ_2 are nonunital, then $\tau_1 \overset{s}{\sim} \tau_2 \Leftrightarrow \tau_1 \overset{w}{\sim} \tau_2$.

3. *If A is nonunital and $\tau_1 \overset{I}{\sim} \tau_2$, then v^*v does not belong to $\tau_1(A)$ in general.*

If A is unital, $\tau_i(1_A)$ is the unit of $\tau_i(A)$. Then Definition I and Definition IV are equivalent.

Let B be a stable C^*-algebra. Then $\mathrm{Ext}_I(A, B)$, $\mathrm{Ext}_w(A, B)$ and $\mathrm{Ext}_s(A, B)$ are abelian semigroups under suitable sum of extensions. If B is not stable, then we replace B by $B \otimes \mathcal{K}$.

One can define the following sub-semigroups: $\mathrm{Ext}^u_*(A, B)$, $\mathrm{Ext}^e_*(A, B)$, $\mathrm{Ext}^{eu}_*(A, B)$, $\mathrm{Ext}_{*0}(A, B)$, $\mathrm{Ext}^{su}_{*0}(A, B)$, $\mathrm{Ext}^e_{*0}(A, B)$, $\mathrm{Ext}^{esu}_{*0}(A, B)$ and $\mathrm{Ext}^{nu}_*(A, B)$, where $* = w$ or s. The subscript 's' can be omited. Those with subscript '0' are the sets of equivalence classes of trivial extensions; those with superscript 'e' are the sets of equivalence classes of essential extensions. Those with superscript 'u' are the sets of equivalence classes of unital extensions.

Hence, $\mathrm{Ext}^{su}_{*0}(A, B)$ is the set of equivalence classes of strongly unital trivial extensions and $\mathrm{Ext}^e_{*0}(A, B)$ is the set of equivalence classes of essential trivial extensions. Similarly, $\mathrm{Ext}^{esu}_{*0}(A, B)$ is the set of equivalence classes of essential and strongly unital trivial extensions and $\mathrm{Ext}^{nu}_*(A, B)$ is the set of equivalence classes of nonunital extensions. The superscript 'a' means absorbing and 'ua' means unital-absorbing, such as $\mathrm{Ext}^a(A, B)$ and $\mathrm{Ext}^{ua}(A, B)$, etc.

Let $\mathcal{E}\mathrm{xt}(A, B)$ be the set of all extensions of A by B. We can define subsets of $\mathcal{E}\mathrm{xt}(A, B)$ analogously. We omit the notation $\mathcal{K}$ in all semigroups above in the case of $B = \mathcal{K}$ in order to abbreviate notations.

One can check the following relationships among the above semigroups when $B = \mathcal{K}$.

1. If A is nonunital, then $\mathrm{Ext}_I(A) = \mathrm{Ext}_w(A) = \mathrm{Ext}_s(A)$.
2. If A is unital, then

(1) $\mathrm{Ext}_s(A) = \mathrm{Ext}^u_s(A) \cup \mathrm{Ext}^{nu}_s(A)$ and $\mathrm{Ext}^u_s(A) \cap \mathrm{Ext}^{nu}_s(A) = \emptyset$.

(2) $\mathrm{Ext}_w(A) = \mathrm{Ext}^u_w(A) \cup \mathrm{Ext}^{nu}_w(A)$ and $\mathrm{Ext}^u_w(A) \cap \mathrm{Ext}^{nu}_w(A) = \emptyset$.

(3) $\mathrm{Ext}^{nu}_s(A) = \mathrm{Ext}^{nu}_w(A) = \mathrm{Ext}^{nu}_I(A)$ and $\mathrm{Ext}^{nu}_I(A) \to \mathrm{Ext}_I(A)$ is isomorphic.

(4) $\mathrm{Ext}^u_w(A) \to \mathrm{Ext}_I(A)$ is isomorphic, but $\mathrm{Ext}_w(A)$ and $\mathrm{Ext}_I(A)$ may not be the same.

(5) $\mathrm{Ext}^u_s(A) \to \mathrm{Ext}_I(A)$ is a unital surjective homomorphism, and it is the same as the homomorphism $\mathrm{Ext}^u_s(A) \to \mathrm{Ext}^u_w(A)$.

All the above homomorphisms map equivalence class to another equivalence class of the same extension. We now only prove that the homomorphism in (3) is surjective while the other proofs are similar.

Let $\sigma_1 : A \to \mathcal{Q}$ be a unital extension. Take $q \in \mathcal{Q}\backslash\{1\}$. Then there is a partial isometry v such that $vv^* = q$ and $v^*v = 1$. Put $\sigma_2 = Adv \circ \sigma_1$. Then we have

$$\sigma_2(ab) = v(\sigma_1(a))v^*v(\sigma_1(b))v^* = \sigma_2(a)\sigma_2(b)$$

and $\sigma_2(1) = q \neq 1$. Thus σ_2 is nonunital and $\sigma_1 \overset{I}{\sim} \sigma_2$. This shows that the homomorphism in (3) is surjective.

The following definitions of $\mathrm{Ext}_w(A, B)$ and $\mathrm{Ext}_s(A, B)$ are given in [1]:

$$\begin{aligned} E\mathrm{xt}_*(A, B) &= \mathrm{Ext}_*(A, B)/\mathrm{Ext}_{*0}(A, B), \\ E\mathrm{xt}^u_*(A, B) &= \mathrm{Ext}^u_*(A, B)/\mathrm{Ext}^{su}_{*0}(A, B), \\ E\mathrm{xt}^e_*(A, B) &= \mathrm{Ext}^e_*(A, B)/\mathrm{Ext}^e_{*0}(A, B), \\ E\mathrm{xt}^{eu}_*(A, B) &= \mathrm{Ext}^{eu}_*(A, B)/\mathrm{Ext}^{esu}_{*0}(A, B), \end{aligned}$$

where $* = s, w$. All the above quotient semigroups are abelian monoids. Write $E\mathrm{xt}(A, B) = E\mathrm{xt}_s(A, B)$. Two extensions are called stably strongly (weakly) equivalent if they stand for the same element in $E\mathrm{xt}(A, B)$ $[E\mathrm{xt}_w(A, B)]$ (denoted by ss (sw)).

Let $e_i : 0 \to B \to E_i \to A \to 0$ be two extensions of A by B with Busby invariants τ_i for $i = 1, 2$. One may see [1] for the definitions of the following isomorphisms.

DEFINITION V (Strong isomorphism)

Two extensions e_1 and e_2 are called strongly isomorphic if there is an isomorphism $\eta : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle id} & & \\ e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

DEFINITION VI (Unitary isomorphism)

Two extensions e_1 and e_2 are called unitarily isomorphic if there is an unitary $u \in M(B)$ and an isomorphism $\phi : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle Adu} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle id} & & \\ e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

DEFINITION VII (Isomorphism)

Two extensions e_1 and e_2 are called (weakly) isomorphic, denoted by $e_1 \cong e_2$, if there are isomorphisms $\alpha : B \to B,\ \eta : E_1 \to E_2$ and $\gamma : A \to A$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \gamma} & & \\ e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Let $E\text{xt}_{wi}(A)$ denote the set of all isomorphic equivalence classes. If there are homomorphisms $\alpha : B \to B,\ \eta : E_1 \to E_2$ and $\gamma : A \to A$ such that the above diagram commutes, we call (α, η, γ) an extension homomorphism from e_1 to e_2.

Relationships.

(1) It is obvious that V $\Longrightarrow$ VI $\Longrightarrow$ VII.
(2) Definition III $\Longleftrightarrow$ Definition VI (This has been proved through analysis method (see [9] for example), and we will give an algebraic proof in Theorem 3.1.)
(3) II $\not\Rightarrow$ VII, VII $\not\Rightarrow$ II.

In page 67 of [16], the author thinks that weak unitary equivalence implies weak isomorphism. However, this is not true from the following Examples 1 and 2.

(4) In general, the fact $e_1 \cong e_2$ and $e_1' \cong e_2'$ does not imply $e_1 \oplus e_1' \cong e_2 \oplus e_2'$. It follows that $E\text{xt}_{wi}(A)$ is not a semigroup under the above addition. Thus the statements on this question in page 68 of [16] are not accurate. One can see Example 2 for details.

Example 1. Let $A = M_n$. We consider extensions of A by $\mathcal{K}$. Let $\tau_1 : M_n \to \mathcal{Q}$ be a unital extension of M_n which is not strongly unital trivial, and τ_2 a strongly unital trivial extension of M_n. Since all the unital extensions of M_n are weakly unitarily equivalent, $\tau_1 \overset{w}{\sim} \tau_2$. But τ_1 is not weakly isomorphic to τ_2, otherwise τ_1 is strongly unitarily trivial since weak isomorphism preserves strong unitary triviality.

Example 2. Let $A = C(\mathbb{T})$. Consider the essential unital extension of A by $B = \mathcal{K}$: $0 \to \mathcal{K} \to E \to A \to 0$. We have

$$E\text{xt}(A) \cong \text{Hom}(K_1(A),\ K_0(B)) \cong \mathbb{Z}$$

by UCT. Since $K_1(B) = 0$, $\text{Ext}^{eu}(A) \cong E\text{xt}(A, B) = \mathbb{Z}$.

Let e_n and e_m be the essential unital extensions of A with index n and m, respectively. It follows from the above that e_n is strongly equivalent to e_m if and only if $n = m$. Furthermore, e_n and e_m are weakly isomorphic if and only if $|n| = |m|$. Therefore, if $n,\ m \geq 0$ and $n \neq m$, then $e_n \not\cong e_m$. Thus $e_2 \cong e_{-2}$ and $e_3 \cong e_3$. Since $e_2 \oplus e_3 \overset{s}{\sim} e_5,\ e_{-2} \oplus e_3 \overset{s}{\sim} e_1$ and $e_5 \not\cong e_1$, $e_2 \oplus e_3 \not\cong e_{-2} \oplus e_3$.

Theorem 3.1. *Let $e_i \in \mathcal{E}\mathrm{xt}^e(A, B)$ with Busby invariant τ_i for $i = 1, 2$. Then τ_1 and τ_2 are strongly unitarily equivalent if and only if e_1 and e_2 are unitarily isomorphic.*

Proof. Suppose that e_1 and e_2 are unitarily isomorphic. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Adu^*} & & \downarrow{\scriptstyle \phi^{-1}} & & \downarrow{\scriptstyle id} & & \\
e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \tau_1} & & \\
e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Adu} & & \downarrow{\scriptstyle Adu} & & \downarrow{\scriptstyle Ad\pi(u)} & & \\
e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0.
\end{array}$$

Let $\lambda : e_2 \to e_1$, $\xi : e_1 \to e_0$ and $\zeta : e_0 \to e_0$ be the homomorphisms between extensions in the above diagram. Then

$$\zeta \circ \xi \circ \lambda = (id, Adu \circ \eta_1 \circ \phi^{-1}, Ad\pi(u) \circ \tau_1) : e_2 \to e_0$$

is an extension homomorphism. Since η_2 and τ_2 are the unique homomorphisms, the following diagram commutes:

$$\begin{array}{ccccccccc}
e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \tau_2} & & \\
e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0.
\end{array}$$

Then $\tau_2 = Ad\pi(u) \circ \tau_1$ and $\eta_2 = Adu \circ \eta_1 \circ \phi^{-1}$, and hence τ_1 and τ_2 are strongly unitarily equivalent.

Conversely, suppose that τ_1 and τ_2 are strongly unitarily equivalent. For each $x \in E_1$, we have

$$\pi(Adu \circ \eta_1(x)) = Ad\pi(u) \circ \tau_1 \circ \psi_1(x) = \tau_2 \circ \psi_1(x).$$

Then $Adu \circ \eta_1(E_1) \subset \pi^{-1}(im\tau_2) = \eta_2(E_2)$.

Set $\phi = \eta_2^{-1} \circ Adu \circ \eta_1$. Since the following diagram

$$\begin{array}{ccccccccc}
e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \tau_1} & & \\
e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Adu} & & \downarrow{\scriptstyle Adu} & & \downarrow{\scriptstyle Ad\pi(u)} & & \\
e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

commutes and $\tau_2 = Ad\pi(u) \circ \tau_1$, the following diagram commutes:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Adu} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0. \end{array}$$

Thus e_1 and e_2 are unitarily isomorphic. □

Theorem 3.2. *Two extensions τ_1 and τ_2 are weakly equivalent if and only if there exists an element $v \in M(B)$ such that $\pi(v)$ is unitary in $\mathcal{Q}$, $im(Adv \circ \eta_1) \subset im\eta_2$, and the following diagram commutes*:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Adv} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0 \end{array}$$

where $\phi = \eta_2^{-1} \circ Adv \circ \eta_1$.

Proof.

($\Longrightarrow$) Suppose that there is a unitary $u \in \mathcal{Q}$ such that $\tau_2 = Adu \circ \tau_1$. Take $v \in M(B)$ such that $\pi(v) = u$, so we have the following commutative diagram:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \tau_1} & & \\ e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Adv} & & \downarrow{\scriptstyle Adv} & & \downarrow{\scriptstyle Adu} & & \\ e_0 : 0 & \longrightarrow & B & \longrightarrow & M(B) & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

Since $u(im\tau_1)u^* \subset im\tau_2$, $\pi(Adv(\pi^{-1}(im\tau_1))) \subset im\tau_2$. Hence $Adv(\pi^{-1}(im\tau_1)) \subset \pi^{-1}(im\tau_2)$ and $Adv(\eta_1(E_1)) \subset \eta_2(E_2)$. Therefore $\phi = \eta_2^{-1} \circ Adv \circ \eta_1 : E_1 \to E_2$ is well-defined. It follows that the following diagram is commutative:

$$\begin{array}{ccccccccc} e_1 : 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\psi_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Adv} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ e_2 : 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\psi_2} & A & \longrightarrow & 0. \end{array}$$

($\Longleftarrow$) Suppose that there is an element $v \in M(B)$ such that $\pi(v)$ is a unitary in $\mathcal{Q}$ and $\psi_1 = \psi_2 \circ \eta_2^{-1} \circ Adv \circ \eta_1$. Put $u = \pi(v)$. Since $\tau_i \circ \psi_i = \pi \circ \eta_i$, we have

$$\tau_2 \circ \psi_1 = \pi \circ Adv \circ \eta_1 = Adu \circ (\pi \circ \eta_1) = Adu \circ (\tau_1 \circ \psi_1).$$

Then $\tau_2 \circ \psi_1 = (Adu \circ \tau_1) \circ \psi_1$. Since ψ_1 is surjective, $\tau_2 = Adu \circ \tau_1$. □

COROLLARY 3.3

If $B = \mathcal{K}$, then $\tau_1 \overset{w}{\sim} \tau_2$ if and only if there exists an isometry $v \in B(H)$ such that $(Adv,\ \eta_2^{-1} \circ Adv \circ \eta_1,\ id) :\ e_1 \to e_2$ is an extension homomorphism or $(Adv,\ \eta_1^{-1} \circ Adv \circ \eta_2,\ id) :\ e_2 \to e_1$ is an extension homomorphism.

Proof. Since $\tau_2 = Adu \circ \tau_1$ if and only if $\tau_1 = Adu^* \circ \tau_2$, and either u or u^* can lift to an isometry, then the result follows from Theorem 3.2. □

We have the following equivalent conditions for weak isomorphism.

Theorem 3.4 [16, 17]. *Let $e_i : 0 \to B \to E_i \overset{\psi_i}{\to} A \to 0$ be essential extensions with Busby invariant τ_i for $i = 1, 2$. Suppose that σ_i are the inclusion mapping from E_i into $M(B)$ for $i = 1, 2$. Then the following statements are equivalent*:

(1) *The extensions e_1 and e_2 are isomorphic.*
(2) *There exist isomorphisms $\phi : M(B) \to M(B)$ and $\beta : E_1 \to E_2$ such that $\phi(B) = B$ and $\phi \circ \sigma_1 = \sigma_2 \circ \beta$. Moreover, $\tau_1 = \psi^{-1} \circ \tau_2 \circ \gamma$, where $\psi : \mathcal{Q} \to \mathcal{Q}$ is the isomorphism induced by ϕ and $\gamma : A \to A$ is the isomorphism induced by β.*
(3) *There is an isomorphism $\phi : M(B) \to M(B)$ such that $\phi(B) = B$ and $(\psi \circ \tau_1)(A) = \tau_2(A)$, where $\psi : \mathcal{Q} \to \mathcal{Q}$ is the isomorphism induced by ϕ.*

Proof.

(1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) follow from [17, Proposition 3.4].
(2) $\Longrightarrow$ (3). Since $\psi \circ \tau_1 = \tau_2 \circ \gamma$, $\psi \circ \tau_1(A) = \tau_2 \circ \gamma(A)$. Then $\psi \circ \tau_1(A) = \tau_2(A)$ since γ is an isomorphism. □

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References

[1] Blackadar B, K-theory for operator algebras, Second edition, Mathematical Sciences Research Institute Publications, 5 (Cambridge: Cambridge University Press) (1998)
[2] Brown L G, Extensions and the structure of C^*-algebras, *Symposia Mathematica* **20** (1976) 539–566
[3] Brown L G, Douglas R G and Fillmore P A, Extensions of C^*-algebras and K-homology, *Ann. Math. (2)* **105(2)** (1977) 265–324
[4] Clifford A and Preston G, The Algebraic Theory of Semigroups, AMS, 190 Hope Street, Providence, Rhode Island (1961)
[5] Elliott G A, On the classification of C^*-algebras of real rank zero, *J. Reine Angew. Math.* **443** (1993) 179–219
[6] Elliott G A, A classification of certain simple C^*-algebras, II, *J. Ramanujan Math. Soc.* **12(1)** (1997) 97–134

[7] Elliott G A and Gong G, On the classification of C^*-algebras of real rank zero, II, *Ann. Math. (2)* **144(3)** (1996) 497–610

[8] Handelman D, Extensions for AF C^*-algebras and dimension groups, *Trans. AMS* **27(2)** (1982) 537–573

[9] Jensen K K and Thomsen K, Elements of *KK*-theory (Boston, Basel, Berlin: Birkhauser) (1991)

[10] Lin H, On the classification of C^*-algebras of real rank zero with zero K_1, *J. Operator Theory* **35(1)** (1996) 147–178

[11] Lin H, A classification theorem for infinite Toeplitz algebras, Operator algebras and operator theory (Shanghai, 1997) pp. 219–275, Contemp. Math. 228, Amer. Math. Soc. (RI: Providence) (1998)

[12] Lin H, Classification of simple $C*$-algebras of tracial topological rank zero, *Duke Math. J.* **125(1)** (2004) 91–119

[13] Lin H, Classification of simple C^*-algebras and higher dimensional noncommutative tori, *Ann. Math.* **157(2)** (2003) 521–544

[14] Lin H, Unitary equivalences for essential extensions of C^*-algebras, *Proc. Am. Math. Soc.* **137(10)** (2009) 3407–3420

[15] Phillips N C, A classification theorem for nuclear purely infinite simple C^*-algebras, *Doc. Math.* **5** (2000) 49–114

[16] Wegge-Olsen N E, *K*-Theory and C^*-Algebras (Oxford, New York, Tokyo: Oxford University Press) (1994)

[17] Wei C, Equivalence relations of extensions of *AF*-algebras, *Proc. Indian Acad. Sci. (Math. Sci.)* **120(2)** (2010) 209–215

[18] Wei C, Universal coefficient theorem for the stable Ext-groups, *J. Funct. Anal.* **258** (2010) 650–664

[19] Wei C, On the ideals and quotients of *AT*-algebras, *Proc. Indian Acad. Sci. (Math. Sci.)* **118(4)** (2008) 517–524