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Minimal degrees of faithful quasi-permutation representations for direct products of *p*-groups

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Abstract. In [2], the algorithms of $c(G)$, $q(G)$ and $p(G)$, the minimal degrees of faithful quasi-permutation and permutation representations of a finite group G are given. The main purpose of this paper is to consider the relationship between these minimal degrees of non-trivial p-groups H and K with the group $H \times K$.

Keywords. Quasi-permutation representations; p-groups; character theory.

1. Introduction

By quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of G by complex quasi-permutation matrices. Then it is easy to see that

$$c(G) \le q(G) \le p(G),$$

where G is a finite group.

In Theorem 2 and Corollary 2 of [7], it is shown that if H and K are non-trivial nilpotent groups, then $p(H \times K) = p(H) + p(K)$. The purpose of this article is to get similar results on $q(G)$ and $c(G)$, for all non-trivial p-groups H and K, that is

$$q(H \times K) = q(H) + q(K) \quad \text{and} \quad c(H \times K) = c(H) + c(K).$$

By Theorem 3.2 of [3], $p(G) = q(G)$ for each p-group and if $p \neq 2$, then $c(G) = q(G) = p(G)$. Now as $p(H \times K) = p(H) + p(K)$, $q(H \times K) = q(H) + q(K)$. Also in the case p odd, $c(H \times K) = c(H) + c(K)$. Hence it remains to prove the last equation, in the case $p = 2$. For this purpose, we will show that the number of Galois conjugacy classes of complex irreducible characters of a p-group G, needed in the algorithm of $c(G)$ given in Theorem 2.1 is equal to the minimal number of generators of its center.

We mention that the last two equations do not remain true, in the case of arbitrary nilpotent groups H and K. For example, if $G = C_2 \times C_3$, then $c(G) = q(G) = 4$, however $c(C_2) = q(C_2) = 2$ and $c(C_3) = q(C_3) = 3$. In general, the above equations are not true for abelian groups with a direct summand C_6. In fact it is a well-known result that for a non-trivial abelian group $G \cong \prod_{i=1}^{r} C_{m_i}$, where each m_i is a prime power, we have $c(G) = q(G) = T(G) - n$, where $T(G) = \sum_{i=1}^{r} m_i$ and n is maximal such that G has a direct summand C_6^n (see [1]).

2. The main results

We begin by recalling the formula of $c(G)$, which is valid for all finite groups.

Let G be a finite group. Let $\mathcal{C}_i$ for $0 \leq i \leq r$ be the Galois conjugacy classes over $\mathbb{Q}$ of irreducible complex characters of the group G. For $0 \leq i \leq r$, suppose that ψ_i is a representative of the class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Write $\Psi_i = \sum \mathcal{C}_i$ and $K_i = \ker \psi_i$. Clearly $K_i = \ker \Psi_i$. For $I \subseteq \{0, 1, \ldots, r\}$, put $K_I = \bigcap_{i \in I} K_i$. Also, if the n_i's are non-negative integers and $I \subseteq \{1, \ldots, r\}$, then we will use the notation $m(\chi)$ for $\chi = \sum_{i \in I} n_i \Psi_i$ to denote $m(\chi) = -\min\{\sum_{i \in I} n_i \Psi_i(g) : g \in G\}$.

Theorem 2.1. *Let G be a finite group. Then in the above notation*

$$c(G) = \min\Big\{\xi(1) + m(\xi) : \xi = \sum_{i \in I} \Psi_i,\ \ K_I = 1 \ \textit{for}\ \ I \subseteq \{1, \ldots, r\}\ \ \textit{and}\ \ K_J \neq 1\ \textit{if}\ \ J \subset I\Big\}.$$

Proof. See Lemma 2.2 of [3]. □

We express a lemma on p-groups before the principal results.

Lemma 2.2. Let $\mathfrak{A} = \{A_1, \ldots, A_n\}$ be a collection of subgroups of an abelian p-group A such that

(a) $\bigcap_{i=1}^{n} A_i = 1$,
(b) *for all* $1 \leq j \leq n$, $\bigcap_{\substack{i=1 \\ i \neq j}}^{n} A_i \neq 1$,
(c) *for all* $1 \leq j \leq n$, A/A_j *is cyclic*;

then $n = d(A)$, the minimal number of generators of A.

Proof. See Lemma 2 of [6]. □

Theorem 2.3. *Let G be a p-group whose center $Z(G)$ is minimally generated by d elements. Let $c(G) = \xi(1) + m(\xi)$ and $\xi = \sum_{i \in I} \Psi_i$. Let Ψ_i's satisfy the conditions of the algorithm $c(G)$. Then*

(a) $m(\xi) = \frac{1}{p-1} \sum_{i \in I} \Psi_i(1)$,
(b) $|I| = d$.

Proof.

(a) For every $k \in I$, let $I_k = I - \{k\}$. By Theorem 2.1, $C_k = \bigcap_{i \in I_k} \ker \psi_i \neq 1$, so choose $z_k \in C_k \bigcap Z(G)$ of order p. Since $K_I = 1$, we have $z_k \notin \ker \psi_k$. Let $z = \Pi_{i \in I} z_i$. Clearly, $o(z) = p$ and $z \notin \bigcup_{i \in I} \ker \psi_i$. Thus, $\psi_i(z) = \epsilon_i \psi_i(1)$ for all $i \in I$, where ϵ_i is a complex p-th root of 1 and $\epsilon_i \neq 1$. Hence, ϵ_i is the primitive p-th root of unity. Set $H_i = \mathrm{Gal}_{\mathbb{Q}}(\mathbb{Q}(\epsilon_i))$. Since $\epsilon_i^{p-1} + \cdots + \epsilon_i = -1$ and $|H_i| = p - 1$, it follows that $\sum_{\sigma\in H_i}\psi_i(z)^\sigma = -\psi_i(1)$. As $\mathbb{Q}(\epsilon_i)\subseteq\mathbb{Q}(\psi_i)$, we set $\Gamma_i = \mathrm{Gal}_{\mathbb{Q}}(\psi_i)$ and $\Gamma_i' = \mathrm{Gal}_{\mathbb{Q}(\epsilon_i)}(\mathbb{Q}(\psi_i))$. Note that $\mathbb{Q}(\psi_i)$ is a finite degree Galois extension of $\mathbb{Q}$ and the Galois group Γ_i is abelian. So the restriction map $\phi : \Gamma_i \to H_i$ defined by $\phi(\sigma) = \sigma|_{\mathbb{Q}(\epsilon_i)}$ induces an isomorphism $\tilde{\phi} : \Gamma_i/\Gamma_i' \cong H_i$. By definition of ϕ and $\tilde{\phi}$, it follows that $\Psi_i(z) = \sum_{\sigma\in\Gamma_i}\psi_i(z)^\sigma = -d_i\psi_i(1)$, where $d_i = |\Gamma_i'| = \frac{|\Gamma_i|}{p-1}$. Therefore, $\sum_{i\in I}\Psi_i(z) = -\sum_{i\in I} d_i\psi_i(1) = -\frac{1}{p-1}\sum_{i\in I}|\Gamma_i|\psi_i(1) = -\frac{1}{p-1}\sum_{i\in I}\Psi_i(1)$. Since $m(\xi) = -\min\{\sum_{i\in I}\Psi_i(g) : g \in G\}$, we have $m(\xi) \geq \frac{1}{p-1}\sum_{i\in I}\Psi_i(1)$. In the case $p = 2$, since $\frac{1}{p-1} = 1$ and $\sum_{i\in I}\Psi_i(z) = -\sum_{i\in I}\Psi_i(1)$, by definition of $m(\xi)$ and the fact that $\Psi_i(g) \leq \Psi_i(1)$ for each $i \in I$ and $g \in G$, it follows that $m(\xi) = \sum_{i\in I}\Psi_i(1)$. Therefore, let $p > 2$ and $m(\xi) > \frac{1}{p-1}\sum_{i\in I}\Psi_i(1)$. Then by Theorem 3 of [4], there exist subgroups $G_{1i} < G_{2i}$ in G such that $|G_{2i} : G_{1i}| = p$ and $\Psi_i = {\lambda_{1i}}^G - {\lambda_{2i}}^G$, where λ_{hi} is the principal character of G_{hi}. Hence,

$$\xi = \sum_{i\in I}\Psi_i = \sum_{i\in I}({\lambda_{1i}}^G - {\lambda_{2i}}^G) = \sum_{i\in I}{\lambda_{1i}}^G - \sum_{i\in I}{\lambda_{2i}}^G.$$

Since ξ is faithful, so is $\xi + \sum_{i\in I}{\lambda_{2i}}^G$ and thus $\sum_{i\in I}{\lambda_{1i}}^G$. However, by Lemma 5.11 of [5], $\ker {\lambda_{1i}}^G = (G_{1i})_G$, where $(G_{1i})_G = \bigcap_{x\in G}{G_{1i}}^x$ is the core of G_{1i} in G. Hence, $1 = \ker(\sum_{i\in I}{\lambda_{1i}}^G) = \bigcap_{i\in I}(G_{1i})_G$. Therefore,

$$\begin{aligned} c(G) &= \xi(1) + m(\xi) > \frac{1}{p-1}\sum_{i\in I}\Psi_i(1) + \sum_{i\in I}\Psi_i(1) = \frac{p}{p-1}\sum_{i\in I}\Psi_i(1) \\ &= \frac{p}{p-1}\sum_{i\in I}(|G : G_{1i}| - |G : G_{2i}|) = \frac{p}{p-1}\sum_{i\in I}\frac{p-1}{p}|G : G_{1i}| \\ &= \sum_{i\in I}|G : G_{1i}|. \end{aligned} \tag{1}$$

Thus, $c(G) > \sum_{i\in I}|G : G_{1i}|$. On the other hand, let $\Omega_i = \{G_{1i}x : x \in G\}$ and $\Omega = \bigcup_{i\in I}\Omega_i$. So $|\Omega_i| = |G : G_{1i}|$. Let ρ denote the permutation representation of G corresponding to the action of G on Ω by right multiplication. We know that $\ker\rho = \bigcap_{\omega\in\Omega}\mathrm{Stab}_G(\omega)$. But

$$\bigcap_{\omega\in\Omega}\mathrm{Stab}_G(\omega) = \bigcap_{i\in I}\left(\bigcap_{x\in G}{G_{1i}}^x\right) = \bigcap_{i\in I}(G_{1i})_G = 1.$$

So ρ is a faithful permutation (hence quasi-permutation) representation of G with degree $\rho(1) = \sum_{i\in I}|\Omega_i| = \sum_{i\in I}|G : G_{1i}|$. Therefore, by definition of $c(G)$, it follows that $c(G) \leq \sum_{i\in I}|G : G_{1i}|$, which is a contradiction. Thus $m(\xi) = \frac{1}{p-1}\sum_{i\in I}\Psi_i(1)$.

(b) Put $K_i = \ker\Psi_i = \ker\psi_i$, $i \in I$. We show that the set $\{K_i \bigcap Z(G) : i \in I\}$ satisfies the hypothesis of Lemma 2.2. Then it follows that $|I| = d$. Clearly, the subgroups $K_i \bigcap Z(G)$ have the properties (a) and (b) in Lemma 2.2. Also

$$Z(G)/\left(K_i \bigcap Z(G)\right) \cong Z(G)K_i/K_i \leq Z(G/K_i).$$

But by Lemma 2.27 of [5], $Z(G/K_i)$ is a cyclic group. So $Z(G)/(K_i \bigcap Z(G))$ is cyclic. Therefore Lemma 2.2 can be applied. Hence the proof is complete. □

By combining Theorems 2.3 and the next one, we have a programme for finding the minimal degree of a faithful representation of direct products of p-groups by complex quasi-permutation matrices.

Theorem 2.4. *Let $G = H \times K$, where H and K are non-trivial p-groups. Let $m = d(Z(H))$ and $n = d(Z(K))$ and suppose that $\mathfrak{G} = \{G_1, \ldots, G_{m+n}\}$ is a collection of normal subgroups of G such that*

(a) $\bigcap_{i=1}^{m+n} G_i = 1$,

(b) *for all* $1 \le j \le m+n$, $\bigcap_{\substack{i=1 \\ i\neq j}}^{m+n} G_i \neq 1$;

then the subgroups G_i can be reordered such that

$$\left(\bigcap_{i=1}^{m} G_i\right) \bigcap H = 1 \quad \textit{and} \quad \left(\bigcap_{i=1}^{n} G_{m+i}\right) \bigcap K = 1.$$

Proof. It is obvious that

$$d(Z(G)) = d(Z(H)) + d(Z(K)) = m + n.$$

For all $1 \le i \le m+n$, put $\hat{G}_i = \left(\bigcap_{\substack{j=1 \\ j\neq i}}^{m+n} G_j\right) \bigcap Z(G)$. Then by hypothesis, $\hat{G}_i \neq 1$. Also

$$\hat{G}_i \bigcap \prod_{\substack{j=1 \\ j\neq i}}^{m+n} \hat{G}_j \le \hat{G}_i \bigcap G_i = \left(\bigcap_{j=1}^{m+n} G_j\right) \bigcap Z(G) = 1.$$

So $\hat{G}_1 \times \cdots \times \hat{G}_{m+n}$ is a direct product of non-trivial subgroups of $Z(G)$. Since $d(Z(G)) = m + n$, it follows that each $\hat{G}_i$ is a non-trivial cyclic group. Thus $\hat{G}_i \bigcap \Omega_1(Z(G)) = \langle z_i \rangle$, say, a cyclic group and in the elementary abelian group $\Omega_1(Z(G))$, one also deduces that

(i) $G_i \bigcap \Omega_1(Z(G)) = \langle z_j \,:\, 1 \le j \le m+n,\ j \neq i \rangle, \quad 1 \le i \le m+n$

(ii) $\Omega_1(Z(H)) \times \Omega_1(Z(K)) = \Omega_1(Z(G)) = \langle z_i \,:\, 1 \le i \le m+n \rangle$.

By Note 1 after Theorem 1 of [7], we conclude that the G_i can be reordered so that

(iii) $\langle z_i \,:\, 1 \le i \le m \rangle \bigcap \Omega_1(Z(K)) = 1$ and $\langle z_{m+i} \,:\, 1 \le i \le n \rangle \bigcap \Omega_1(Z(H)) = 1$.
So from (i), it follows that

$$\left(\bigcap_{i=1}^{m} G_i\right) \bigcap \Omega_1(Z(G)) = \langle z_{m+i} \,:\, 1 \le i \le n \rangle$$

and

$$\left(\bigcap_{i=1}^{n} G_{m+i}\right) \bigcap \Omega_1(Z(G)) = \langle z_i \,:\, 1 \le i \le m \rangle.$$

Hence by (iii), we deduce

$$\left(\bigcap_{i=1}^{m} G_i\right) \bigcap Z(H) = 1 \quad \text{and} \quad \left(\bigcap_{i=1}^{n} G_{m+i}\right) \bigcap Z(K) = 1.$$

The result is now apparent. □

Theorem 2.5. *Let $G = H \times K$ with H and K non-trivial 2-groups. Then*

$$c(H \times K) = c(H) + c(K).$$

Proof. Suppose that $m = d(Z(H))$ and $n = d(Z(K))$. By Theorem 2.1, $c(G) = m(\xi) + \xi(1)$, where $\xi = \sum_{i \in I} \Psi_i$, $\bigcap_{i=1}^{m+n} \ker \Psi_i = 1$ and $\bigcap_{\substack{i=1 \\ i \neq j}}^{m+n} \ker \Psi_i \neq 1$ for all $1 \leq j \leq m+n$. Also by Theorem 2.3, $|I| = d(Z(G)) = m+n$.

Put $G_i = \ker \Psi_i$, $1 \leq i \leq m+n$. By Theorem 2.4, we conclude that G_i can be reordered so that

$$\bigcap_{i=1}^{m} (H \bigcap G_i) = 1 \quad \text{and} \quad \bigcap_{i=1}^{n} (K \bigcap G_{m+i}) = 1.$$

For each $1 \leq i \leq m+n$, let ψ_i be a representative of the Galois conjugacy class of the irreducible characters of G with the sum Ψ_i. Then $\psi_i = \lambda_i \times \phi_i$ with $\lambda_i \in \mathrm{Irr}(H)$ and $\phi_i \in \mathrm{Irr}(K)$. Hence

$$\ker \lambda_i = H \bigcap G_i \quad \text{and} \quad \ker \phi_i = K \bigcap G_i.$$

For each $1 \leq i \leq m+n$, put

$$\Lambda_i = \sum_{\sigma \in \mathrm{Gal}_{\mathbb{Q}}(\lambda_i)} \lambda_i^{\sigma}, \quad \Phi_i = \sum_{\sigma \in \mathrm{Gal}_{\mathbb{Q}}(\phi_i)} \phi_i^{\sigma}$$

and

$$\xi_1 = \sum_{i=1}^{m} \Lambda_i, \quad \xi_2 = \sum_{i=1}^{n} \Phi_{i+m}.$$

By Theorem 2.1, it follows that

$$c(H) \leq m(\xi_1) + \sum_{i=1}^{m} \Lambda_i(1) \quad \text{and} \quad c(K) \leq m(\xi_2) + \sum_{i=1}^{n} \Phi_{i+m}(1).$$

As $\mathbb{Q}(\lambda_i) \subseteq \mathbb{Q}(\psi_i)$, we get $|\mathrm{Gal}_{\mathbb{Q}}(\lambda_i)| \leq |\mathrm{Gal}_{\mathbb{Q}}(\psi_i)|$. Hence

$$\Lambda_i(1) = |\mathrm{Gal}_{\mathbb{Q}}(\lambda_i)|\lambda_i(1) \leq |\mathrm{Gal}_{\mathbb{Q}}(\psi_i)|\psi_i(1) = \Psi_i(1).$$

Similarly $\Phi_i(1) \leq \Psi_i(1)$. Since $p = 2$, Theorem 2.3(a) specifically says that $m(\xi) = \sum_{i=1}^{m+n} \Psi_i(1)$, and so $\xi(1) + m(\xi) = 2\sum_{i=1}^{m+n} \Psi_i(1)$. Thus

$$\begin{aligned} c(G) &= 2\sum_{i=1}^{m+n} \Psi_i(1) = 2\sum_{i=1}^{m} \Psi_i(1) + 2\sum_{i=m+1}^{m+n} \Psi_i(1) \\ &\geq 2\sum_{i=1}^{m} \Lambda_i(1) + 2\sum_{i=m+1}^{m+n} \Phi_i(1) \geq c(H) + c(K). \end{aligned}$$

The reverse inequality holds obviously for all finite groups. Hence the proof is complete. □

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