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Vertex pancyclicity and new sufficient conditions

KEWEN ZHAO and YUE LIN

Department of Mathematics, Qiongzhou University, Sanya, Hainan 572022, People's Republic of China
E-mail: kwzqzu@yahoo.cn

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Abstract. For a graph G, $\delta(G)$ denotes the minimum degree of G. In 1971, Bondy proved that, if G is a 2-connected graph of order n and $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$. In 2001, Xu proved that, if G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + \delta(G) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$. In this paper, we introduce a new sufficient condition of generalizing degree sum and neighborhood union and prove that, if G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + d(w) \geq n$ for any three vertices x, y, w of $d(x, y) = 2$ and wx or $wy \notin E(G)$ in G, then G is 4-vertex pancyclic or G belongs to two classes of well-structured exceptional graphs. This result also generalizes the above results.

Keywords. Hamiltonian graphs; vertex pancyclic; degree sum; neighborhood union; sufficient conditions.

1. Introduction

We generalize two well-known degree sum and neighborhood union for characterizing Hamiltonian graphs, in particular for vertex-pancyclic. We first give a few definitions and some notation. We consider only finite undirected graphs with no loops or multiples. We denote by $\delta(G)$ the minimum degree of G, and K_h the complete graph of order h. If u is a vertex and H is a subgraph of G, then let $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ be the vertex set of H that are adjacent to vertex u, and if S is a vertex set or subgraph of G, then set $N_H(S) = \cup_{u \in S} N_H(u)$ and set $N[u] = N(u) \cup \{u\}$. Let $G - H$ and $G[S]$ denote the subgraphs of G induced by $V(G) - V(H)$ and S, respectively. If $C_m = x_1x_2 \cdots x_m x_1$ is a cycle of order m, let $N^+_{C_m}(u) = \{x_{i+1} : x_i \in N_{C_m}(u)\}$, $N^-_{C_m}(u) = \{x_{i-1} : x_i \in N_{C_m}(u)\}$, and $N^{\pm}_{C_m}(u) = N^+_{C_m}(u) \cup N^-_{C_m}(u)$, where subscripts are taken modulo m. For a graph G of order n, Ore [7] introduced degree sum condition $d(u) + d(v) \geq n$ for Hamiltonian graphs; Faudree *et al.* [3] introduced neighborhood union $NC = \min\{|N(x) \cup N(y)| : x, y \in V(G), xy \notin E(G)\}$; and Lindquester [6] introduced neighborhood union of each pair of vertices at distance 2 as follows: $NC2 = \min\{|N(x) \cup N(y)| : x, y \in V(G), d(x, y) = 2\}$. In this paper, we introduce new and sufficient condition for generalizing degree sum and neighborhood union as follows: $DNC2 = \min\{|N(x) \cup N(y)| + d(w) : x, y, w \in V(G), d(x, y) = 2, wx \text{ or } wy \notin E(G)\}$. For graphs A and B the joining operator $A \vee B$ of A and B is the graph constructed from A and B by adding all the edges joining the vertices of A and B.

If no ambiguity can arise we sometimes write $N(u)$ instead of $N_G(u)$, δ instead of $\delta(G)$, etc.

If a graph G has a Hamiltonian cycle (a cycle containing every vertex of G), then G is called Hamiltonian. A graph G is said to be pancyclic if G contains cycles of every length k, $3 \leq k \leq n$. For each vertex u of G, if u is contained in cycles of G of every length k, $r \leq k \leq n$, then graph G is called r-vertex pancyclic. Terminologies and notations can be found in [2].

Ore [7] obtained the following well-known Hamiltonian result on sum condition.

Theorem 1.1 [7]. *If G is a graph of order n and $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is Hamiltonian.*

Bondy [1] considered Ore's condition for pancyclic graphs.

Theorem 1.2 [1]. *If G is a 2-connected graph of order n and $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

Faudree *et al.* [3] proved the following result on neighborhood union.

Theorem 1.3 [3]. *If G is a 2-connected graph of order n and $|N(x) \cup N(y)| + \delta \geq n$ for each pair of non-adjacent vertices x, y in G, then G is Hamiltonian.*

Xu [8] proved the following pancyclic result.

Theorem 1.4 [8]. *If G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + \delta \geq n$ for each pair of non-adjacent vertices x, y in G, then G is pancyclic or $G = K_{n/2,n/2}$.*

Lin and Song [5] also proved the following vertex pancyclic result on neighborhood union.

Theorem 1.5 [5]. *If G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + \delta \geq n$ for each pair of non-adjacent vertices x, y of $d(x, y) = 2$ in G, then G is 4-vertex-pancyclic or $G = K_{n/2,n/2}$.*

Recently, some sufficient conditions for the characterizing of Hamiltonian and pancyclic graphs which motivate the work of this paper was shown in [4,9].

In this paper, we introduce a new and sufficient condition and prove the following result which generalizes the above results.

Theorem 1.6. *If G is a 2-connected graph of order $n \geq 6$ and $|N(x) \cup N(y)| + d(w) \geq n$ for any three vertices x, y, w in G of $d(x, y) = 2$ and wx or $wy \notin E(G)$, then G is a 4-vertex pancyclic or $G \in \{K_{n/2,n/2}, G_h : (K_1 \cup K_{n-h-1})\}$.*

Here G_h is some graph of order h, $h = 2, 3$. $G_h : (K_1 \cup K_{n-h-1})$ is the graph obtained from each vertex of graph G_h adjacent to some vertices of two disjoint complete graphs K_1 and K_{n-h-1}. More detailed descriptions can be found in the proof of Lemma 2.1.

Note. If the condition of Theorem 1.2 holds, i.e., each pair of nonadjacent vertices of a graph G satisfy the condition of Theorem 1.2, then any three vertices x, y, w with $xw \notin E(G)$ or $yw \notin E(G)$ satisfy $|N(x) \cup N(y)| + d(w) \geq n$. Since graphs $G_h : (K_1 \cup K_{n-h-1})$

for $h = 2, 3$ are pancyclic and clearly G satisfies condition of Theorem 1.2 which has 3-cycle, Theorem 1.6 implies Theorem 1.2. Also, if each pair of nonadjacent vertices of a graph G satisfy the condition of Theorem 1.4 or Theorem 1.5, then any three vertices x, y, w of $d(x, y) = 2$ satisfy $|N(x) \cup N(y)| + d(w) \geq n$, and thus Theorem 1.6 implies Theorems 1.4 and 1.5 too.

2. Proof of the main result

Obviously, Theorem 1.6 can be obtained immediately by Lemmas 2.1 and 2.6, so we only need to prove Lemmas 2.1 and 2.6.

Lemma 2.1. *If G is a* 2*-connected graph of order $n \geq 6$ and $DNC2 \geq n$, then for each vertex u in G, G has C_3, C_4 or C_4, C_5 all containing u or $G \in \{G_2 : (K_1 \cup K_{n-3}), G_3 : (K_1 \cup K_{n-4}), K_{n/2,n/2}\}$, where G_2 is a subgraph of order* 2, *$G_2 : (K_1 \cup K_{n-3})$ is the graph obtained from each vertex of G_2 adjacent to some vertices of two disjoint complete graphs K_1 and K_{n-3}. G_3 is a subgraph of order* 3 *with $|E(G_3)| = 1$. The description of $G_3 : (K_1 \cup K_{n-4})$ can be found in the following proof.*

Proof. Let u be any vertex of G. Then we consider the following two cases:

Case 1. $d(u) \geq 3$. Let $x, y, z \in N(u)$. Consider the following:

Subcase 1.1. There exist two adjacent vertices in $N(u)$. In this case, G has C_3 containing u. Then, we consider the following two subcases:

Subcase 1.1.1. The induced subgraph $G[N(u)]$ contains path of order 3. In this case, G has C_4 containing u, so the proof of Lemma 2.1 is complete.

Subcase 1.1.2. $G[N(u)]$ does not contain any path of order 3. If $N(u)$ has two vertices who have common neighbor in $G - N[u]$, then clearly, G has C_4 containing u, and so the proof is complete.

Consider the case where each pair of vertices of $N(u)$ have no common neighbor in $G - N[u]$. Without loss of generality, assume $x, y, z \in N(u)$ satisfying $xy \notin E(G)$, together without a path of order 3 in $G[N(u)]$ such that $zx \notin E(G)$ or $zy \notin E(G)$. Without loss of generality, assume $zy \notin E(G)$. (1). If $zx \notin E(G)$, then we easily check that $|N(x) \cup N(y)| + |N(z)| \leq n-1$, a contradiction. (2). If $zx \in E(G)$, clearly, $|N(u)| = 3$. Otherwise, if $|N(u)| \geq 4$, we can check that $|N(x) \cup N(y)| + |N(z)| \leq n - 1$, a contradiction. Then, we have that $G - N[u]$ is a complete subgraph. Otherwise, if there exist two vertices $w, t \in V(G - N[u])$ with $d(w, t) = 2$, then, we have $|N(w) \cup N(t)| + |N(z)| \leq n - 1$, a contradiction. We denote the graph without C_4 of containing u by $G_3 : (K_1 \cup K_{n-4})$, where $V(G_3) = \{x, y, z\}$, $V(K_1) = \{u\}$. Each vertex of K_{n-4} is adjacent and only adjacent to one of $V(G_3)$.

Subcase 1.2. $N(u)$ does not have two adjacent vertices. In this case, by the condition of Lemma 2.1 that $|N(x) \cup N(y)| + |N(z)| \geq n$, we can check that $N(u)$ has two vertices who have common neighbor in $G - N[u]$, so G has C_4 containing u. If G has no C_5 containing u, then we consider the following:

Let $v, w, z \in N(u)$, since $d(w, z) = 2$, by condition of Lemma 2.1 that $|N(w) \cup N(z)| + d(v) \geq n$, we can check $|V(G - N[u])| \geq |N(u)| - 1$. Consider the following two cases:

(1) When $|V(G-N[u])| = |N(u)|-1$. In this case, by $|N(w) \cup N(z)|+d(v) \geq n$ for any three vertices w, z, v in $N(u)$, each vertex of $N(u)$ must be adjacent to every vertex of $G - N(u)$ for example, if $v \in N(u)$ is not adjacent to some vertex of $G - N(u)$, let $w, z \in N(u) \setminus \{v\}$, and then we can check $|N(w) \cup N(z)| + d(v) \leq n - 1$, a contradiction, so $G \in N(u) \vee (G - N(u))$. Since G has no C_5 containing u, $G - N(u)$ is an empty subgraph, and this implies that $G = K_{n/2,n/2}$.

(2) When $|V(G - N[u])| \geq |N(u)|$.

(2)–(1) There exists some v of $N(u)$ that is adjacent to two x, y of $G - N[u]$ with $xy \notin E(G)$. In this case, since G is 2-connected, $G - N[u]$ is not an empty subgraph. Also, since G has no C_5 containing u, so $G - N[u]$ is not a complete subgraph. Thus, there exist two vertices of distance 2 in $V(G - N[u])$. Let x, y be two vertices of distance 2 in $V(G - N[u])$. By $|N(x) \cup N(y)| + d(u) \geq n$, we can check that there are at most $|N(u)| - 1$ vertices of $G - N[u]$ that are not adjacent to x and y, so there are at least $|V(G - N[u])| - (|N(u)| - 1)$ vertices of $G - N[u]$ that are adjacent to x or y.

Since G has no C_5 containing u, so clearly, vertex w is not adjacent to any of $N(x) \cup N(y)$ for each $w \in N(u) \setminus \{u, v\}$. Here v is the notation represented in case (2)–(1).

By the condition of Lemma 2.1, we have that there exists at least a vertex of $N(u) \setminus \{v\}$ that is adjacent to at least two vertices of $V(G - N[u])$. Otherwise, let $w, z \in N(u) \setminus \{v\}$. We can check that $|N(w) \cup N(v)| + d(z) \leq n - 1$, a contradiction. So without loss of generality, let $w \in N(u) \setminus \{v\}$ is adjacent to at least two vertices q, r of $V(G - N[u])$. Consider the following:

(i) *When* $qr \in E(G)$. Since G has no C_5 containing u, so both q, r are not adjacent to both v, z. Here $z \in N(u) \setminus \{v, w\}$, so we have $d(w) \leq |V(G - N[u])| - |N(x) \cup N(y)| + |\{u\}| \leq |V(G - N[u])| - (|V(G - N[u])| - (|N(u)| - 1)) + |\{u\}| \leq |N(u)|$. And we have $|N(t) \cup N(z)| + d(w) \leq (|V(G - N[u])| - |\{q, r\}|) + |\{u\}| + |N(u)| \leq |V(G - N(u) \cup \{u\})| + |N(u)| \leq n - 1$, a contradiction.

(ii) *When* $qr \notin E(G)$. Here also, we can check $d(w) \leq |V(G - N[u])| - |N(x) \cup N(y)| + |\{u\}| \leq |V(G - N[u])| - (|V(G - N[u])| - (|N(u)| - 1)) + |\{u\}| \leq |N(u)|$. Then, by the condition of Lemma 2.1, $|N(q) \cup N(r)| + d(u) \geq n$ and by $|V(G - N[u])| \geq |N(u)|$, we can see that q or r must be adjacent to at least a vertex of $V(G - N[u])$, i.e., there exist two adjacent vertices h, k of $G - N[u]$ with h or k adjacent to w. Without loss of generality, assume k is adjacent to w. Then, since G has no C_5 containing u, both h is not adjacent to both v, z. Then we have $|N(v) \cup N(z)| \leq (|V(G - N[u])| - |\{h\}|) + |\{u\}|$, together with $d(w) \leq |N(u)|$. So we have $|N(v) \cup N(z)| + d(w) \leq (|V(G - N[u])| - 1) + |\{u\}| + |N(u)| \leq |V(G - N(u) \cup \{u\})| + |N(u)| \leq n - 1$, a contradiction.

(2)–(2) $xy \in E(G)$ for any v of $N(u)$ that is adjacent to two x, y of $G - N[u]$. In this case, since G has no C_5 containing u, v of $N(u)$ is adjacent to x, y of $G - N[u]$, and w is not adjacent to both x and y for each $w \in N(u) \setminus \{v\}$. Thus, let $w, z, v \in N(u)$. We can check $|N(w) \cup N(z)| + d(v) \leq n - 1$, a contradiction.

Case 2. $d(u) = 2$. In this case, if $G - N(u) \cup \{u\}$ contains two nonadjacent vertices, then we can choose two vertices x, y in $G - N(u) \cup \{u\}$ of distance 2, but we check $|N(x) \cup N(y)| + |N(u)| \leq n - 1$. This contradicts the condition of the lemma. This contradiction shows that $G - N(u) \cup \{u\}$ is a complete subgraph K_{n-3}, and this implies that $G \in G_2 : (K_1 \cup K_{n-3})$. □

To prove Lemma 2.6, we need the following lemmas.

Lemma 2.2. *If G is a 2-connected graph of order $n \geq 6$, and $DNC_2 \geq n$, C_m is a cycle of order m, u is a vertex of $G - C_m$, $|N_{C_m}(u)| \geq 2$, then the following two hold*:

(1) *If $x_{i+1}, x_{j+1} \in N^+_{C_m}(u)$ and $d(x_{i+1}, x_{j+1}) = 2$ and x_{i+1}, x_{j+1} are not adjacent to any of $N[u] \setminus (C_m)$, then there exists $x_k \in N_{C_m}(u)$ satisfying $x_{i+1}x_{k+1}$ or $x_{j+1}x_{k+1} \in E(G)$.*
(2) *If there exist $x_{i+1}, x_{j+1} \in N^+_{C_m}(u)$ satisfying $d(x_{i+1}, x_{j+1}) \geq 3$ and $\{x_{i+1}, x_{i+2}, \ldots, x_{j-1}\} \cap N_{C_m}(u) = \emptyset$ and x_{j+1} is not adjacent to any of $N[u] \setminus (C_m)$, then there exists at least a vertex x_k in $P = x_{j+1}x_{j+2} \cdots x_i$ such that $x_k \in N(x_{j+1})$ with $x_{k-1}x_{i+1}$ or $x_{k-1}u \in E(G)$.*

Clearly, (1) and (2) imply that $V(C_m) \cup \{u\}$ structures a C_{m+1}.

Proof. Suppose Lemma 2.2(1) is false, i.e., for any $x \in N(u) \cup \{u\}$, when $x \notin V(C_m)$, x is not adjacent to x_{i+1}, x_{j+1}. When $x = x_k \in V(C_m)$, x_{k+1} is not adjacent to x_{i+1}, x_{j+1}. This implies that $|N(x_{i+1}) \cup N(x_{j+1})| \leq n - |N(u) \cup \{u\}|$, so $|N(x_{i+1}) \cup N(x_{j+1})| + d(u) \leq n - 1$, a contradiction. □

Therefore, (1) holds, and we can construct cycle $C_{m+1} = x_i u x_k x_{k-1} \cdots x_{i+1} x_{k+1} x_{k+2} \cdots x_i$ if $x_{k+1}x_{i+1} \in E(G)$ for some $x_k \in N_{C_m}(u)$ or cycle $C_{m+1} = x_j u x_k x_{k-1} \cdots x_{j+1} x_{k+1} x_{k+2} \cdots x_j$ if $x_{k+1}x_{j+1} \in E(G)$ for some $x_k \in N_{C_m}(u)$, the two cycles consisting of u and C_m.

If (2) is false, i.e., for any $x \in N(x_{j+1})$, when $x \notin V(C_m)$, x is not adjacent to vertex u and since $d(x_{i+1}, x_{j+1}) \geq 3$, x is also not adjacent to x_{i+1}. When $x = x_k$ in path $P = x_{j+1}x_{j+2} \cdots x_i$ then $x_{k-1}x_{i+1}, x_{k-1}u \notin E(G)$, i.e., none of $N^-_P(x_{j+1})$ is adjacent to x_{i+1}, u. When x in path $R = x_{i+1}x_{i+2} \cdots x_{j-1}$, $d(x_{i+1}, x_{j+1}) \geq 3$ and $\{x_{i+1}, x_{i+2}, \cdots, x_{j-1}\} \cap N_{C_m}(u) = \emptyset$, so x is not adjacent to u, x_{i+1}, i.e., none of $N^+_R(x_{j+1})$ is adjacent to x_{i+1}, u. Also x_{i+1}, u are not adjacent to x_{i+1}, u. Clearly in the three sets $N^-_P(x_{j+1})$, $N^+_R(x_{j+1})$ and $\{u, x_{i+1}\}$, there are no two sets that have any common vertex, and $|N^+_R(x_{j+1})| + |N^-_P(x_{j+1})| = |N_{Cm}(x_{j+1})| - 1$. Hence we can check $|N(u) \cup N(x_{i+1})| \leq n - |N^+_R(x_{j+1})| - |N^-_P(x_{j+1})| - |\{x_{i+1}, u\}| \leq n - d(x_{j+1}) - 1$. This implies that $|N(u) \cup N(x_{i+1})| + d(x_{j+1}) \leq n - 1$, a contradiction.

Therefore, (2) holds, and we can construct cycle $C_{m+1} =: x_i u x_j x_{j-1} \cdots x_{i+1} x_{k-1} x_{k-2} \cdots x_{j+1} x_k x_{k+1} \cdots x_i$ or $x_j u x_{k-1} x_{k-2} \cdots x_{j+1} x_k x_{k+1} \cdots x_j$, respectively.

Similarly, it is also easy to obtain the following result by considering the reverse direction on C_m from Lemma 2.2.

Lemma 2.3. *If G is a 2-connected graph of order $n \geq 6$, and $DNC_2 \geq n$, C_m is a cycle of order m, u is a vertex of $G - C_m$, $|N_{C_m}(u)| \geq 2$, then the following two hold*:

(1) *If $x_{i-1}, x_{j-1} \in N^-_{C_m}(u)$ and $d(x_{i-1}, x_{j-1}) = 2$ and x_{i-1}, x_{j-1} are not adjacent to any of $N[u] \setminus (C_m)$, then there exists $x_k \in N_{C_m}(u)$ satisfying $x_{i-1}x_{k-1}$ or $x_{j-1}x_{k-1} \in E(G)$.*

(2) *If there exist* $x_{i-1}, x_{j-1} \in N^-_{C_m}(u)$ *satisfying* $d(x_{i-1}, x_{j-1}) \geq 3$ *and* $\{x_{j+1}, x_{j+2}, \ldots, x_{i-1}\} \cap N_{C_m}(u) = \emptyset$ *and* x_{j-1} *is not adjacent to any of* $N[u] \setminus (C_m)$, *then there exists at least a vertex* x_k *in* $P = x_{j+1}x_{j+2}\cdots x_i$ *such that* $x_k \in N(x_{j+1})$ *with* $x_{k+1}x_{i+1}$ *or* $x_{k+1}u \in E(G)$.

To prove Lemma 2.6, we also need the following two propositions:

PROPOSITION 2.4

Let $C_{m+1} =: y_1y_2\cdots y_{m+1}y_1$ *be the cycle of order* $m+1$ *obtained from* (1) *or* (2) *of Lemma* 2.2, *if* $v \in V(G - C_{m+1})$ *is adjacent to some* y_h *in* $V(C_{m+1}) \cap V(C_m)$, *when* $y_h \in \{x_i, x_{i+1}, x_j, x_{j+1}, x_{k-1}, x_k, x_{k+1}\}$ *as described in Lemma 2.2. Then clearly* y_{h+1} *or* $y_{h-1} \in N^{\pm}_{C_m}(v)$. *When* $y_h \notin \{x_i, x_{i+1}, x_j, x_{j+1}, x_k, x_{k+1}\}$, *then clearly* $y_{h+1}, y_{h-1} \in N^{\pm}_{C_m}(v)$.

PROPOSITION 2.5

Let $C_{m+1} =: y_1y_2\cdots y_{m+1}y_1$ *be the cycle of order* $m+1$ *obtained from* (1) *or* (2) *of Lemma* 2.2 *consisting of* u *and* C_m, *where* $u, y \in V(G-C_m)$ *and the condition of Lemma* 2.2 *holds. If* $x_j \in N_{C_m}(y)$ *satisfying this path* $x_{j-1}x_jx_{j+1}$ *of* C_m *is also a path of* C_{m+1}, *and if* $y_h \in N_{C_{m+1}}(y) \setminus \{x_j\}$ *satisfies* y_{h+1} *or* $y_{h-1} \in N^{\pm}_{C_m}(y)$, *then we can obtain* C_{m+2} *consisting of* y *and* C_{m+1}.

That is, under hypothesis of Proposition 2.4, there must exist $y_k, y_h \in N_{C_{m+1}}(y)$ satisfying both y_{k+1}, y_{h+1} or both y_{k-1}, y_{h-1} which are not adjacent to any $N_{G-C_{m+1}}(y)$. (For example, let $x_{j-1}x_jx_{j+1} = y_{i-1}y_iy_{i+1}$, if $y_{h+1} \in N^{\pm}_{C_m}(y)$, then y_{i+1}, y_{h+1} are not adjacent to any $N(y) \setminus V(C_{m+1})$; if $y_{h-1} \in N^{\pm}_{C_m}(y)$, then y_{i-1}, y_{h-1} are not adjacent to any $N(y) \setminus V(C_{m+1})$.) Thus, the cycle C_{m+1} and graph G must satisfy the condition of Lemma 2.2, so we can obtain C_{m+2} consisting of y and C_{m+1} immediately by Lemma 2.2.

Lemma 2.6. *If* G *is a* 2*-connected graph of order* $n \geq 6$ *and* u *is a vertex of* G, *if* $DNC2 \geq n$ *and* C_m, C_{m+1} *are two cycles of containing* u, *then* G *has* C_{m+2} *containing* u.

Proof. Assume on the contrary, that G has no C_{m+2} containing u. Under the hypothesis, clearly, for each vertex x of $G-C_m$, none of $N^{\pm}_{C_m}(x)$ are adjacent to any of $N(x)\setminus V(C_m)$. (Otherwise, if some vertex of $N^{\pm}_{C_m}(x)$ is adjacent to some vertex of $N(x) \setminus V(C_m)$, for example, if $x_{i+1} \in N^+_{C_m}(u)$ or $x_{i-1} \in N^-_{C_m}(u)$ is adjacent to $y \in N(x) \setminus V(C_m)$, then we obtain cycle $x_ixyx_{i+1}x_{i+2}\cdots x_i$ or cycle $x_ixyx_{i-1}x_{i-2}\cdots x_i$, respectively, that are two C_{m+2} consisting of $V(C_m) \cup \{x, y\}$, so this C_{m+2} contains u, a contradiction.) Then we consider the following cases:

Case 1. There exist two distinct vertices x, y in $G - C_m$ such that $|N_{C_m}(x)| \geq 2$ and $|N_{C_m}(y)| \geq 2$.

Subcase 1.1. $N_{C_m}(x) = N_{C_m}(y) = \{x_i, x_j\}$. In this case, we have $xy \in E(G)$. Otherwise, if $xy \notin E(G)$, let $x_k \in V(C_m) \setminus \{x_i, x_j\}$ satisfying $x_k \in \{x_{i+1}, x_{j+1}\}$. Then we can check $|N(x) \cup N(y)| \leq n - |N[x_k] \setminus \{x_i, x_j\}| - |\{x, y\}| = n - |N[x_k]| = n - d(x_k) - 1$. This implies that $|N(x) \cup N(y)| + d(x_k) \leq n - 1$, a contradiction.

Thus, $xy \in E(G)$ holds. Then by Lemma 2.2(1) or Lemma 2.2(2), we can construct C_{m+1} consisting of $V(C_m) \cup \{x\}$. Clearly C_{m+1} contains u and it contains edge xx_i or edge xx_j. Since y is adjacent to x, x_i and x_j, we have C_{m+2} containing u.

Subcase 1.2. $N_{C_m}(x) \neq N_{C_m}(y)$ or $\max\{|N_{C_m}(x)|, |N_{C_m}(y)|\} \geq 3$.

Subcase 1.2.1. $xy \in E(G)$.

Subcase 1.2.1.1. $N_{C_m}(x) \cap N_{C_m}(y) \neq \emptyset$. In this case, let $x_j \in N_{C_m}(x) \cap N_{C_m}(y)$. Then we choose $x_i \in N_{C_m}(x)$ of satisfying $\{x_{i+1}, x_{i+2}, \cdots, x_{j-1}\} \cap N_{C_m}(x) = \emptyset$.

(I) If $d(x_{i+1}, x_{j+1}) \geq 3$, by Lemma 2.2(2), there exists $x_k \in N_{C_m}(x_{j+1})$ satisfying $x_{i+1}x_{k-1}$ or $xx_{k-1} \in E(G)$. When $x_{i+1}x_{k-1}$, then $C_{m+2} = x_i xyx_j x_{j-1} \cdots x_{i+1}x_{k-1}x_{k-2} \cdots x_{j+1}x_k x_{k+1} \cdots x_i$ is a cycle of order $m + 2$ containing u, a contradiction. When xx_{k-1}, then $C_{m+2} = x_j yxx_{k-1}x_{k-2} \cdots x_{j+1}x_k x_{k+1} \cdots x_j$ contains u.

(II) If $d(x_{i+1}, x_{j+1}) = 2$, by Lemma 2.2(1), there exists $x_k \in N_{C_m}(x)$ satisfying $x_{j+1}x_{k+1}$ or $x_{i+1}x_{k+1} \in E(G)$. When $x_{j+1}x_{k+1} \in E(G)$, then $C_{m+2} = x_j yxx_k x_{k-1} \cdots x_{j+1}x_{k+1}x_{k+2} \cdots x_j$ contains u. When $x_{i+1}x_{k+1} \in E(G)$, we first construct $C_{m+1} = y_1 y_2 \cdots y_{m+1} y_1$ consisting of x and C_m via Lemma 2.2(1). Clearly, since $x_j \in N_{C_m}(x) \cap N_{C_m}(y) \setminus \{x_i\}$, the path $x_{j-1}x_j x_{j+1}$ of C_m is also a path of C_{m+1}, together with Proposition 2.5 and since $|N_{C_m}(y)| \geq 2$, there must exist vertex $y_r \in N_{C_{m+1}}(y) \setminus \{x_j\}$ together with x_j Without loss of generality, assume x_j of C_m to be y_h of C_{m+1} satisfying $\{y_{h+1}, y_{h+2}, \ldots, y_{r-1}\} \cap N_{C_{m+1}}(y) = \emptyset$ with both y_{r+1}, y_{h+1} or both y_{r-1}, y_{h-1} are not adjacent to any $N(y) \setminus V(C_{m+1})$, by Lemma 2.2(1) or Lemma 2.2(2) and we obtain C_{m+2} containing u.

Subcase 1.2.1.2. $N_{C_m}(x) \cap N_{C_m}(y) = \emptyset$.

Subcase 1.2.1.2.1. There exist consecutive x_i, x_{i+1} on C_m satisfying $x_i, x_{i+1} \in N_{C_m}(x)$ or $N_{C_m}(y)$.

Without loss of generality, assume $x_i, x_{i+1} \in N_{C_m}(x)$, so we obtain $C_{m+1} = x_i xx_{i+1}x_{i+2} \ldots x_i$. Then for each $x_j \in N_{C_m}(y)$ path $x_{j-1}x_j x_{j+1}$ of C_m is also a path of C_{m+1}, similar to the subcase 1.2.1.1. By Lemma 2.2(1) or Lemma 2.2(2), we can obtain C_{m+2} containing u.

Subcase 1.2.1.2.2. There are no consecutive x_i, x_{i+1} on C_m such that $x_i, x_{i+1} \in N_{C_m}(x)$ or $N_{C_m}(y)$.

In this case using x and C_m we first construct C_{m+1}.

By the hypothesis of G that has no C_{m+2} of containing u and it is no subcase 1.2.1.2.1, together with xy as an edge, for any $x_i \in N_{C_m}(x)$, y is not adjacent to x_{i-1}, x_i, x_{i+1}. Since $|N_{C_m}(y)| \geq 2$, there exists at least a vertex $x_j \in N_{C_m}(y)$ satisfying path $x_{j-1}x_j x_{j+1}$ of C_m which is also a path of C_{m+1}. By (1) or (2) of Lemma 2.2, we can obtain C_{m+2} containing u.

Case 1.2.2. $xy \notin E(G)$.

Subcase 1.2.2.1. There exist consecutive x_i, x_{i+1} on C_m satisfying $x_i, x_{i+1} \in N_{C_m}(x)$ or $N_{C_m}(y)$.

Without loss of generality, assume $x_i, x_{i+1} \in N_{C_m}(x)$, so we have $C_{m+1} = x_i xx_{i+1}x_{i+2} \cdots x_i$. Then if y is adjacent to two consecutive vertices of C_{m+1}, we have C_{m+2} containing u. Otherwise, since this is not Case 1.1, there exists at least a vertex $x_j \in N_{C_m}(y)$ satisfying path $x_{j-1}x_j x_{j+1}$ of C_m which is also the path of C_{m+1}. By Proposition 2.4 and Lemma 2.2, we have C_{m+2} containing u.

Subcase 1.2.2.2. There are no consecutive x_i, x_{i+1} on C_m such that $x_i, x_{i+1} \in N_{C_m}(x)$ or $N_{C_m}(y)$. By Lemma 2.2, we first construct C_{m+1} by x and C_m containing u. Then (1) if y is adjacent to some consecutive two vertices on C_{m+1}, we have C_{m+2} containing u. (2) If y is not adjacent to any consecutive two vertices on C_{m+1}, then we consider the following:

(2–1) When $x_i \in N_{C_m}(x) \cap N_{C_m}(y) \neq \emptyset$, since this is not Case 1.2.2.1, so $xx_{i+1}, yx_{i+1} \notin E(G)$, and by $|N(x) \cup N(y)| + d(x_{i+1}) \geq n$, since any one of $\{x, y\}$ and x_{i+1} do not have common neighbor in $G - C_m$ (otherwise, we have C_{m+2} containing u immediately), so x, y are at least adjacent to three vertices of C_m, together with $|N_{C_m}(x)| \geq 2$ and $|N_{C_m}(y)| \geq 2$, so there must exist $x_h, x_k \in V(C_m) \setminus \{x_i\}$ that are adjacent to x, y, respectively. By $|N(x) \cup N(y)| + d(x_{i+1}) \geq n$, we have $x_{i+1}x_{h+1} \in E(G)$. Since y is not adjacent to any consecutive two vertices on C_{m+1}, so clearly, there exists at least $x_j \in N_{C_m}(y)$ satisfying path $x_{j-1}x_jx_{j+1}$ of C_m which is also the path of C_{m+1}. By Proposition 2.5 and Lemma 2.2, we have C_{m+2} containing u.
(2–2) When $N_{C_m}(x) \cap N_{C_m}(y) = \emptyset$, there must exist $x_h, x_k \in N_{C_m}(x)$ satisfying $\{x_{h+1}, x_{h+2}, \ldots, x_{k-1}\} \cap N_{C_m}(x) = \emptyset$ and $x_j \in N_{C_m}(y)$ satisfying $x_j \in \{x_{h+1}, x_{h+2}, \ldots, x_{k-1}\} \setminus \{x_{h+1}\}$. Then by Lemma 2.2 we first construct C_{m+1} consisting of $V(C_m) \cup \{x\}$ containing u. Then clearly path $x_{j-1}x_jx_{j+1}$ of C_m is also the path of C_{m+1}. By Proposition 2.5 and Lemma 2.2, we have C_{m+2} containing u.

Case 2. There exists a vertex x in $G - C_m$ such that $|N_{C_m}(x)| \geq 2$.

In this case, under the assumption that G does not have any C_{m+2} containing u, we have claim (I) that if $y \in V(G - C_m)$ and $N_{C_m}(y) = \{x_i\}$, then induced subgraph $G[C_m - x_i]$ is a complete subgraph of order $m - 1$.

That is, since G is 2-connected, so there must at least exist a vertex $y \in V(G - C_m)$ such that $|N_{C_m}(y)| = 1$. Let $N_{C_m}(y) = \{x_i\}$, then we first prove $x_{i-1}x_{i+1} \in E(G)$, Otherwise, if $x_{i-1}x_{i+1} \notin E(G)$, since there does not exist C_{m+2} containing u, so if $w \in N[y] \setminus \{x_i\}$ then w is not adjacent to x_{i+1}, x_{i-1}. Hence we can check $|N(x_{i+1}) \cup N(x_{i-1})| \leq n - |N[y] \setminus \{x_i\}| - |\{x_{i+1}, x_{i-1}\}|$, a contradiction.

Similarly, we have $x_{i-1}x_{i+2} \in E(G)$, if $x_{i-1}x_{i+2} \notin E(G)$. Since G has no C_{m+2} containing u, $w \in N[y] \setminus \{x_i\}$ and w is not adjacent to x_{i+2}, x_{i-1} (for example, if $wx_{i+2} \in E(G)$, we have $C_{m+2} = x_{i-1}x_{i+1}x_iywx_{i+2}x_{i+3} \cdots x_{i-1}$ containing u). Hence we have $|N(x_{i+1}) \cup N(x_{i-1})| \leq n - |N[y] \setminus \{x_i\}| - |\{x_{i+1}, x_{i-1}\}|$, a contradiction.

Now, we use induction for the following proof. That is, under the assumption $x_{i-1}x_{i+r} \in E(G)$, we have $x_{i-1}x_{i+r+1} \in E(G)$.

Thus, $x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{i-3}, x_{i-2}$ are all adjacent to x_{i-1}. Then, clearly for each pair x_h, x_k in $C_m \setminus \{x_i\}$, $d(x_h, x_k) \leq 2$, if $x_hx_k \notin E(G)$. If $w \in N[y] \setminus \{x_i\}$ then w is not adjacent to x_h, x_k (for example, $wx_h \in E(G)$, together with $x_{i-1}x_{i+h-1} \in E(G)$, so we have $C_{m+2} = x_{i-1}x_{i+h-1}x_{i+h-2} \cdots x_iywx_{i+h}x_{i+h+1} \cdots x_{i-1}$ containing u). Hence we have $|N(x_h) \cup N(x_k)| \leq n - |N[y] \setminus \{x_i\}| - |\{x_h, x_k\}|$, a contradiction.

Therefore, induced subgraph $G[V(C_m) \setminus \{x_i\}]$ is a complete subgraph of order $m - 1$.

Let $P = y_1y_2 \cdots y_k$ be a path of $G - C_m$ whose two end-vertices y_1, y_k adjacent to two vertices x_i, x_j on C_m with order k of path P is as small as possible. Without loss of generality, assume $N_{C_m}(y_1) = \{x_i\}$, so $G[C_m - x_i]$ is a complete subgraph of order $m - 1$. Then we consider the following cases on order k of path P.

Case 2.1. $k = 2$. In this case, if $|j - i| = 1$ or $m - 1$, then we obtain $C_{m+2} = x_i y_1 y_k x_{i+1} x_{i+2} \cdots x_i$ containing u, a contradiction. If $2 \le |j-i| \le m-2$, by Claim (I), we have $x_{i+1}x_{j+1} \in E(G)$ so we obtain $C_{m+2} = x_i y_1 y_k x_j x_{j-1} \cdots x_{i+1} x_{j+1} x_{j+2} \cdots x_i$ containing u, a contradiction.

Case 2.2. $k = 3$.

Subcase 2.2.1. u is at least adjacent to one of y_1, y_k.

Note. u is the vertex that will contain C_{m+2}.

Since $u \in V(C_m)$ it is equal to $u \in \{x_i, x_j\}$. When $m = 3$, $C_{m+2} = C_5 = x_i y_1 y_2 y_3 x_j x_i$ containing u, a contradiction. When $m \ge 4$ $|j-i| = 1$ without loss of generality, assume $j \ge i$. By Claim (I), $x_{i-1}x_{i-3} \in E(G)$, so we have $C_{m+2} = x_i y_1 y_2 y_3 x_j x_{j+1} \cdots x_{i-3} x_{i-1} x_i$ not containing x_{i-2}, and containing u, a contradiction. If $|j - i| \ge 2$, clearly $x_{i+1}x_{j+2} \in E(G)$, so we have $C_{m+2} = x_i y_1 y_2 y_3 x_j x_{j-1} \cdots x_{i+1} x_{j+2} x_{j+3} \cdots x_i$ not containing x_{j+1}, and containing u, a contradiction.

Subcase 2.2.2. u is not adjacent to y_1, y_k. In this case we consider the following cases on length m of C_m.

When $m = 3$, $C_m = C_3 = x_i x_j u x_i$, so for any $w \in N[u] \setminus \{x_i, x_j\}$, w is not adjacent to y_1, y_3 (for example, if $wy_1 \in E(G)$, then $C_5 = u x_j x_i y_1 w u$ is a cycle of order 5 with containing u). Hence we have $|N(y_1) \cup N(y_3)| \le n - |N[u] \setminus \{x_i, x_j\}| - |\{y_1, y_3\}|$, a contradiction.

When $m \ge 4$. u is adjacent to x_i, together with $G[C_m - x_i]$ which is a complete subgraph. Then it is easy to obtain C_{m+2} containing u, if u is not adjacent to x_i. When $m = 4$, we have $|N(y_1) \cup N(y_3)| \le n - |N(u) \setminus \{x_j\}| - |\{y_1, y_3\}|$, a contradiction. When $m \ge 5$, since $G[C_m - x_i]$ is complete subgraph, it is easy to obtain C_{m+2} containing u, a contradiction.

Case 2.3. $k \ge 4$. Let x_i, x_j in C_m be adjacent to y_1, y_k, respectively, and $|N_{C_m}(y_1)| = |\{x_i\}|$. Let $x_t \in V(C_m) \setminus \{x_i, x_j\}$. So for any $w \in N[x_t] \setminus \{x_i, y_4\}$, $w \notin V(C_m)$, since $k \ge 4$. Then w is not adjacent to y_1, y_3. (Otherwise, if w is adjacent to y_1, in this case we have path $P = wy_1$ of order 2 of $G - C_m$ whose two end-vertices w, y_1 are adjacent to two vertices of C_m, respectively. This contradicts our choice that k is as small as possible. When w is adjacent to y_3, then $wy_3y_4 \cdots y_k$ is a path of order less than k of $G - C_m$ whose two end-vertices w, y_k are adjacent to two vertices of C_m, respectively, a contradiction.) When $w \in V(C_m) \setminus \{x_i\}$, then w is not adjacent to y_1, y_3. (This is because $N_{C_m}(y_1) = \{x_i\}$, so $wy_1 \notin E(G)$. If $wy_3 \in E(G)$, then $y_1y_2y_3$ is a path of order less than k of $G - C_m$ whose two end-vertices y_1, y_3 are adjacent to two vertices x_i, w of C_m, respectively, a contradiction.) Hence we have $|N(y_1) \cup N(y_3)| \le n - |N[x_t] \setminus \{x_i, y_4\}| - |\{y_1, y_3\}|$, a contradiction.

This completes the proof of Lemma 2.6. □

Proof of Theorem 1.6. Under the condition of Theorem 1.6, by Lemma 2.1, G contains cycles C_3, C_4 or C_4, C_5 all containing u for each vertex u in G or $G \in \{G_2 : (K_1 \cup K_{n-3}), G_3 : (K_1 \cup K_{n-4}), K_{n/2,n/2}\}$. By Lemma 2.6, if G has cycles C_m, C_{m+1} containing u, then G has C_{m+2} containing u. Thus, if $G \in \{G_2 : (K_1 \cup K_{n-3}), G_3 : (K_1 \cup K_{n-4}), K_{n/2,n/2}\}$, then G has cycles $C_4, C_5, \ldots, C_n$ all containing u, i.e., G is 4-vertex pancyclic. This completes the proof of Theorem 1.6.

3. Conclusion

Edge-pancyclic graphs have also been studied as a subject in graph theory for the past few years. However, the condition $DNC2 \geq n$ of Theorem 1.6 has not been used for studying edge-pancyclic graph. Clearly, condition $DNC2 \geq n$ that has been used for edge-pancyclic graphs is more difficult than for vertex-pancyclic graphs. Moreover, it is also interesting to generalize the condition $DNC2 \geq n$ of Theorem 1.6 for the research of vertex-pancyclic graphs. For example, what is the result if we consider the condition $|N(x) \cup N(y)| + d(w) \geq n$ for any three vertices x, y, w in G of $d(x, y) = d(y, w) = 2$?

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